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A NOTE ON STRONG CONVERGENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES UNDER SUB-LINEAR EXPECTATION

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ABSTRACT. The sub-linear expectation space is a nonlinear expectation space having advantages of modeling the uncertainty of probability and statistics. Strong convergence for non-additive probabilities or nonlinear expectation are challenging issues which have raised progressive interest recently. In the sub-linear expectation space, we use capacity and sub-linear expectation to replace probability and expectation of classical probability theory. Recently Zhang has proved very important theorems in the sub-linear expectation space which can be looked upon as being extensions of Kolmogorov's three series theorem in classical probability theory. Zhang and Lin obtained Marcinkiewicz's strong law of large numbers for sums of independent random variables under sub-linear expectation space. In addition, they have given a theorem about strong convergence of a random series for independent random variables under sub-linear expectation.

In this paper we investigate the strong convergence for sums of independent random variables under sub-linear expectation and gives also almost surely convergence of sums of independent random variables in capacity. They are extensions of strong convergence of a random series and almost surely convergence of sums for independent random variables in the framework of sub-linear expectation. By using Zhang's result, the proof rests on the methods of proof due to Petrov's result concerning the almost sure convergence of series of independent random variable. As an application Marcinkiewicz's strong law of large number for nonlinear expectation are obtained.

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1. INTRODUCTION

Limit theorems play on an important role in the development of probability theory and statistics. They were widely used in many fields, which only hold in the case of model certainty. The proofs of these limit theorems depend basically on the additivity of probability measures and mathematical expectation in some occasions of model certainty. However, there are uncertainties, such as risk measures, super-hedge pricing and modeling uncertainty in finance. The model certainty assumption is not realistic in many areas of applications since the uncertainty phenomena can not be modeled using additive probabilities or expectations in the model certainty. So the question arises naturally whether the strong convergence of sums of random variables can be still maintained in sub-linear expectation.

Recently, motivated by the risk measures, super-hedge pricing and modeling uncertainty in finance, Peng([9]) initiated the notion of independent and identically distributed random variables under sub-linear expectations. The general framework of the sub-linear expectation space in a general function space by relaxing the linear property of the linear expectation to the sub-additivity and positive homogeneity was introduced by Peng([8-11]). The sub-linear expectation does not depend on the probability measure, provides a very flexible framework to model distribution uncertainty problems and produces many interesting properties different from those of the linear expectations. Under Peng's framework, many limit theorems have been investigating. Recently, Peng([9]) obtained a new central limit theorem under sub-linear expectation and Zhang([17]) studied on Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm and also Zhang and Lin([20]) investigated on Marcinkiewics's strong law of large numbers for nonlinear expectation. Many authors investigated on limit theorems under nonlinear expectation in the wide fields such as the strong law of large number([1,4,7,18]), the law of the iterated logarithm([2,16,17]) and the convergence of the infinite series of random variables([15,19]). For the convergence of the sums of random variables, Zhang([19]) gave three series theorem on the sufficient and necessary conditions for the almost sure convergence of the infinite series $\sum_{n=1}^{\infty} X_n$ under the sub-linear expectation.

In this paper we investigate the strong convergence for sums of independent random variables under sub-linear expectation. Our results is basically form by Petrov's results([12-14]) on the strong convergence of sums of random variables. Our aim is that Petrov's results still hold on some assumptions under sub-linear expectation. Recently, Xu and Zhang([15]) and Zhang([19]) has proved very important theorems in the sub-linear expectation space which can be looked upon as being extensions of Kolmogorov's three series theorem in classical probability theory. By using Zhang's results, the proof rests on the methods of proof due to Petrov's result concerning the almost sure convergence of the infinite series of independent random variable.

This paper is organized as follows: in Section 2, we summarized some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary propositions that are useful to obtain the main results. In Section 3, we give some lemmas and the main results including the proof.

2. Preliminaries

We use the framework and notations of Peng([8-11]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \forall \mathbf{x},\ \mathbf{y} \in R^n$$

for some $C > 0$, $m \in \mathbf{N}$ depending on φ. Let $C_{b,Lip}(R^n)$ denote the linear space of bounded functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C|x - y|, \quad \forall\ \mathbf{x},\ \mathbf{y} \in R^n,$$

for some $C > 0$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 2.1. ([8-11]) A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$ we have

(i) Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$; whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$

Here $\bar{R} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called *a sub-linear expectation space.*

Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] = -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that

$$\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X], \ \ \widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c \ \text{ and } \ \widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$$

for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Let $X = (X_1, X_2, \cdots, X_n)$ be a given n-dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. We define a functional on $C_{l,Lip}(R^n)$ by

$$\mathbb{F}_X[\varphi] := \widehat{\mathbb{E}}[\varphi(X)] : \varphi \in C_{l,Lip}(R^n) \to R.$$

$\mathbb{F}_X$ is called *the distribution of X under $\widehat{\mathbb{E}}$.*

We adopt the following notion of independence and identical distribution for sublinear expctation which is initiated by Peng([8-9,11]).

Definition 2.2. (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. X_1 and X_2 are called *identically distributed*, denoted by $\mathbf{X}_1 =^{\mathrm{d}} \mathbf{X}_2$, if

$$\widehat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)], \quad \forall \varphi \in C_{l,Lip}(R^n),$$

whenever the sub-linear expectation are finite.

Definition 2.3. (Independent) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be *independent to another random vector* $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ *under* $\widehat{\mathbb{E}}$ if

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}\varphi(x, \mathbf{Y})|_{x=\mathbf{X}}], \ \forall \varphi \in C_{l,Lip}(R^m \times R^n),$$

whenever $\overline{\varphi}(x) = \widehat{\mathbb{E}}[|\varphi(x, \mathbf{Y})|] < \infty$ for all x and $\widehat{\mathbb{E}}[|\overline{\varphi}(\mathbf{X})|] < \infty$.

Definition 2.4. (IID random variables) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *independent*, if X_{i+1} is independent to $(X_1, X_2, \cdots, X_i)$ for each $i \geq 1$. It is said to be *identically distributed*, if $X_i =^{\mathrm{d}} X_1$ for each $i \geq 1$.

In Peng([8-11]), the space of the test function φ is $C_{l,Lip}(R^n)$. When the considered random variables have finite moments of each order, i.e., $\widehat{\mathbb{E}}[\varphi(X)] < \infty$ for each $\varphi \in C_{l,Lip}(R^n)$, the test function φ in the definition is limit in the space of bounded Lipschitz function $C_{b,Lip}(R^n)$, since there exists $\varphi_k \in C_{b,Lip}(R^n)$ such that $\varphi_k \downarrow \varphi$ $(\varphi_k(x) = \sup_{y \in R^n} \{\varphi(y) - k|x-y|\})$.

Definition 2.5. (I) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called *a capacity* if $\mathbb{V}(\emptyset) = 0, \mathbb{V}(\Omega) = 1$ and $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{F}$.

(II) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called to be *countably sub-additive* if

$$\mathbb{V}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{V}(A_n) = 1, \ \forall A_n \in \mathcal{F}.$$

Let $(\Omega, \mathcal{H}.\widehat{\mathbb{E}})$ be a sub-linear space. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) = 1 - \mathcal{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement se of A. Then

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g],$$

if $f \leq I_A \leq g, \ f, g \in \mathcal{H}$. It is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X+Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that

$$\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \backslash B) \geq \mathbb{V}(A^c) - \mathbb{V}(B) \ \text{ and } \ \widehat{\mathbb{E}}[-X-Y] \leq \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[-Y].$$

Further, if X is not in $\mathcal{H}$, we define $\widehat{\mathbb{E}}$ by $\widehat{\mathbb{E}}[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leq Y, Y \in \mathcal{H}\}$. Then $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$.

In this paper we only consider the capacity generated by a sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, we define a capacity:

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \forall A \in \mathcal{F}$$

and also define the conjugate capacity:

$$\mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}.$$

It is clear that $\mathbb{V}$ is a sub-additive capacity and $\mathcal{V}(A) = \widehat{\mathcal{E}}[I_A]$.

The following representation theorem for sub-linear expectation is very useful(see Peng([9-11]) for the Proof):

Proposition 2.1. *Let* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *be a sublinear expectation space.*
(I) *There exists a family of finitely additive probability measures* $\{P_\theta : \theta \in \Theta\}$ *defined on* $(\Omega, \mathcal{F})$ *such that for each* $X \in \mathcal{H}$

$$\widehat{\mathbb{E}}[X] = \max_{\theta \in \Theta} E_{P_\theta}[X].$$

(II) *For any fixed random variables* $X \in \mathcal{H}$*, there exists a family of probability measures* $\{\mu_\theta\}_{\theta\in\Theta}$ *defined on* $(R, \mathcal{B}(R))$ *such that for each* $\varphi \in C_{l,Lip}(R)$*,*

$$\widehat{\mathbb{E}}[\varphi(X)] = \sup_{\theta\in\Theta} \int_R \varphi(x)\mu_\theta(dx).$$

The following proposition can be found in Proposition 2.1 in Chen([3]).

Proposition 2.2. (Chebyshev's inequality) *Let* X *be a real measurable random variable in sub-linear expectation space* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$*. Let* $f(x) > 0$ *be a nondecreasing function on* R*. Then for any* x*,*

$$\mathbb{V}(X \geq x) \leq \frac{\widehat{\mathbb{E}}[f(x)]}{f(x)}.$$

Let $f(x) > 0$ *be an even function and nondecreasing on* $(0, \infty)$*. Then for any* $x > 0$*,*

$$\mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}[f(x)]}{f(x)}.$$

We need the following notation. As in [13-14], the set of functions $\psi(x)$ such that each $\psi(x)$ is positive and nondecreasing for $x > x_0$ for some x_0 and the series $\sum 1/(n\psi(n))$ is convergent will be denoted by Ψ_c. Functions x^α and $\log^{1+\alpha} x$ for any $\alpha > 0$ are examples of the class of functions Ψ_c.

For any random variable X and constant c, define $X^c = (-c) \vee (X \wedge c)$.

3. Main Results

In this section, we give the main results. We first recall some related important lemmas in sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Then we give a theorem about the convergence of a random series, from which we can deduce a theorem similar to Kolmogorov's three series theorem in classical probability theory.

Lemma 3.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables on* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$*. Suppose that* $\mathbb{V}$ *is countably sub-additive. Then* $\sum_{n=1}^\infty$ *converges almost surely in capacity if the following three conditions hold for some* $c > 0$*;*

(S_1) $\sum_{n=1}^\infty \mathbb{V}(|X_n| > c) < \infty$,
(S_2) $\sum_{n=1}^\infty \widehat{\mathbb{E}}[X_n^c]$ *and* $\sum_{n=1}^\infty \widehat{\mathbb{E}}[-X_n^c]$ *are both convergent,*
(S_3) $\sum_{n=1}^\infty \widehat{\mathbb{E}}[(X_n^c - \widehat{\mathbb{E}}[X_n^c])^2)] < \infty$ *and* $\sum_{n=1}^\infty \widehat{\mathbb{E}}[(X_n^c + \widehat{\mathbb{E}}[-X_n^c])^2)] < \infty$
Conversely, if $\mathbb{V}$ *is continuous and* $\sum_{n=1}^\infty X_n$ *is convergent almost surely in capacity* $\mathbb{V}$*, then* $(S_1), (S_2), (S_3)$ *will hold for all* $c > 0$

Proof. The proof of Lemma 3.1 can be found in Zhang([19]). □

The following theorem is concerned about the strong convergence of a random series under sub-linear expectation in view of Chung's result([5]) and Petrov's results([12-14]). It is used in applications such as Chung's strong law of large numbers under sub-linear expectation and Marcinkiewicz's strong law of large numbers under sub-linear expectation.

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, and let $\mathbb{V}$ be countably sub-additive. Let $\{a_n, n \geq 1\}$ be a positive increasing sequence tending to infinity. Let $\{g_n(x), n \geq 1\}$ be a sequence of locally Lipschhitz, even functions, positive and non-decreasing for $x > 0$. For each n, let one of the following two conditions hold:*

(a) $x/g_n(x)$ is non-decreasing for $x > 0$,

(b) $x/g_n(x)$ and $g_n(x)/x^2$ are non-increasing for $x > 0$.

If the series

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[g_n(X_n)]}{g_n(a_n)} < \infty \tag{1}$$

is satisfied. Moreover, suppose that

$$\widehat{\mathbb{E}}[X_n] = 0 = \widehat{\mathcal{E}}[X_n], \quad n = 1, 2, \cdots$$

when condition (b) is satisfied. Then the series

$$\sum_{n=1}^{\infty} \frac{X_n}{a_n} \tag{2}$$

converges almost surely in capacity $\mathbb{V}$.

Proof. It would be sufficient for our purpose to get the conditions $(S_1), (S_2), (S_3)$ of Lemma 3.1. By Proposition 2.2, we have

$$\mathbb{V}(|X_n| \geq a_n) \leq \frac{\widehat{\mathbb{E}}[g_n(X_n)]}{g_n(a_n)}.$$

Hence by (1),

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_n| \geq a_n) \leq \sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[g_n(X_n)]}{g_n(a_n)} < \infty.$$

It is easy to prove that $\sum_{n=1}^{\infty} \mathbb{V}(|X_n| \geq a_n) < \infty$ is equivalent to for any $c > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_n| \geq ca_n) < \infty.$$

Therefore it follows that condition (S_1) in Lemma 3.1 is proved.

Next we will prove condition (S_2) in Lemma 3.1. Define

$$X_n^c = (-a_n) \vee (X_n \wedge a_n) \quad \text{for} \quad n = 1, 2, \cdots .$$

We just need to prove

$$\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n^c] < \infty.$$

Because of considering $\{-X_n, n \geq 1\}$ instead of $\{X_n, n \geq 1\}$ in Lemma 3.1, we can obtain

$$\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[-X_n^c] < \infty.$$

By Proposition 2.1, for any fixed random variable $X \in \mathcal{H}$, there exists a family of probability measures $\{\mu_\theta\}_{\theta\in\Theta}$ defined on $(R, \mathcal{B}(R))$ such that for each $\varphi \in C_{l,Lip}(R)$

$$\widehat{\mathbb{E}}[\varphi(X)] = \sup_{\theta\in\Theta} \int_R \varphi(x)\mu_\theta(dx).$$

Hence if condition (a) is satisfied, then

$$\begin{aligned}
|\widehat{\mathbb{E}}[X_n^c]| &\leq \sup_{\theta\in\Theta} \left| \int_{|x|\leq a_n} x\mu_{n,\theta}(dx) \right| \\
&\leq \sup_{\theta\in\Theta} \int_{|x|\leq a_n} \frac{a_n}{g_n(a_n)} g_n(|x|)\mu_{n,\theta}(dx) \\
&\leq \frac{a_n}{g_n(a_n)} \sup_{\theta\in\Theta} \int_{|x|\leq a_n} g_n(x)\mu_{n,\theta}(dx) \\
&\leq \frac{a_n}{g_n(a_n)} \widehat{\mathbb{E}}[g_n(X_n)].
\end{aligned}$$

Using the fact that $\widehat{\mathbb{E}}[X_n] = 0 = \widehat{\mathcal{E}}[X_n]$ for $n = 1, 2, \cdots$ and $X_n - X_n^c \leq (|X_n| - a_n)^+ \leq |X_n| \cdot I_{\{|X_n|>a_n\}}$ and also $|X_n| \cdot I_{\{|X_n|>a_n\}} \in C_{l,Lip}(R)$ since $X_n^c = (-a_n) \vee (X_n \wedge a_n)$, we have

$$\begin{aligned}
\widehat{\mathbb{E}}[X_n^c] &= \widehat{\mathbb{E}}[X_n^c] - \widehat{\mathbb{E}}[X_n] \;\leq\; \widehat{\mathbb{E}}[X_n^c - X_n] \;\leq\; \widehat{\mathbb{E}}[|X_n - X_n^c|] \\
&\leq \widehat{\mathbb{E}}[|X_n| \cdot I\{|X_n| > a_n\}]
\end{aligned}$$

and also

$$\widehat{\mathcal{E}}[X_n^c] \leq \widehat{\mathcal{E}}[|X_n| \cdot I\{|X_n| > a_n\}].$$

If condition (b) holds, the only modification is that we have

$$\begin{aligned}
|\widehat{\mathbb{E}}[X_n^c]| &\leq \sup_{\theta\in\Theta} \int_{|x|>a_n} |x|\mu_{n,\theta}(dx) \\
&\leq \sup_{\theta\in\Theta} \int_{|x|>a_n} \frac{a_n}{g_n(a_n)} g_n(|x|)\mu_{n,\theta}(dx) \\
&\leq \frac{a_n}{g_n(a_n)} \sup_{\theta\in\Theta} \int_{|x|>a_n} g_n(x)\mu_{n,\theta}(dx) \\
&\leq \frac{a_n}{g_n(a_n)} \widehat{\mathbb{E}}[g_n(X_n)],
\end{aligned}$$

since now $g_n(a_n)/a_n \leq g_n(x)/x$ for $|x| \geq a_n$ by assumption (b). Thus we obtain

$$\sum_{n=1}^{\infty} \widehat{\mathbb{E}}\left[\frac{X_n^c}{a_n}\right] < \infty.$$

Finally, we just prove condition (S_3) in Lemma 3.1. Suppose that $g_n(x)$ satisfies condition (a). Then for $|x| \le a_n$, we have

$$\frac{x^2}{a_n^2} \le \frac{g_n^2(x)}{g_n^2(a_n)} \le \frac{g_n(x)}{g_n(a_n)}.$$

But if n is such that (b) holds, then for this same region

$$\frac{x^2}{g_n(x)} \le \frac{a_n^2}{g_n(a_n)}$$

for $|x| \le a_n$, we have

$$\frac{x^2}{a_n^2} \le \frac{g_n(x)}{g_n(a_n)}$$

for all n. Therefore

$$\begin{aligned}\widehat{\mathbb{E}}[X_n^{c\,2}] &= \sup_{\theta\in\Theta}\int_{|x|\le a_n} x^2\mu_{n,\theta}(dx)\\ &\le \frac{a_n^2}{g_n(a_n)}\sup_{\theta\in\Theta}\int_{|x|\le a_n} g_n(x)\mu_{n,\theta}(dx)\\ &\le \frac{a_n^2}{g_n(a_n)}\widehat{\mathbb{E}}[g_n(X_n)]\end{aligned}$$

By (1), we have

$$\sum_{n=1}^{\infty}\widehat{\mathbb{E}}\left[\left(\frac{X_n^c}{a_n}\right)^2\right] < \infty.$$

Therefore, from Lemma 3.1, it follows that the series (2) is convergent a.s in capacity $\mathbb{V}$, and the proof is complete. □

The following lemma 3.3 generalizes the Abel-Dini Theorem (see [6]) and is easily proved in the same way.

Lemma 3.3. *Let $\{a_n\}$ be a sequence of non-negative numbers, $A_n = \sum_{k=1}^n a_k$ and $A_n \to \infty$. Then the series $\sum_{n=1}^{\infty}(a_n/A_n\psi(A_n))$ is convergent for any function $\psi \in \Psi_c$.*

The following theorem gives a criterion for the almost surely convergence of sums of independent random variables in capacity. It is to generalize Petrov's Theorem([13-14]) under sub-linear expectation.

Theorem 3.4. *Let $\{X_n, n \ge 1\}$ be a sequence of independent random variables in the sublinear expectation space, and let $\mathbb{V}$ be countably sub-additive. Let $g(x)$ be a locally Lipschitz sequence of even functions, positive and strictly increasing for $x > 0$ and is such that $g(x) \to \infty$ as $x \to \infty$. Let either of the following two conditions hold:*

(a) $x/g(x)$ is non-decreasing for $x > 0$,

(b) $x/g(x)$ and $g(x)/x^2$ are non-increasing for $x > 0$.

Further, let

$$\widehat{\mathbb{E}}[g(X_n)] < \infty, \quad n = 1, 2, \cdots \tag{3}$$

and

$$A_n = \sum_{k=1}^{n} \widehat{\mathbb{E}}[g(X_k)] \to \infty. \tag{4}$$

Moreover, suppose that

$$\widehat{\mathbb{E}}[X_n] = 0 = \widehat{\mathcal{E}}[X_n], \quad n = 1, 2, \cdots \tag{5}$$

when condition (b) *is satisfied. Then*

$$S_n = o(g^{-1}(A_n\psi(A_n))) \qquad \text{a.s.} \quad \mathbb{V}$$

for any function $\psi(x) \in \Psi_c$*. Here* $g^{-1}(\cdot)$ *stands for the function inverse to* $g(\cdot)$*.*

Proof. We shall use the idea of the proof of the theorem in [13,14]. Let $\psi(x) \in \Psi_c$. By the hypotheses of g in the theorem, it guarantee the existence of a positive inverse function $g^{-1}(x)$ for all sufficiently large x. Let $a_n = g^{-1}(A_n\psi(A_n))$, then $a_n \uparrow \infty$ as $n \to \infty$. From (4) and Lemma 3.3, it follows that

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[g(X_n)]}{A_n\psi(A_n)} < \infty.$$

On the other hand, since the equation $g(a_n) = A_n\psi(A_n)$, by using Theorem 3.2, the series $\sum X_n/a_n$ is convergent a.s. in capacity $\mathbb{V}$. By Kronecker's lemma, we have

$$S_n/g^{-1}(A_n\psi(A_n)) \to 0 \qquad \text{a.s.} \quad \mathbb{V},$$

which completes the proof. □

Remark. It is open problem that : *Let* $g(x)$ *be an even continuous function, positive and strictly increasing for* $x > 0$ *and such that* $g(0) = 0$ *and* $g(x) \to \infty$ *as* $x \to \infty$*. For any function* $\psi(x) \in \Psi_d$*, there exists a sequence of independent random variables* $\{X_n, n \geq 1\}$ *satisfying condition* $(3), (4), (5)$ *and*

$$\limsup_{n\to\infty} \frac{|S_n|}{g^{-1}(A_n\psi(A_n))} > 0 \qquad \text{a.s.} \quad \mathbb{V},$$

where $S_n = \sum_{i=1}^n X_i$*.*

We have the following corollaries. It is to state the consequence of Theorem 3.2 corresponding $g_n(x) = x$ and $a_n = n^{1/p}$, $n = 1, 2, \cdots$. The following Corollary 3.5 is very similar to Marcinkiew's strong law of large numbers for nonlinear expectations(see [15],[20]).

Corollary 3.5. *Suppose* $\{X_n, n \geq 1\}$ *is a sequence of independent and identical random variable in the sub-linear expectation space with* $\widehat{\mathbb{E}}[X_1] = 0 = \widehat{\mathcal{E}}[X_1]$ *and* $\mathbb{V}$ *is countably sub-additive. If the series*

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbb{E}}[X_n^2]}{n^{2/p}} < \infty.$$

(I) For $0 < p < 1$*, then*

$$\frac{S_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad \mathbb{V}$$

(II) For $1 < p < 2$, *suppose* $\lim_{a\to\infty} \widehat{\mathbb{E}}[(|X_1| - a)^+] = 0$, *then*

$$\frac{S_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad \mathbb{V}$$

It is considerably simple to state the consequence of Theorem 3.2 and Theorem 3.4 corresponding to the $g(x) = |x|^p$ for order of growth of sums of independent random variables under sub-linear expectation.

Corollary 3.6. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables under the sub-linear expectation space and* $\mathbb{V}$ *is countably sub-additive. Let* $S_n = \sum_{k=1}^n X_k$ *and let*

$$\widehat{\mathbb{E}}[|X_n|^p] < \infty, \; n = 1, 2, \cdots$$

for some positive $p \leq 2$. *Put*

$$A_n = \sum_{k=1}^{n} \widehat{\mathbb{E}}[|X_n|^p].$$

If

$$A_n \to \infty \qquad as \quad n \to \infty.$$

Then if $0 < p < 1$, *we have* $S_n = o((A_n\psi(A_n))^{1/p})$ a.s. *in capacity* $\mathbb{V}$ *for any function* $\psi(x) \in \Psi_c$, *if* $1 \leq p \leq 2$, *the assertion is true under the condition* (5).

4. Conclusions

This paper proves a theorem about strong convergence of a random series for independent random variables under sub-linear expectation, and gives also a criterion for almost surely convergence of sums of independent random variables in capacity. They are extensions of strong convergence of a random series and almost surely convergence of sums for independent random variables in the framework of sub-linear expectation.

Limit theorems for non-additive probabilities or non-linear expectations are challenging issues which have raised progressive interest recently. Our results include Marcinkiewicz's strong law of large numbers for nonlinear expectations and gives a criterion for order of growth of sums of independent random variables under sub-linear expectation. It is very useful in finance when there is ambiguity. But the constraint conditions in this paper are very strong, such as the condition $\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[g_n(X_n)]/g_n(a_n) < \infty$ and independence under sub-linear expectation. How can we weaken the constraint conditions? Can we find the conditions to hold the open problem in Remark? We will investigate them in the future work.

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