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Control of the wireless power transfer circuit

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ABSTRACT. We consider one of the most popular wireless power transfer circuits, formalize it as the optimal control problem and apply Pontryagin maximum principle for obtaining its solution.

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KEYWORDS AND PHRASES. Wireless power transfer, optimal control problem, Pontryagin maximum principle.

1. Introduction

Most of devices that are driven by electric power are usually fed via electric wires connected to AC power supply. On the other hand, some of electric devices are required or designed to be fed without electric wires — in the way of called wireless power transfer (WTP).

Many papers concerning wireless power transfer use resonant phenomena [2, 5, 9, 10]. In general, if input frequency is adjusted to the resonant frequency, the amplitude of current or voltage will be maximal [9]. Since the power of resistive load is determined by its current or voltage, the power will be maximal at the resonance. As for wireless power transfer, it is one of significant specifications to transfer power as much as possible. Then it is naturally to try to maximize transmitted power with resonant phenomena. On the other hand, efficiency is another significant specification on wireless power transfer. However, efficiency is defined as a ratio with two different powers, and it cannot be concluded that efficiency is always maximal when either power is maximal. Therefore, efficiency has no straightforward relation with resonance in contrast with the relation between resonance and power.

In this paper, we consider another approach to control of the wireless power transfer circuit based on maximum principle. We explore a typical for wireless power transfer sample of the circuit, which can be formalized as the optimal control problem: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1,3,4,8]. Depending on objective function and constraints, we apply corresponding maximum principle and obtain optimal process in the class of piecewise continuous control functions. We examine and analysis it in comparison with resonant case.

2. Statement of the problem

Optimal control theory began to take shape as a mathematical discipline in the 1950's and is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature. The theory of optimal control absorbed all previous achievements in the calculus of variations, and was enriched with new results and new content. The central results of the theory – the Pontryagin Maximum Principle became widely known in the scientific and engineering community, and is now widely used in various academic fields [1, 3, 6, 7, 8].

In this paper, we study a typical circuit for WPT depicted below [2]. Despite of the placement of two coils in the figure, we assume they have a common central axis.

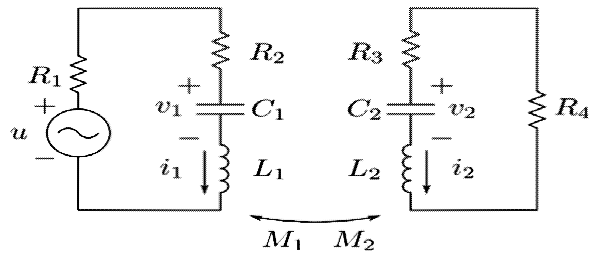

Figure 1. A Wireless Power Transfer Circuit

The resistor R_1 is supposed to represent an internal impedance of the power supply, and R_4 - the load. R_2, R_3, C_1, C_2 are parasitic factors of transmitting and receiving coils. This circuit is mathematically modelled as the following system of state equations:

$$\dot{x} = Ax + Bu, \quad \text{where } A = \frac{1}{\Delta}\begin{pmatrix} 0 & 0 & \frac{\Delta}{C_1} & 0 \\ 0 & 0 & 0 & \frac{\Delta}{C_2} \\ -L_2 & M_2 & -(R_1+R_2)L_2 & (R_3+R_4)M_2 \\ M_1 & -L_1 & (R_1+R_2)M_1 & -(R_3+R_4)L_1 \end{pmatrix}, \quad B = \frac{1}{\Delta}\begin{pmatrix} 0 \\ 0 \\ L_2 \\ -M_1 \end{pmatrix},$$

$$\Delta = L_1 L_2 - M_1 M_2.$$

Here $x = \begin{pmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{pmatrix}$ is the state vector. v_1, v_2, i_1, i_2 are corresponding voltage and amperage (current). Control u is the input that we define as a piecewise continuous function with the range in compact U. If we take $U=[-1, 1]$, then sinusoidal input $u = \sin \omega t$, used in many applications, is a particular case of such control. Both $x = x(t)$ and $u = u(t)$ are functions of time $t \in [0, t_1]$. t_1 is fixed or not fixed moment of time.

The basic characteristic of the wireless power transfer circuit is its efficiency

$$\eta = \frac{\overline{P}_4}{\overline{P}_1},$$

where $\overline{P}_1 = \frac{1}{t_1}\int_0^{t_1} (u(t) - R_1 i_1(t)) i_1(t)\,dt$ and $\overline{P}_4 = \frac{1}{t_1}\int_0^{t_1} i_2^2(t) R_4(t)\,dt$ are average power of input and output accordingly. In terms of state variables the efficiency has the form

$$\eta = \frac{\int_0^{t_1} x_4^2 R_4 dt}{\int_0^{t_1} (u - R_1 x_3) x_3 dt}.$$

Then, depending on objective function and restrictions of the model, we can formulate different optimal control problems. In particular, we have

Problem 1.

$$\eta = \frac{\int_0^{t_1} x_4^2 R_4 dt}{\int_0^{t_1} (u - R_1 x_3) x_3 dt} \to \max,$$

$$\dot{x} = Ax + Bu, \; x(t_0) = x^0, \; u(t) \in U, \; t \in [0, t_1].$$

Problem 2.

$$\begin{aligned} &\int_0^{t_1} x_4^2 R_4 dt \to \max, \\ &\dot{x} = Ax + Bu, \\ &x(t_0) = x^0, \; (u - R_1 x_3) x_3 \le \alpha, \\ &u(t) \in U, \; t \in [0, t_1]. \end{aligned}$$

Problem 3.

$$\begin{aligned} &x_4^2(t_1) \to \max, \\ &\dot{x} = Ax + Bu, \\ &x(t_0) = x^0, \; u(t) \in U, \; t \in [0, t_1]. \end{aligned}$$

Problem 4.

$$\begin{aligned} &t_1 \to \min, \\ &\dot{x} = Ax + Bu, \\ &x(t_0) = x^0, \; \int_0^{t_1} x_4^2 R_4 dt = \beta \int_0^{t_1} (u - R_1 x_3) x_3 dt, \\ &u(t) \in U, \; t_1 \ge 0, \end{aligned}$$

and others.

3. Solution of the Simplest Problem

We consider problem 3. For fixed t_1 we classify it as the simplest problem [1]:

$$J = \Phi(x(t_1)) \to \min$$

$$\dot{x} = f(x,u,t),$$
$$x(t_0) = x^0, \quad (1)$$
$$u \in U,\ t \in [t_0, t_1].$$

The pair $x(t), u(t)$ that satisfies all conditions of the simplest problem except, possibly, the first condition is said to be a *process*. A process $x(t), u(t)$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t)$, the following inequality holds

$$\Phi(x(t_1)) \le \Phi(\tilde{x}(t_1)).$$

Function $u(t)$ is called *optimal control* and $x(t)$ - *optimal trajectory*. We seek optimal control in the class of piecewise continuous on segment $[t_0, t_1]$ functions. The goal consists in determining the optimal process of the problem (1).

The necessary conditions of optimality of the process $x(t)$, $u(t)$ for (1) is given by Pontryagin Maximum Principle [1, 4].

Theorem (Maximum Principle). Let $x(t)$, $u(t)$ be a solution of the simplest problem of optimal control (1). Then there exists continuous solution $\psi(t)$ of the conjugate Cauchy problem

$$\dot{\psi} = -H_x(\psi, x, u, t) \quad (2)$$
$$\psi(t_1) = -\Phi_{x(t_1)}(x(t_1))$$

such that $H(\psi(t), x(t), u(t), t) = \max\limits_{v \in U} H(\psi(t), x(t), v, t), \quad t_0 \le t \le t_1$.

Here $H(\psi, x, u, t) = \sum\limits_{j=1}^{n} \psi_j f_j(x,u,t)$ is Hamiltonian. Since mathematical model in the problem (1) for wireless power transfer circuit is linear, then Hamiltonian becomes

$$H(\psi, x, u, t) = \psi^T A x + \psi^T B u$$

and Cauchy problem (2) arrives at

$$\dot{\psi} = -A^T \psi, \quad (3)$$
$$\psi(t_1) = (0, 0, 0, 2x_4(t_1)).$$

Solution $\psi(t)$ of this problem depends on $x_4(t_1)$ and contains polynomial, exponential, sine and cosine functions. General solution of the conjugate system, obtained by WolframAlpha program, has the form

$$\psi_1(t) = k_1\left(-\sum_w \frac{be^{tw}w^2 - be^{tw}fw + ae^{tw}gw}{4w^3 - 3fw^2 - 3pw^2 + 2dhw - 2gmw + 2anw + 2fpw - dfh + cgh + bmn - anp}\right)$$

$$+k_2\left(\sum_w \frac{-e^{tw}w^3 + e^{tw}fw^2 + e^{tw}pw^2 - de^{tw}hw + e^{tw}gmw - e^{tw}fpw + de^{tw}fh - ce^{tw}gh}{-4w^3 + 3fw^2 + 3pw^2 - 2dhw + 2gmw - 2anw - 2fpw + dfh - cgh - bmn + anp}\right)$$
$$+k_3 h\left(\sum_w \frac{-be^{tw}f + ae^{tw}g + be^{tw}w}{4w^3 - 3fw^2 - 3pw^2 + 2dhw - 2gmw + 2anw + 2fpw - dfh + cgh + bmn - anp}\right)$$

$$-k_4\left(\sum_w \frac{-ae^{tw}w^2-be^{tw}mw+ae^{tw}pw+bce^{tw}h-ade^{tw}h}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right),$$

$$\psi_2(t)=k_1\left(-\sum_w \frac{-de^{tw}w^2+de^{tw}fw-ce^{tw}gw+bce^{tw}n-ade^{tw}n}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$+k_2n\left(\sum_w \frac{de^{tw}m-ce^{tw}p+ce^{tw}w}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$-k_3\left(\sum_w \frac{e^{tw}w^3-e^{tw}fw^2-e^{tw}pw^2-e^{tw}gmw+ae^{tw}nw+e^{tw}fpw+be^{tw}mn-ae^{tw}np}{-4w^3+3fw^2+3pw^2-2dhw+2gmw-2anw-2fpw+dfh-cgh-bmn+anp}\right)$$

$$-k_4\left(\sum_w \frac{ce^{tw}w^2+de^{tw}mw-ce^{tw}pw}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right),$$

$$\psi_3(t)=k_1\left(\sum_w \frac{bne^{tw}w-ge^{tw}w^2}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$-k_2n\left(\sum_w \frac{dhe^{tw}-pwe^{tw}+e^{tw}w^2}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$-k_3h\left(\sum_w \frac{bne^{tw}-gwe^{tw}}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$-k_4\left(\sum_w \frac{e^{tw}w^3-pe^{tw}w^2+dhwe^{tw}}{-4w^3+3fw^2+3pw^2-2dhw+2gmw-2anw-2fpw+dfh-cgh-bmn+anp}\right),$$

$$\psi_4(t)=k_1\left(\sum_w \frac{-e^{tw}w^3+fe^{tw}w^2-anwe^{tw}}{-4w^3+3fw^2+3pw^2-2dhw+2gmw-2anw-2fpw+dfh-cgh-bmn+anp}\right)$$

$$-k_2n\left(\sum_w \frac{che^{tw}-mwe^{tw}}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$-k_3h\left(\sum_w \frac{ane^{tw}-fwe^{tw}+w^2e^{tw}}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right)$$

$$+k_4\left(\sum_w \frac{chwe^{tw}-me^{tw}w^2}{4w^3-3fw^2-3pw^2+2dhw-2gmw+2anw+2fpw-dfh+cgh+bmn-anp}\right).$$

Here $a = \frac{L_2}{L_1L_2 - M_1M_2}$, $b = \frac{M_1}{L_1L_2 - M_1M_2}$, $c = \frac{M_2}{L_1L_2 - M_1M_2}$, $d = \frac{L_1}{L_1L_2 - M_1M_2}$,

$n = \frac{1}{C_1}$, $f = (R_1 + R_2)\frac{L_2}{L_1L_2 - M_1M_2}$, $g = (R_1 + R_2)\frac{M_1}{L_1L_2 - M_1M_2}$,

$h = \frac{1}{C_2}$, $m = (R_3 + R_4)\frac{M_2}{L_1L_2 - M_1M_2}$, $p = (R_3 + R_4)\frac{L_1}{L_1L_2 - M_1M_2}$

and k_1, k_2, k_3, k_4 are arbitrary constants. Summation is taken by parameter w that is a solution of the algebraic equation

$$-w^4 + fw^3 + pw^3 - dhw^2 + gmw^2 - anw^2 - fpw^2 + dfhw - cghw - bmnw + anpw + bchn - adhn = 0 .$$

Optimal control $u^{opt}(t)$ derives from the following extreme problem:

$$\psi(t)^T Bv \to \max_{v\in U}, \quad 0 \le t \le t_1 . \tag{4}$$

Let $U = [-1,1]$. Then solving the problem (4), yields

$$u^{opt}(t) = sign(\psi^T(t)B) = \begin{cases} 1, & if\, \psi^T(t)B \ge 0 \\ -1, & if\, \psi^T(t)B < 0 \end{cases}, \quad 0 \le t \le t_1 .$$

And optimal trajectory $x^{opt}(t) = \Im^{-1}\left\{(sI - A)^{-1}\left(B\Im\{u^{opt}(t)\} + x^0\right)\right\}$ is the solution of the initial-value problem

$$\dot{x} = Ax + Bu^{opt}(t),$$
$$x(t_0) = x^0 .$$

Here $\Im\{f(t)\}$ is the Laplace transform of the function $f(t)$.

Illustrating example

We consider the wireless power transfer circuit with parameters [2]

elements	values	elements	values
R_1	50 Ω	L_2	10 μH
R_2	0.1 Ω	M_1	0.5 μH
R_3	0.1 Ω	M_2	0.5 μH
R_4	50 Ω	C_1	1 nF
L_1	10 μH	C_2	1 nF

If we take $t_0 = 0$ and $t_1 = 1$ then Problem 3 arrives at the following simplest problem:

$x_4^2(1) \to \max$,

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -0.1x_1 + 0.005x_2 - 5.023x_3 + 0.251x_4 + 0.1u \\ \dot{x}_4 = 0.005x_1 - 0.1x_2 + 0.251x_3 - 5.023x_4 - 0.005u \end{cases}, \quad x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |u| \le 1, \; t \in [0,1].$$

We form Hamiltonian

$$H(\psi, x, u, t) = \psi_1 x_3 + \psi_2 x_4 + \psi_3(-0.1x_1 + 0.005x_2 - 5.023x_3 + 0.251x_4 + 0.1u)$$
$$+\psi_4(0.005x_1 - 0.1x_2 + 0.251x_3 - 5.023x_4 - 0.005u)$$

and conjugate initial-value problem (3)

$$\begin{cases} \dot{\psi}_1 = 0.1\psi_3 - 0.005\psi_4 \\ \dot{\psi}_2 = -0.005\psi_3 + 0.1\psi_4 \\ \dot{\psi}_3 = -\psi_1 + 5.023\psi_3 - 0.251\psi_4 \\ \dot{\psi}_4 = -\psi_2 - 0.251\psi_3 + 5.023\psi_4 \end{cases}, \quad \psi(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2x_4(1) \end{pmatrix}.$$

Its solution is

$$\psi_1(t) = x_4(1)(0.0196641e^{0.0199847t} - 0.0196786e^{0.0199915t} + 0.000173343e^{4.75201t} -$$
$$0.000104849e^{5.25402t})$$

$$\psi_2(t) = x_4(1)(-0.0196641e^{0.0199847t} - 0.0196786e^{0.0199915t} + 0.000173343e^{4.75201t} +$$
$$0.000104849e^{5.25402t})$$

$$\psi_3(t) = x_4(1)(0.00374268e^{0.0199847t} - 0.00414112e^{0.0199915t} + 0.00867081e^{4.75201t} -$$
$$0.00524645e^{5.25402t})$$

$$\psi_4(t) = x_4(1)(-0.00374268e^{0.0199847t} - 0.00414112e^{0.0199915t} + 0.00867081e^{4.75201t} -$$
$$0.00524645e^{5.25402t})$$

Maximization of Hamiltonian yields

$$H(\psi, x, u, t) = \ldots + \psi^T(t)Bu \to \max_{|u|\le 1}, \quad 0 \le t \le 1$$

or

$$(0.1\psi_3(t) - 0.005\psi_4(t))u \to \max_{|u|\le 1}, \quad 0 \le t \le 1.$$

Substituting $\psi_3(t)$ and $\psi_4(t)$ gives

$$x_4(1)(0.0003929814e^{0.0199847t} - 0.0003934064e^{0.0199915t} + 0.00082372695e^{4.75201t} - 0.00049841275e^{5.25402t}) \to \max_{|u|\le 1}, \quad 0 \le t \le 1.$$

Solving the latter extreme problem, we obtain

$$u^{opt}(t) = \begin{cases} 1, & if\ x_4(1) > 0 \\ -1, & if\ x_4(1) < 0 \end{cases}, \quad 0 \le t \le 1.$$

Note that we exclude the case $x_4(1) = 0$ as not usable. Optimal control takes boundary values +1 or -1 and does not contain break points on interval $0 \le t \le 1$. Optimal trajectory, corresponding to $u^{opt}(t)$ is

$$x_1(t) = e^{-10.046t}(0.00190911e^{4.79198t} - 4.33681\times 10^{-19}e^{4.81197t} + 0.00211237e^{5.29399t} - 2.1684\times 10^{-19}e^{5.29399t} - 8.67362\times 10^{-19}e^{5.31398t} - 6.50521\times 10^{-19}e^{5.31398t} - 0.502112e^{10.026t} - 0.501909e^{10.026t} + 1.11022\times 10^{-16}e^{10.046t} + 0.497888e^{10.046t} + 0.502112e^{10.046t} + 1.11022\times 10^{-16}e^{10.046t} + 2.1684\times 10^{-19}e^{10.548t} + 2.1684\times 10^{-19}e^{14.778t} - 2.1684\times 10^{-19}e^{15.28t})$$

$$x_2(t) = e^{-10.046t}(-0.00190911e^{4.79198t} + 3.25261\times 10^{-19}e^{4.81197t} + 0.00211237e^{5.29399t} - 4.33681\times 10^{-19}e^{5.29399t} - 1.0842\times 10^{-18}e^{5.31398t} - 0.502112e^{10.026t} + 0.501909e^{10.026t} + 1.66533\times 10^{-16}e^{10.046t} - 0.502112e^{10.046t} + 0.502112e^{10.046t} - 1.11022\times 10^{-16}e^{10.046t} + 4.33681\times 10^{-19}e^{10.548t} + 2.1684\times 10^{-19}e^{14.778t} + 4.33681\times 10^{-19}e^{14.778t} + 2.1684\times 10^{-19}e^{15.28t})$$

$$x_3(t) = e^{-10.046t}(-0.0100305e^{4.79198t} + 8.67362\times 10^{-19}e^{4.81197t} + 0.010038e^{5.29399t} - 5.20417\times 10^{-18}e^{5.31398t} + 8.67362\times 10^{-19}e^{9.54399t} + 0.010038e^{10.026t} + 0.0100305e^{10.026t} - 2.60209\times 10^{-18}e^{10.046t} + 0.010038e^{10.046t} - 0.010038e^{10.046t} - 8.67362\times 10^{-19}e^{10.548t} - 1.01644\times 10^{-20}e^{14.778t} + 6.77626\times 10^{-21}e^{15.28t})$$

$$x_4(t) = e^{-10.046t}(0.0100305e^{4.79198t} - 8.67362\times 10^{-19}e^{4.81197t} - 0.010038e^{5.29399t} + 3.46945\times 10^{-18}e^{5.31398t} + 2.60209\times 10^{-18}e^{5.31398t} - 8.67362\times 10^{-19}e^{9.54399t} + 0.010038e^{10.026t} - 0.0100305e^{10.026t} - 8.67362\times 10^{-19}e^{10.046t} + 0.010038e^{10.046t} - 0.01038e^{10.046t} + 1.73472\times 10^{-18}e^{10.046t} - 8.67362\times 10^{-19}e^{10.548t} - 6.77626\times 10^{-21}e^{14.778t} - 3.38813\times 10^{-21}e^{15.28t})$$

if $x_4(1) > 0$ and

$$x_1(t) = -e^{-10.046t}(0.00190911e^{4.79198t} - 4.33681\times10^{-19}e^{4.81197t} + 0.00211237e^{5.29399t} - 2.1684\times10^{-19}e^{5.29399t} - 8.67362\times10^{-19}e^{5.31398t} - 6.50521\times10^{-19}e^{5.31398t} - 0.502112e^{10.026t} - 0.501909e^{10.026t} + 1.11022\times10^{-16}e^{10.046t} + 0.497888e^{10.046t} + 0.502112e^{10.046t} + 1.11022\times10^{-16}e^{10.046t} + 2.1684\times10^{-19}e^{10.548t} + 2.1684\times10^{-19}e^{14.778t} - 2.1684\times10^{-19}e^{15.28t})$$

$$x_2(t) = -e^{-10.046t}(-0.00190911e^{4.79198t} + 3.25261\times10^{-19}e^{4.81197t} + 0.00211237e^{5.29399t} - 4.33681\times10^{-19}e^{5.29399t} - 1.0842\times10^{-18}e^{5.31398t} - 0.502112e^{10.026t} + 0.501909e^{10.026t} + 1.66533\times10^{-16}e^{10.046t} - 0.502112e^{10.046t} + 0.502112e^{10.046t} - 1.11022\times10^{-16}e^{10.046t} + 4.33681\times10^{-19}e^{10.548t} + 2.1684\times10^{-19}e^{14.778t} + 4.33681\times10^{-19}e^{14.778t} + 2.1684\times10^{-19}e^{15.28t})$$

$$x_3(t) = -e^{-10.046t}(-0.0100305e^{4.79198t} + 8.67362\times10^{-19}e^{4.81197t} + 0.010038e^{5.29399t} - 5.20417\times10^{-18}e^{5.31398t} + 8.67362\times10^{-19}e^{9.54399t} + 0.010038e^{10.026t} + 0.0100305e^{10.026t} - 2.60209\times10^{-18}e^{10.046t} + 0.010038e^{10.046t} - 0.010038e^{10.046t} - 8.67362\times10^{-19}e^{10.548t} - 1.01644\times10^{-20}e^{14.778t} + 6.77626\times10^{-21}e^{15.28t})$$

$$x_4(t) = -e^{-10.046t}(0.0100305e^{4.79198t} - 8.67362\times10^{-19}e^{4.81197t} - 0.010038e^{5.29399t} + 3.46945\times10^{-18}e^{5.31398t} + 2.60209\times10^{-18}e^{5.31398t} - 8.67362\times10^{-19}e^{9.54399t} + 0.010038e^{10.026t} - 0.0100305e^{10.026t} - 8.67362\times10^{-19}e^{10.046t} + 0.010038e^{10.046t} - 0.01038e^{10.046t} + 1.73472\times10^{-18}e^{10.046t} - 8.67362\times10^{-19}e^{10.548t} - 6.77626\times10^{-21}e^{14.778t} - 3.38813\times10^{-21}e^{15.28t})$$

if $x_4(1) < 0$.

This solution has been obtained by WolframAlpha program. Objective function gets the value $x_4^2(1) = 1.36084\times10^{-7}$. Note that objective function, corresponding to control $\tilde{u}(t) = \sin(1.07\times10^7)t$ that matches to maximum efficiency in resonance [2], gives the value less than $u^{opt}(t)$.

Conclusion

We considered the wireless power transfer circuit and represented it as the simplest optimal control problem. Using Pontryagin maximum principle we obtained optimal control that is more effective than solution corresponding to maximum efficiency in resonance.

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