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SYMMETRIC IDENTITIES OF TYPE 2 BERNOULLI AND EULER POLYNOMIALS UNDER S_3

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Abstract. In this paper, we give some identities of symmetry for the type 2 Bernoulli and Euler polynomials under symmetry group of degree 3 arising from the p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction

For a long time, special functions have been considered the particular province of pure and applied mathematics, and many special functions have been appeared as solutions of differential equations or integrals of elementary functions (see [1]). The *Bernoulli polynomials* and the *Euler polynimials* are defined by the generating functions to be

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \text{ and } \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \ |t|<2\pi, \tag{1}$$

respectively (see [9, 10]). In the special case, if we put $x=0$, then $B_n = B_n(0)$ and $E_n = E_n(0)$ are called the *Bernoulli numbers* and *Euler numbers*, respectively.

The Bernoulli and Euler numbers are very important roles in the pure and applied mathematics, and have been generalized by many researchers (see [2, 10, 11, 12, 13, 14, 15]). In particular, Kim and Kim [2, 4] defined *type* 2 *Bernoulli and Euler polynomials* as follows

$$\frac{t}{e^t-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \ \frac{2}{e^t+e^{-t}}e^{xt} = \sum_{n=0}^{\infty} e_n(x)\frac{t^n}{n!}, \tag{2}$$

respectively, and found relations between some special functions or numbers and those polynomials. In the special case $x=0$, $b_n(0) := b_n$ and $e_n(0) := e_n$ are called the *type* 2 *Bernoulli numbers* and *type* 2 *Euler numbers* respectively.

Also, Kim and Kim defined *type* 2 *Changhee polynomials* as follows

$$\frac{2}{(1+t)-(1+t)^{-1}}(1+t)^x = \sum_{n=0}^{\infty} Ch_n^*(x)\frac{t^n}{n!}, \tag{3}$$

where $x=0$, $Ch_n^* = Ch_n^*(0)$ are called the *type* 2 *Changhee numbers*.

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Now, we observe

$$\begin{aligned}\sum_{n=0}^{\infty} e_n(x)\frac{t^n}{n!} &= \frac{2}{e^t+e^{-t}}e^{xt} \\ &= \left(\sum_{m=0}^{\infty} e_m\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} x^l\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}e_m x^{n-m}\right)\frac{t^n}{n!},\end{aligned} \tag{4}$$

(see [2]). Therefore, we obtain the following result.

$$e_n(x) = \sum_{m=0}^{n}\binom{n}{m}e_m x^{n-m}. \tag{5}$$

By replacing t by $\log(1+t)$ in (2),

$$\begin{aligned}\sum_{k=0}^{\infty} e_k\frac{1}{k!}(\log(1+t))^k &= \sum_{k=0}^{\infty} e_k\sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} e_k S_1(n,k)\right)\frac{t^n}{n!},\end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned}\frac{2}{e^{\log(1+t)}+e^{-\log(1+t)}} &= \frac{2}{(1+t)-(1+t)^{-1}} \\ &= \sum_{n=0}^{\infty} Ch_n^*\frac{t^n}{n!}.\end{aligned} \tag{7}$$

Therefore, by (6) and (7), we obtain the following result.

$$Ch_n^* = \sum_{k=0}^{n} E_k^* S_1(n,k), \text{ (see [4]).} \tag{8}$$

Also, we obtain the inversion formula of (8)

$$e_n = \sum_{k=0}^{n} Ch_k^* S_2(n,k), \text{ (see [4]).} \tag{9}$$

In this paper, we give some identities of symmetry for the type 2 Bernoulli and Euler polynomials under symmetry group of degree 3 arising from the p-adic q-integral on $\mathbb{Z}_p$.

2. Type 2 Bernoulli polynomials

For a given prime number p, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. The *bosonic p-adic integral on $\mathbb{Z}_p$* are defined as follows:

$$\begin{aligned} I_0(f) =& \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N\to\infty} f(x)\mu_0\left(x + p^N\mathbb{Z}_p\right) \\ =& \lim_{N\to\infty} \frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x). \end{aligned} \tag{10}$$

(see [3, 5, 6]). By (10), we have

$$I_0(f_n) - I_0(f) = \sum_{a=0}^{n-1} f'(a), \tag{11}$$

where $f_n = f(x+n)$ for each positive integer n (see [3, 5, 6]).
If we put $f(y) = e^{(2y+x+1)t}$, then by (11),

$$\frac{1}{2}\int_{\mathbb{Z}_p} e^{(2y+x+1)t} d\mu_0(y) = \frac{t}{e^t - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{12}$$

Note that

$$\begin{aligned} \sum_{k=0}^{\infty}\left(b_k(2n) - b_k\right)\frac{t^k}{k!} =& t\sum_{l=0}^{n-1} e^{(2l+1)t} \\ =& \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n-1}(n+1)(2l+1)^n\right)\frac{t^{n+1}}{(n+1)!}, \end{aligned} \tag{13}$$

and so

$$\sum_{l=0}^{k-1}(2l+1)^l = \frac{1}{k+1}\left(b_{k+1}(2n) - b_{k+1}\right) \tag{14}$$

for each nonnegative integer k (see [7]).
In addition, by the definition of type 2 Bernoulli polynomials, we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} =& \left(\sum_{n=0}^{\infty} b_n\frac{t^n}{n!}\right)\left(\frac{t^n}{n!}x^n\right) \\ =& \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} b_m x^{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{15}$$

By (12) and (15), we have

$$b_k(x) = \frac{1}{2}\int_{\mathbb{Z}_p}(2y+x+1)^k d\mu_0(y) = \sum_{m=0}^{k}\binom{k}{m} b_m x^{k-m},\ (k \geq 0). \tag{16}$$

The equation (11) yields the following:

$$\int_{\mathbb{Z}_p} e^{(2(x+n)+1)t} d\mu_0(x) - \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x) = 2t\sum_{a=0}^{n-1} e^{(2a+1)t}. \tag{17}$$

By (11) and (17), we get

$$\int_{\mathbb{Z}_p} e^{(2x+2n+1)t} d\mu_0(x) - \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x) = \frac{2nt \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x)}{\int_{\mathbb{Z}_p} e^{2nxt} d\mu_0(x)}. \tag{18}$$

If we put $T_k(n) = \sum_{a=0}^{n}(2a+1)^k$ for each nonnegative integer n, then, by (16) and (17), we get

$$\begin{aligned}\frac{w_1w_2w_3 \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt} d\mu_0(x)} &= \sum_{a=0}^{w_1w_2w_3-1} e^{(2a+1)t} \\ &= \sum_{n=0}^{\infty} \sum_{a=0}^{w_1w_2w_3-1} (2a+1)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} T_n(w_1w_2w_3-1)\frac{t^n}{n!},\end{aligned} \tag{19}$$

where w_1, w_2, w_3 are positive integers.

From now on, we consider the following equation for finding the symmetric properties of type 2 Bernoulli polynomials under symmetric group of order 3.

(20)

$$\begin{aligned}& I(w_1, w_2, w_3) \\ =& \frac{(w_1w_2w_3)^2}{2} \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{\left(2\sum_{i=1}^{3} w_ix_i + w_1w_2w_3x + \sum_{i=1}^{3} w_i\right)t} d\mu_0(x_1)d\mu_0(x_2)d\mu_0(x_3)}{\left(\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt} d\mu_0(x)\right)^2}\end{aligned}$$

From (12), (19) and (20), we have

(21)

$$\begin{aligned}& I(w_1, w_2, w_3) \\ =& \frac{w_2w_3}{2} \int_{\mathbb{Z}_p} e^{(2x_1+w_2w_3x+1)w_1t} d\mu_0(x_1) \frac{w_1w_3 \int_{\mathbb{Z}_p} e^{(2x_2+1)w_2t} d\mu_0(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt} d\mu_0(x)} \frac{w_1w_2 \int_{\mathbb{Z}_p} e^{(2x_3+1)w_3t} d\mu_0(x_3)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt} d\mu_0(x)} \\ =& w_2w_3 \left(\sum_{n=0}^{\infty} b_n(w_2w_3x)\frac{(w_1t)^n}{n!}\right) \left(\sum_{n=0}^{\infty} T_n(w_1w_3-1)\frac{(w_2t)^n}{n!}\right) \left(\sum_{n=0}^{\infty} T_n(w_1w_2-1)\frac{(w_3t)^n}{n!}\right) \\ =& \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n}\sum_{l=0}^{m} \binom{n}{m}\binom{m}{l} b_l(w_2w_3x) w_1^l w_2^{m-l} w_3^{n-m} T_{n-m}(w_1w_2-1)T_{m-l}(w_1w_3-1)\right)\frac{t^n}{n!}.\end{aligned}$$

By (20), we know that $I(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)})$ have the same value for each $\sigma \in S_3$, and thus, by (21), we obtain the following theorem.

Theorem 2.1. *Let w_1, w_2, w_3 be positive integers. For each $\sigma \in S_3$ and each nonnegative integer n,*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{l=0}^{m} \binom{n}{m}\binom{m}{l} b_l\left(w_{\sigma(2)}w_{\sigma(3)}x\right) w_{\sigma(1)}^l w_{\sigma(2)}^{m-l} w_{\sigma(3)}^{n-m} \\ &\times T_{n-m}(w_{\sigma(1)}w_{\sigma(2)}-1)T_{m-l}(w_{\sigma(1)}w_{\sigma(3)}-1)\end{aligned}$$

have the same value.

If we put $x=0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let w_1, w_2, w_3 be positive integers. For each $\sigma \in S_3$ and each nonnegative integer n,*

$$\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}b_l w_{\sigma(1)}^{l} w_{\sigma(2)}^{m-l} w_{\sigma(3)}^{n-m} T_{n-m}(w_{\sigma(1)}w_{\sigma(2)}-1)T_{m-l}(w_{\sigma(1)}w_{\sigma(3)}-1)$$

have the same value.

From (21), we get

(22)

$$\begin{aligned}
&I(w_1,w_2,w_3)\\
=&\frac{w_2w_3}{2}\int_{\mathbb{Z}_p} e^{(2x_1+w_2w_3x+1)w_1t}d\mu_0(x_1)\frac{w_1w_3\int_{\mathbb{Z}_p}e^{(2x_2+1)w_2t}d\mu_0(x_2)}{\int_{\mathbb{Z}_p}e^{2w_1w_2w_3xt}d\mu_0(x)}\frac{w_1w_2\int_{\mathbb{Z}_p}e^{(2x_3+1)w_3t}d\mu_0(x_3)}{\int_{\mathbb{Z}_p}e^{2w_1w_2w_3xt}d\mu_0(x)}\\
=&\frac{w_2w_3}{2}e^{w_1w_2w_3xt}\int_{\mathbb{Z}_p} e^{(2x_1+1)w_1t}d\mu_0(x_1)\sum_{l=0}^{w_1w_3-1}e^{(2l+1)w_2t}\sum_{m=0}^{w_1w_2-1}e^{(2m+1)w_3t}\\
=&\frac{w_2w_3}{2}e^{w_1w_2w_3xt}\sum_{l=0}^{w_1w_3-1}\sum_{m=0}^{w_1w_2-1}\int_{\mathbb{Z}_p} e^{\left(2x_1+1+(2l+1)\frac{w_2}{w_1}+(2m+1)\frac{w_3}{w_1}\right)w_1t}d\mu_0(x_1)\\
=&w_2w_3\sum_{l=0}^{w_1w_3-1}\sum_{m=0}^{w_1w_2-1}\sum_{n=0}^{\infty}b_n\left(w_2w_3x+(2l+1)\frac{w_2}{w_1}+(2m+1)\frac{w_3}{w_1}\right)\frac{(w_1t)^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{w_1w_3-1}\sum_{m=0}^{w_1w_2-1}w_1^n w_2w_3 b_n\left(w_2w_3x+(2l+1)\frac{w_2}{w_1}+(2m+1)\frac{w_3}{w_1}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

By (20) and (22), we obtain the following theorem.

Theorem 2.3. *For each positive integers w_1, w_2, w_3, each $\sigma \in S_3$ and each nonnegative integer n,*

$$\sum_{l=0}^{w_{\sigma(1)}w_{\sigma(3)}-1}\sum_{m=0}^{w_{\sigma(1)}w_{\sigma(2)}-1} w_{\sigma(1)}^n w_{\sigma(2)}w_{\sigma(3)} b_n\left(w_{\sigma(2)}w_{\sigma(3)}x+(2l+1)\frac{w_{\sigma(2)}}{w_{\sigma(1)}}+(2m+1)\frac{w_{\sigma(3)}}{w_{\sigma(1)}}\right)$$

have the same value.

In the special case of the Theorem 2.3, if we put $x=0$, then we obtain the following corollary.

Corollary 2.4. *For each positive integers w_1, w_2, w_3, each $\sigma \in S_3$ and each nonnegative integer n,*

$$\sum_{l=0}^{w_{\sigma(1)}w_{\sigma(3)}-1}\sum_{m=0}^{w_{\sigma(1)}w_{\sigma(2)}-1} w_{\sigma(1)}^n w_{\sigma(2)}w_{\sigma(3)} b_n\left((2l+1)\frac{w_{\sigma(2)}}{w_{\sigma(1)}}+(2m+1)\frac{w_{\sigma(3)}}{w_{\sigma(1)}}\right)$$

have the same value.

3. Type 2 Euler polynomials

Let p be given as a fixed odd prime number, and let $C(\mathbb{Z}_p)$ be the set of all continuous functions on $\mathbb{Z}_p$. The *fermionic integral on* $\mathbb{Z}_p$ is also defined by Kim to be

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-1}\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [5, 6, 7, 14]).}\end{aligned} \tag{23}$$

By (23), we know that

$$I_{-1}(f_n)+(-1)^{n-1}I_{-1}(f)=2\sum_{l=0}^{n-1}f(l)(-1)^{n-1-l} \tag{24}$$

where $f_n=f(x+n)$ and n is a positive integer.

If we put $f(y)=e^{(2y+x+1)}$, then

$$\int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu_{-1}(y)=\frac{2}{e^t+e^{-t}}e^{xt}=\sum_{n=0}^{\infty}e_n(x)\frac{t^n}{n!}. \tag{25}$$

Note that, by the definition of the type 2 Euler polynomials,

$$\begin{aligned}\sum_{k=0}^{\infty}\left(e_k(2n)-e_k\right)\frac{t^k}{k!} &=\frac{2}{e^t+e^{-t}}\left(e^{2nt}-1\right)\\ &=2\sum_{l=0}^{n-1}(-1)^{n+l+1}e^{(2l+1)t}\\ &=\sum_{k=0}^{\infty}\left(2\sum_{l=0}^{n-1}(-1)^{n+l+1}(2l+1)^k\right)\frac{t^k}{k!},\end{aligned} \tag{26}$$

where n is a positive integer. By (26), we have

$$\sum_{l=0}^{n-1}(-1)^{n+l+1}(2l+1)^k=\frac{1}{2}\left(e_k(2n)-e_k\right), \tag{27}$$

where k a nonnegative integer and n is a positive integer.

From the equation (24), we have

$$\int_{\mathbb{Z}_p} e^{(2y+2n+1)t}d\mu_{-1}(y)+\int_{\mathbb{Z}_p} e^{(2y+1)t}d\mu_{-1}(y)=2\sum_{l=0}^{n-1}e^{(2l+1)t}(-1)^l \tag{28}$$

where n is a positive odd integer. Hence, if we put $R_k(n)=\sum_{l=0}^{n}(-1)^l(2l+1)^k$, then

$$\int_{\mathbb{Z}_p} e^{(2y+2n+1)t}d\mu_{-1}(y)+\int_{\mathbb{Z}_p} e^{(2y+1)t}d\mu_{-1}(y)=\frac{2\int_{\mathbb{Z}_p}e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}e^{2nxt}d\mu_{-1}(x)}, \tag{29}$$

and

$$\begin{aligned}\frac{2\int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{2nxt}d\mu_{-1}(x)} =&2\sum_{l=0}^{n-1} e^{(2l+1)t}(-1)^l \\ =&2\sum_{k=0}^{\infty}\left(\sum_{l=0}^{n-1}(2l+1)^k(-1)^l\right)\frac{t^k}{k!} \\ =&2\sum_{k=0}^{\infty} R_k(n-1)\frac{t^k}{k!},\end{aligned} \tag{30}$$

where n is a positive odd integer.

From now on, we assume that $w_1, w_2 w_3$ are positive even integers, and let

(31)

$$\begin{aligned}&J(w_1,w_2,w_3)\\ =&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{(2(w_1x_1+w_2x_2+w_3x_3)+w_1w_2w_3x+w_1w_2w_3)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3)}{\left(\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt}d\mu_{-1}(x)\right)^2}.\end{aligned}$$

By (29) and (31), we have

(32)

$$\begin{aligned}&J(w_1,w_2,w_3)\\ =&\frac{1}{4}\int_{\mathbb{Z}_p} e^{(2x_1+1+w_2w_3x)w_1t}d\mu_{-1}(x_1)\frac{2\int_{\mathbb{Z}_p} e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt}d\mu_{-1}(x)}\frac{2\int_{\mathbb{Z}_p} e^{(2x_3+1)w_3t}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt}d\mu_{-1}(x)}\\ =&\frac{1}{4}\left(\sum_{n=0}^{\infty} e_n(w_2w_3x)\frac{(w_1t)^n}{n!}\right)\left(2\sum_{k=0}^{\infty} R_k(w_1w_3-1)\frac{(w_2t)^k}{k!}\right)\left(2\sum_{k=0}^{\infty} R_k(w_1w_2-1)\frac{(w_3t)^k}{k!}\right)\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}w_1^l w_2^{m-1}w_3^{n-m}e_l(w_2w_3x)R_{n-m}(w_1w_2-1)R_{m-l}(w_1w_3-1)\right)\frac{t^n}{n!}.\end{aligned}$$

By (31), we know that $J(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)})$ have the same value for each $\sigma \in S_3$, and thus, by (32), obtain the following theorem.

Theorem 3.1. *Let w_1, w_2, w_3 be positive even integers. For each $\sigma \in S_3$ and each nonnegative integer n,*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}{w_{\sigma(1)}}^l {w_{\sigma(2)}}^{m-1}{w_{\sigma(3)}}^{n-m}e_l\left(w_{\sigma(2)}w_{\sigma(3)}x\right)\\ &\times R_{n-m}\left(w_{\sigma(1)}w_{\sigma(2)}-1\right)R_{m-l}\left(w_{\sigma(1)}w_{\sigma(3)}-1\right)\end{aligned}$$

have the same value.

As a special case of the Theorem 3.1, if we put $x = 0$, then we obtain the following corollary.

Corollary 3.2. *Let w_1, w_2, w_3 be positive even integers. For each $\sigma \in S_3$ and each nonnegative integer n,*

$$\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}e_l {w_{\sigma(1)}}^l {w_{\sigma(2)}}^{m-1}{w_{\sigma(3)}}^{n-m}R_{n-m}\left(w_{\sigma(1)}w_{\sigma(2)}-1\right)R_{m-l}\left(w_{\sigma(1)}w_{\sigma(3)}-1\right)$$

have the same value.

From (31), we note that

(33)

$$\begin{aligned}
&J(w_1, w_2w_3)\\
=&\frac{1}{4}\int_{\mathbb{Z}_p} e^{(2x_1+1+w_2w_3x)w_1t}d\mu_{-1}(x_1)\frac{2\int_{\mathbb{Z}_p} e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt}d\mu_{-1}(x)}\frac{2\int_{\mathbb{Z}_p} e^{(2x_3+1)w_3t}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{2w_1w_2w_3xt}d\mu_{-1}(x)}\\
=&\frac{e^{w_1w_2w_3t}}{4}\left(\int_{\mathbb{Z}_p} e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(2\sum_{l=0}^{w_1w_3} e^{(2l+1)w_2t}(-1)^l\right)\left(2\sum_{m=0}^{w_1w_2} e^{(2m+1)w_3t}(-1)^m\right)\\
=&\sum_{l=0}^{w_1w_3}\sum_{m=0}^{w_1w_2}(-1)^{l+m}\int_{\mathbb{Z}_p} e^{\left(2x_1+1+w_2w_3+(2l+1)\frac{w_2}{w_1}+(2m+1)\frac{w_3}{w_1}\right)w_1t}d\mu_{-1}(x_1)\\
=&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{w_1w_3}\sum_{m=0}^{w_1w_2}(-1)^{l+m}w_1^n e_n\left(w_2w_3+(2l+1)\frac{w_2}{w_1}+(2m+1)\frac{w_3}{w_1}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

By the (31) and (33), we obtain the following theorem.

Theorem 3.3. *For each positive integers w_1, w_2, w_3, each $\sigma \in S_3$ and each nonnegative integer n,*

$$\sum_{l=0}^{w_{\sigma(1)}w_{\sigma(3)}}\sum_{m=0}^{w_{\sigma(1)}w_{\sigma(2)}}(-1)^{l+m}{w_{\sigma(1)}}^n e_n\left(w_{\sigma(2)}w_{\sigma(3)}+(2l+1)\frac{w_{\sigma(2)}}{w_{\sigma(1)}}+(2m+1)\frac{w_{\sigma(3)}}{w_{\sigma(1)}}\right)$$

have the same value.

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References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions, in: Encyclopedia of Mathematics and its Applications*, vol. 71, Cambridge University Press, Cambridge, 1999.
[2] G. W. Jang and T. Kim, *A note on type* 2 *degenerate Euler and Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **29** (2019), no. 1, 147-159.
[3] D. S. Kim, N. Lee, J. Na and H. K. Park, *Abundant symmetry for higher-order Bernoulli polynomials (I)*, Adv. Stud. Contemp. Math. (Kyungshang), **23** (2013), 461-482.
[4] T. Kim and D. S. Kim, *A note on type* 2 *Changhee and Daehee polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113** (2019), no. 3, 2763-2771.
[5] T. Kim, *On q-analogue of the p-adic log gamma functions and related integral*, J. Number Theory, **76** (1999), no. 2, 320-329.
[6] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), no. 3, 288-299.
[7] D. S. Kim, H. Y. Kim, D. Kim and T. Kim, *Identities of symmetry for type* 2 *Bernoulli and Euler polynomials*, Symmetry, **2019**, 11, 613.
[8] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
[9] S. Gaboury, R. Tremblay, B. J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials,* Proc. Jangjeon Math. Soc.**17** (2014), no. 1, 115-123.
[10] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51-88.
[11] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332-350.

[12] H. Srivastava, *Some generalizations and basic (or q-)extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci., **5** (2011), 390-444.
[13] C. S. Ryoo, *On the extended q-Euler numbers and polynomials with weak weight* α, Far East J. Math. Sci., **70** (2012), no. 2, 365-373.
[14] T. Kim, *A study on the q-Euler numbers and the fermionic q-integrals of the product of several type q-Bernstein polynomials on* $\mathbb{Z}_p$, Adv. Stud. Contemp. Math., **23** (2013), 5-11.
[15] J. W. Park and S. H. Rim, *On the modified q-Bernoulli polynomials with weight*, Proc. Jangjeon Math. Soc., **17** (2014), no. 2, 231-236.

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