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A NOTE ON SOME IDENTITIES OF BERNOULLI NUMBERS ARISING FROM RIEMANN INTEGRALS

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ABSTRACT. In this paper, we represent the generating function of Bernoulli numbers and its reciprocal as Riemann integrals on the unit interval. From these integral representations and in an elementary way, we derive several identities involving Bernoulli numbers, Bell numbers and Stirling numbers of the first and second kinds.

1. INTRODUCTION

The aim of this paper is to consider two basic Riemann integrals on unit intervals from which we derive several identities involving Bernoulli numbers, Bell numbers and Stirling numbers of the first and second kinds. In more detail, our results are as follows.

Firstly, we observe that the generating function of the Bernoulli numbers is given by a Riemann integral on the unit interval. From this observation, we get explicit expressions for Bernoulli and higher-order Bernoulli numbers, and an identity involving Bernoulli numbers and Stirling numbers of the first kind.

Secondly, we note that the reciprocal of the generating function of the Bernoulli numbers is also given by a Riemann integral on the unit interval. Then, from this note, we obtain an expression for Stirling numbers of the second kind, representations of the integral of the Bell polynomial on the unit interval in terms of Stirling numbers of the second kind and also of Bell numbers and Bernoulli numbers, and the value of higher-order Bernoulli polynomials at 1 in terms of Stirling numbers of the first and second kinds.

It is well known that the higher-order Bernoulli numbers are defined by

$$\left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!}, \quad (r \in \mathbb{N}), \quad (\text{see } [12]). \tag{1}$$

When $r=1$, $B_n = B_n^{(1)}$, $(n \geq 0)$, are called the Bernoulli numbers.

The Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0), \quad (\text{see } [1-13]), \tag{2}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

As an inversion formula of (2), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see } [1-13]). \tag{3}$$

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The Bell polynomials are given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x)\frac{t^n}{n!}, \quad (\text{see } [3]). \tag{4}$$

By (3) and (4), we get

$$\mathrm{Bel}_n(x) = \sum_{l=0}^{n} S_2(n,l)x^l, \quad (n \geq 0). \tag{5}$$

2. Integral representations for the generating function of Bernoulli numbers

First, we note that

$$\int_0^1 \frac{1}{1+x(e^t-1)}dx = \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{6}$$

On the other hand,

$$\begin{aligned} \int_0^1 \frac{1}{1+x(e^t-1)}dx &= \sum_{l=0}^{\infty}(-1)^l(e^t-1)^l \int_0^1 x^l dx = \sum_{l=0}^{\infty} \frac{(-1)^l l!}{l+1}\frac{1}{l!}\left(e^t-1\right)^l \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l+1} l! \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(-1)^l}{l+1} l! S_2(n,l)\right)\frac{t^n}{n!}. \end{aligned} \tag{7}$$

From (6) and (7), we have

$$B_n = \sum_{l=0}^{n}(-1)^l \frac{l!}{l+1} S_2(l,n), \quad (n \geq 0).$$

Now, we consider the multivariate integral related the generating function of higher-order Bernoulli numbers which are given by

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{r}\left(\frac{1}{1+x_l(e^t-1)}\right)dx_1 \cdots dx_r = \left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!}. \tag{8}$$

From (7), we note that

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{r}\left(\frac{1}{1+x_l(e^t-1)}\right)dx_1 \cdots dx_r = \sum_{n=0}^{\infty} A_{n,r}\frac{t^n}{n!}, \tag{9}$$

where

$$A_{n,r} = \sum_{\substack{n_1+\cdots+n_r=n \\ n_i \geq 0}} \binom{n}{n_1,\ldots,n_r} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{k_1+\cdots+k_r} \prod_{l=1}^{r}\left(\frac{k_l! S_2(n_l,k_l)}{k_l+1}\right). \tag{10}$$

Thus, by (8)–(10), we have

$$B_n^{(r)} = \sum_{\substack{n_1+\cdots+n_r=n \\ n_i \geq 0}} \binom{n}{n_1,\ldots,n_r} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{k_1+\cdots+k_r} \prod_{l=1}^{r}\left(\frac{k_l! S_2(n_l,k_l)}{k_l+1}\right).$$

It is well known that the Fubini polynomials are given by

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \quad (\text{see } [2,4,7,9,10]). \tag{11}$$

By (6) and (11), we get

$$\int_0^1 F_n(-x)dx = B_n, \quad (n \geq 0). \tag{12}$$

We observe that

$$\int_0^1 \frac{1}{1+x(e^t-1)}dx = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^1 \log^m\big(1+x(e^t-1)\big)dx \tag{13}$$
$$= \sum_{m=0}^{\infty}(-1)^m\sum_{k=m}^{\infty}S_1(k,m)\frac{1}{k!}(e^t-1)^k\int_0^1 x^k dx$$
$$= \sum_{k=0}^{\infty}\sum_{m=0}^{k}(-1)^m\frac{S_1(k,m)}{k+1}\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^m\frac{S_1(k,m)}{k+1}S_2(n,k)\right)\frac{t^n}{n!}.$$

Thus, by (6) and (13), we get

$$B_n = \sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^m\frac{S_2(n,k)S_1(k,m)}{k+1}, \quad (n\geq 0). \tag{14}$$

Replacing t by $\log(1+t)$ in (6), we have

$$\int_0^1 \frac{1}{1+tx}dx = \sum_{l=0}^{\infty}B_l\frac{1}{l!}\big(\log(1+t)\big)^l = \sum_{l=0}^{\infty}B_l\sum_{n=l}^{\infty}S_1(n,l)\frac{t^n}{n!} \tag{15}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}B_lS_1(n,l)\right)\frac{t^n}{n!}.$$

On the other hand,

$$\frac{1}{1+xt} = e^{-\log(1+xt)} = \sum_{l=0}^{\infty}(-1)^l\frac{1}{l!}\big(\log(1+xt)\big)^l \tag{16}$$
$$= \sum_{l=0}^{\infty}(-1)^l\sum_{n=l}^{\infty}S_1(n,l)\frac{x^nt^n}{n!} = \sum_{n=0}^{\infty}x^n\sum_{l=0}^{n}(-1)^lS_1(n,l)\frac{t^n}{n!}.$$

Thus, we note that

$$\int_0^1 \frac{1}{1+xt}dx = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{l=0}^{n}(-1)^lS_1(n,l)\right)\frac{t^n}{n!}. \tag{17}$$

From (15) and (17), we have

$$\frac{1}{n+1}\sum_{l=0}^{n}(-1)^lS_1(n,l) = \sum_{l=0}^{n}S_1(n,l)B_l, \quad (n\geq 0). \tag{18}$$

We note that

$$\int_0^1 \frac{1}{1+xt}dx = \frac{1}{t}\log(1+t) = \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}t^n \tag{19}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}n!\frac{t^n}{n!}.$$

Thus, by (17) and (19), we get

$$1 = \frac{1}{n!}\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l) = \frac{1}{n!}\sum_{l=0}^{n}|S_1(n,l)|, \quad (n\geq 0), \tag{20}$$

where $|S_1(n,l)|$ are the unsigned Stirling numbers of the first kind.

It is easy to show that

$$\int_0^1 e^{xt}dx = \frac{1}{t}\left(e^t-1\right). \tag{21}$$

From (21), we have

$$\begin{aligned}\int_0^1\cdots\int_0^1 e^{(x_1+\cdots+x_k)t}dx_1\cdots dx_k &= \frac{k!}{t^k}\frac{1}{k!}(e^t-1)^k = \frac{k!}{t^k}\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{S_2(n+k,k)}{\binom{n+k}{n}}\frac{t^n}{n!}.\end{aligned} \tag{22}$$

On the other hand,

$$\int_0^1\cdots\int_0^1 e^{(x_1+\cdots+x_k)t}dx_1\cdots dx_k = \sum_{n=0}^{\infty}\left(\sum_{\substack{l_1+\cdots+l_k=n\\ l_i\geq 0}}\binom{n}{l_1,\ldots,l_k}\frac{1}{(l_1+1)(l_2+1)\cdots(l_k+1)}\right)\frac{t^n}{n!}. \tag{23}$$

Thus, by (22) and (23), we get

$$\begin{aligned}S_2(n+k,k) &= \binom{n+k}{n}\sum_{\substack{l_1+\cdots+l_k=n\\ l_i\geq 0}}\binom{n}{l_1,\ldots,l_k}\frac{1}{(l_1+1)(l_2+1)\cdots(l_k+1)},\\ &= \frac{(n+k)!}{k!}\sum_{\substack{l_1+\cdots+l_k=n\\ l_i\geq 0}}\frac{1}{(l_1+1)!(l_2+1)!\cdots(l_k+1)!},\end{aligned} \tag{24}$$

where n,k are non-negative integers.

From (21), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\int_0^1 \mathrm{Bel}_n(x)dx\frac{t^n}{n!} &= \int_0^1 e^{x(e^t-1)}dx = \frac{e^{e^t-1}-1}{e^t-1}\\ &= \frac{t}{e^t-1}\frac{1}{t}\left(e^{e^t-1}-1\right) = \sum_{l=0}^{\infty}B_l\frac{t^l}{l!}\frac{1}{t}\sum_{k=1}^{\infty}\mathrm{Bel}_k\frac{t^k}{k!}\\ &= \sum_{l=0}^{\infty}B_l\frac{t^l}{l!}\sum_{k=0}^{\infty}\frac{\mathrm{Bel}_{k+1}}{k+1}\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{Bel}_{k+1}}{k+1}B_{n-k}\right)\frac{t^n}{n!},\end{aligned} \tag{25}$$

where $\mathrm{Bel}_n=\mathrm{Bel}_n(1)$, $(n\geq 0)$, are called the Bell numbers.

On the other hand,

$$\begin{aligned}\int_0^1 e^{x(e^t-1)}dt &= \sum_{k=0}^{\infty}\int_0^1 x^k dx\frac{1}{k!}\left(e^t-1\right)^k\\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k+1}S_2(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{26}$$

Thus, by (25) and (26), we get

$$\sum_{k=0}^{n}\frac{1}{k+1}S_2(n,k) = \int_0^1 \mathrm{Bel}_n(x)dx = \sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{Bel}_{k+1}}{k+1}B_{n-k}, \quad (n\geq 0). \tag{27}$$

Replacing t by $\log(1+t)$ in (23), we get

$$\begin{aligned}\int_0^1\cdots\int_0^1(1+t)^{x_1+\cdots+x_k}dx_1\cdots dx_k &= \sum_{m=0}^{\infty}\left(\sum_{l_{+1}+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}\frac{1}{(l_1+1)\cdots(l_k+1)}\right)\frac{1}{m!}\big(\log(1+t)\big)^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{l_{+1}+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}\frac{1}{(l_1+1)\cdots(l_k+1)}\right)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{\substack{l_1+\cdots+l_k=m\\ l_i\geq 0}}\binom{m}{l_1,\ldots,l_k}\frac{S_1(n,m)}{(l_1+1)\cdots(l_k+1)}\right)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

On the other hand,

$$\int_0^1\cdots\int_0^1(1+t)^{x_1+\cdots+x_n}dx_1dx_2\cdots dx_k = \left(\frac{t}{\log(1+t)}\right)^k = \sum_{n=0}^{\infty}B_n^{(n-k+1)}(1)\frac{t^n}{n!}, \quad (\text{see [12]}), \tag{29}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α given by

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt} = \sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (\text{see [12]}).$$

Thus, by (28) and (29), we get

$$B_n^{(n-k+1)}(1) = \sum_{m=0}^{n}\sum_{l_1+\cdots+l_k=m}\binom{m}{l_1,\ldots,l_k}\frac{S_1(n,m)}{(l_1+1)\cdots(l_k+1)}. \tag{30}$$

From (22), we can derive the following equation.

$$\begin{aligned}\int_0^1\cdots\int_0^1(1+t)^{x_1+\cdots+x_k}dx_1\cdots dx_k &= \sum_{l=0}^{\infty}\frac{1}{\binom{l+k}{l}}S_2(l+k,k)\frac{1}{l!}\big(\log(1+t)\big)^l \\ &= \sum_{l=0}^{\infty}\frac{1}{\binom{l+k}{k}}S_2(l+k,k)\sum_{n=l}^{\infty}S_1(n,l)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{1}{\binom{l+k}{l}}S_2(l+k,k)S_1(n,l)\right)\frac{t^n}{n!}\end{aligned} \tag{31}$$

Thus, by (29) and (31), we get

$$B_n^{(n-k+1)}(1) = \sum_{l=0}^{n}\frac{1}{\binom{l+k}{k}}S_2(l+k,k)S_1(n,l), \quad (n,k\geq 0). \tag{32}$$

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