

A FURTHER GENERALIZATION OF EXTENDED HURWITZ-LERCH ZETA FUNCTION OF TWO VARIABLES

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ABSTRACT. In the recent years, several extensions of Hurwitz Lerch Zeta function of two variables have been established by several authors in different ways. In this paper, we aim to generalize an extended Hurwitz Lerch Zeta function with the help of extended generalized beta function $B_p^{(\alpha, \beta; m)}(x, y)$. We further investigate its integral representations and various other properties like Mellin Transformation and other transformation formulas. Some interesting special cases of our results are also pointed out.

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1. INTRODUCTION

A few decades ago, various authors have researched extensively about Zeta function and various properties related to them have been investigated. In 2008, Taekyun Kim [10] also introduced various Euler Numbers and Polynomials associated with Zeta Functions. Recently, many authors including Pathan and Daman [15], Parmar et al. [14], gave several extensions of Hurwitz Lerch Zeta function, these are well-known generalizations of Zeta Function. Inspired by these studies, we aim to define another generalization of extended Hurwitz Lerch Zeta function and investigate its various properties. For our purpose, we shall be using various properties and extensions of $B(x, y)$. Some of these facts are listed below:

Riemann Zeta function [10, p. 3, Eq. (11)] is defined as follows:

$$(1.1) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

Apostol [2, p. 89] reported its generalization as follows:

$$(1.2) \quad \varsigma(s, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}, \quad \operatorname{Re}(s) > 1, \quad a \neq \{0, -1, -2, \dots\}.$$

A very well-known generalization of Zeta function was originally reported by Erdelyi et al. [7, p. 27, Eq. 1.11(1)]; this is known as Hurwitz Lerch Zeta

function and is defined as follows:

$$(1.3) \quad \phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(a+m)^s}, \quad (|z| < 1, s \in \mathbb{C}, a \neq \{0, -1, -2, \dots\}; \\ Re(s) > 1 \text{ when } |z| = 1).$$

In 1997, Goyal and Laddha [8, p. 100, Eq. (1.5)] gave another generalization of Hurwitz Lerch Zeta function as follows:

$$(1.4) \quad \phi_{\mu}^*(z, s, a) = \sum_{m=0}^{\infty} \frac{(\mu)_m z^m}{(a+m)^s m!},$$

where $a \neq \{-1, -2, \dots\}$, $\mu \geq 1$ and either $|z| < 1$, $Re(s) > 0$ or $z = 1$ and $Re(s) > \mu$.

Further, Parmar et al. in 2017 [14, p. 179, Eq. (2.1)] gave a generalization of the extended Hurwitz-Lerch Zeta function as:

$$(1.5) \quad \phi_{\lambda, \mu; \nu}^{(\rho, \sigma)}(z, s, a; p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B_p^{(\rho, \sigma)}(\mu + n, \nu - \mu)}{B(\nu, \nu - \mu)} \frac{z^n}{(a+n)^s},$$

where $p \geq 0$, $Re(\rho) > 0$, $Re(\sigma) > 0$; $\lambda, \mu \in \mathbb{C}$; $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$; $Re(s + \nu - \lambda - \mu) > 1$ when $|z| = 1$. $B_p^{(\alpha, \beta)}(x, y)$ is an extended Beta function defined in (1.8).

Recently, Pathan and Daman [15] discussed a generalization of (1.3) and (1.4) for two variables in following form:

$$(1.6) \quad \phi_{\mu, \lambda}^*(z, t, s, a) = \sum_{m, k=0}^{\infty} \frac{(\mu)_m (\lambda)_k z^m t^k}{(a+m+k)^s m! k!} \\ = \sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} \phi_{\mu}^*(z, s, a+k),$$

where $a \neq \{-1, -2, \dots\}$, $\mu, \lambda \geq 1$ and either $|z|, |t| < 1$, $Re(s) > 0$ or $t, z = 1$ and $Re(s) > \mu, \lambda$.

In the most recent research, Lee et al. [11, p. 189, Eq. (1.13)] discussed an advanced generalized Beta function $B_p^{(\alpha, \beta; m)}(x, y)$:

$$(1.7) \quad B_p^{(\alpha, \beta; m)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1 \left(\alpha; \beta; \frac{-p}{t^m (1-t)^m} \right) dt,$$

where $Re(p) > 0$; $\min\{Re(x), Re(y), Re(\alpha), Re(\beta)\} > 0$ and $Re(m) > 0$.

On substituting $m = 1$, the above equation reduces to a very well-known generalized Beta function [13, p. 4602, Eq. (4)]

$$(1.8) \quad B_p^{(\alpha, \beta)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1 \left(\alpha; \beta; \frac{-p}{t(1-t)} \right) dt,$$

where $Re(p) > 0$; $\min\{Re(x), Re(y), Re(\alpha), Re(\beta)\} > 0$.

When $\alpha = \beta$, equation (1.8) reduces to a special case of extended Beta function defined by Chaudhry et al. [6, p. 20, Eq. (1.7)]

$$(1.9) \quad B_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp \left\{ \frac{-p}{t(1-t)} \right\} dt,$$

where $Re(p) > 0$ and $\min\{Re(x), Re(y)\} > 0$.

Further taking $p = 0$, equation (1.9) reduces to a well known Euler's beta function $B(x, y)$ defined by:

$$(1.10) \quad B(x, y) = \begin{cases} \int_0^1 t^x(1-t)^{y-1} & (Re(x) > 0, Re(y) > 0) \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} & (x, y \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases}$$

In 2004, Chaudhry et al. [5, p. 591, Eqs. (2.1), (2.2)] gave an extension of Gauss Hypergeometric function as follows:

$$(1.11) \quad F_p(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

where $Re(c) > Re(b) > 0$; $Re(p) > 0$.

In 2011, an extension of Gauss Hypergeometric Function has been defined and investigated by Lee [11, p. 197, Eq. (6.1)] using the more Generalized Beta function mentioned in (1.7) as follows:

$$(1.12) \quad F_p^{(\alpha, \beta; m)} = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha, \beta; m)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

where $\min\{Re(\alpha), Re(\beta), Re(m)\} > 0$; $Re(c) > Re(b) > 0$; $Re(p) > 0$.

In the sequel to the above extensions, Agarwal et al. [1, p. 406, Eq. (2.2)] defined a new extension of second Appell's Hypergeometric function $F_2(a, b, c; d, e; x, y)$ as follows:

$$(1.13) \quad F_{2,p}^{(\alpha, \beta, \alpha', \beta'; m)}(a, b, c; d, e; x, y; m) = \sum_{r,s=0}^{\infty} (a)_{r+s} \frac{B_p^{(\alpha, \beta; m)}(b+r, d-b) B_p^{(\alpha', \beta'; m)}(c+s, e-c)}{B(b, d-b) B(c, e-c)} \frac{x^r y^s}{r! s!},$$

where $(|x|+|y| < 1; Re(p) \geq 0; \min\{Re(\alpha), Re(\beta), Re(\alpha'), Re(\beta'), Re(m)\} > 0$.

Recently, Batra and Rai [4, p. 1551, Eq. (5)] gave a generalization of Hurwitz–Lerch Zeta function of two variables as follows:

$$(1.14) \quad \phi_{\alpha, \beta, \beta'; \gamma, \gamma'}(z, t, s, a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} \frac{z^m t^n}{(a+m+n)^s},$$

where $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$, $\gamma, a \neq \{0, -1, -2, \dots\}$; $s, z, t \in \mathbb{C}$ and $Re(s) > 0$ when $|z|, |t| < 1$ or $Re(\gamma + s - \alpha - \alpha' - \beta - \beta') > 0$ when $|z|, |t| = 1$.

2. EXTENSION OF HURWITZ LERCH ZETA FUNCTION

Definition 2.1. We generalize the extended Hurwitz Lerch Zeta Function of two variables in the following form:

(2.1)

$$\begin{aligned} & \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \sigma, \rho', \sigma'; r)}(z, t, s, a; p) \\ &= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} B_p^{(\rho, \sigma; r)}(\beta + m, \gamma - \beta) B_p^{(\rho', \sigma'; r)}(\beta' + n, \gamma' - \beta')}{m! n! B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \frac{z^m t^n}{(a+m+n)^s}, \end{aligned}$$

where $p \geq 0$, $\operatorname{Re}(\rho)$, $\operatorname{Re}(\sigma)$, $\operatorname{Re}(\rho')$, $\operatorname{Re}(\sigma') \geq 0$; $\alpha, \beta, \beta' \in \mathbb{C}$; $\gamma, \gamma', a \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s, z, t \in \mathbb{C}$ and $\operatorname{Re}(s) > 0$ when $|z| < 1$, $|t| < 1$; $\operatorname{Re}(s + \gamma + \gamma' - \beta - \beta' - \alpha) > 1$ when $|z| = 1$, $|t| = 1$.

Remark 2.2. Let $r = 1$, $\sigma = \rho$, $\sigma' = \rho'$ and $p = 0$ then (2.1) reduces to (1.14)

$$(2.2) \quad \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \rho, \rho', \rho'; 1)} (z, t, s, a; 0) \\ = \phi_{\alpha, \beta, \beta'; \gamma, \gamma'}^*(z, t, s, a) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} \frac{z^m t^n}{(a + m + n)^s},$$

where $\alpha, \beta, \beta' \in \mathbb{C}$, $\gamma, \gamma', a \neq \{0, -1, -2, \dots\}$; $s, z, t \in \mathbb{C}$ and $\operatorname{Re}(s) > 0$ when $|z|, |t| < 1$ or $\operatorname{Re}(\gamma + \gamma' + s - \alpha - \beta - \beta') > 0$ when $|z|, |t| = 1$.

Following are the limiting cases of equation (2.1)

Case 1.

$$(2.3) \quad \phi_{\beta, \beta'; \gamma, \gamma'}^*(z, t, s, a) = \lim_{\alpha \rightarrow \infty} \left\{ \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \rho, \rho', \rho'; 1)} \left(\frac{z}{\alpha}, \frac{t}{\alpha}, s, a; 0 \right) \right\} \\ = \sum_{m, n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} \frac{z^m t^n}{(a + m + n)^s},$$

where $\alpha, \beta, \beta' \in \mathbb{C}$, $\gamma, \gamma', a \neq \{0, -1, -2, \dots\}$, $s, z, t \in \mathbb{C}$ and $\operatorname{Re}(s) > 0$ when $|z|, |t| < 1$ or $\operatorname{Re}(\gamma + \gamma' + s - \alpha - \beta - \beta') > 0$ when $|z|, |t| = 1$.

Case 2.

$$(2.4) \quad \phi_{\alpha, \beta; \gamma, \gamma'}^*(z, t, s, a) = \lim_{\beta' \rightarrow \infty} \left\{ \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \rho, \rho', \rho'; 1)} \left(z, \frac{t}{\beta}, s, a; 0 \right) \right\} \\ = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n m! n!} \frac{z^m t^n}{(a + m + n)^s},$$

where $\gamma, \gamma', a \neq \{0, -1, -2, \dots\}$ and $\alpha, \beta \in \mathbb{C}$; $s, z, t \in \mathbb{C}$, $\operatorname{Re}(s) > 0$ when $|z|, |t| < 1$ or $\operatorname{Re}(\gamma + \gamma' + s - \alpha - \beta) > 0$ when $|z|, |t| = 1$.

3. INTEGRAL REPRESENTATIONS

Theorem 3.1. The following integral representations holds true:

$$(3.1) \quad \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \sigma, \rho', \sigma'; r)} (z, t, s, a; p) \\ = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} F_{2,p}^{(\rho, \sigma, \rho', \sigma'; r)} (\alpha, \beta, \beta'; \gamma, \gamma'; ze^{-x}, te^{-x}; r) dx,$$

where $|z| + |t| < |e^x|$, $\operatorname{Re}(p) > 0$, $\min\{\operatorname{Re}(\rho), \operatorname{Re}(\sigma), \operatorname{Re}(\rho'), \operatorname{Re}(\sigma'), \operatorname{Re}(r)\} > 0$; $\min\{\operatorname{Re}(s), \operatorname{Re}(a)\} > 0$; $m, n \in \mathbb{N}_0$ when $|z| < 1$, $|t| < 1$ and $\operatorname{Re}(s) > 1$ when $|z|, |t| = 1$.

Proof. Eulerian Integral of Gamma function $\Gamma(s)$ [16, p. 1, Eq. (1)] is given by the following identity:

$$(3.2) \quad \frac{1}{(a + n)^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(a+n)t} dt,$$

where $\min\{\operatorname{Re}(s), \operatorname{Re}(a)\} > 0$; $n \in \mathbb{N}_0$.

From (3.2) and (2.1), further changing the order of summation and integration, we get:

$$(3.3) \quad \begin{aligned} & \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \sigma, \rho', \sigma'; r)} (z, t, s, a; p) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \\ & \quad \times \left(\sum_{m,n=0}^{\infty} \frac{\left((\alpha)_{m+n} B_p^{(\rho, \sigma; r)}(\beta + m, \gamma - \beta) \right)}{m! n! B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} z^m t^n e^{-mx} e^{-nx} \right) dx. \square \end{aligned}$$

Now using the definition of $F_{2,p}^{(\alpha, \beta, \alpha' \beta'; m)}(a, b, c; d, e; x, y; m)$ as stated in (1.13), this leads to the desired result.

Remark 3.2. Using the condition $r = 1$, $p = 0$, $\sigma = \rho$ and $\sigma' = \rho'$ in (3.1) we get the integral representation of (2.2). Further, taking the limit as $\alpha \rightarrow \infty$ and $\beta' \rightarrow \infty$, we get the integral representation of (2.3) and (2.4) which is already stated in [4].

Lemma 3.3. Let us now consider a bounded sequence $\{f(N)\}_{N=0}^\infty$ of essentially arbitrary complex numbers. The summation formula for this bounded sequence was reported by Liu [12] and is stated below:

$$(3.4) \quad \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f(r+s) \frac{x^r y^s}{r! s!}.$$

Theorem 3.4. The following integral representation holds true:

$$(3.5) \quad \begin{aligned} & \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \sigma, \rho', \sigma'; r)} (z, t, s, a; p) \\ &= \frac{\Gamma(\gamma) \Gamma(\gamma')}{\Gamma(s) \Gamma(\beta) \Gamma(\beta') \Gamma(\gamma - \beta) \Gamma(\gamma' - \beta')} \\ & \quad \times \int_0^\infty \int_0^1 \int_0^1 x^{s-1} e^{-ax} \mu^{\beta-1} \vartheta^{\beta'-1} (1-\mu)^{\gamma-\beta-1} (1-\vartheta)^{\gamma'-\beta'-1} \\ & \quad \times (1-z\mu e^{-x} - t\vartheta e^{-x})^{-\alpha} {}_1F_1 \left(\rho; \sigma; \frac{-p}{\mu^r (1-\mu)^r} \right) \\ & \quad \times {}_1F_1 \left(\rho'; \sigma'; \frac{-p}{\vartheta^r (1-\vartheta)^r} \right) d\vartheta d\mu dx, \end{aligned}$$

where $|z|+|t|<|e^x|$, $\operatorname{Re}(p)>0$, $\min\{\operatorname{Re}(\rho), \operatorname{Re}(\sigma), \operatorname{Re}(\rho'), \operatorname{Re}(\sigma'), \operatorname{Re}(r)\}>0$; $\min\{\operatorname{Re}(s), \operatorname{Re}(a)\}>0$; $m, n \in \mathbb{N}_0$ when $|z|<1$, $|t|<1$ and $\operatorname{Re}(s)>1$ when $|z|, |t|=1$.

Proof. From (1.7) and (3.3), further changing the order of integration and summation, we get:

$$\begin{aligned} & \phi *_{\alpha, \beta, \beta'; \gamma, \gamma'}^{(\rho, \sigma, \rho', \sigma'; r)} (z, t, s, a; p) \\ &= \frac{1}{\Gamma(s) B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \\ & \quad \times \int_0^\infty \int_0^1 \int_0^1 x^{s-1} e^{-ax} \mu^{\beta-1} \vartheta^{\beta'-1} (1-\mu)^{\gamma-\beta-1} (1-\vartheta)^{\gamma'-\beta'-1} \end{aligned}$$

$$\begin{aligned} & \times {}_1F_1\left(\rho; \sigma; \frac{-p}{\mu^r(1-\mu)^r}\right) {}_1F_1\left(\rho'; \sigma'; \frac{-p}{\vartheta^r(1-\vartheta)^r}\right) \\ & \times \left(\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (z\mu e^{-x})^m (t\vartheta e^{-x})^n}{m!n!} \right) d\mu d\vartheta dx. \end{aligned}$$

Using Lemma 3.3 in the above equation and after some simplifications, we get the desired result.

Corollary 3.5. *Taking $\xi = \frac{\mu}{1-\mu}$ and $\eta = \frac{\vartheta}{1-\vartheta}$ in (3.5), and simplifying this yields,*

$$\begin{aligned} (3.6) \quad & \phi *_{(\alpha,\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} (z, t, s, a; p) \\ & = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(s)\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')} \\ & \times \int_0^\infty \int_0^1 \int_0^1 x^{s-1} e^{-ax} \xi^{\beta-1} \eta^{\beta'-1} (1+\xi)^{\alpha-\gamma} (1+\eta)^{\alpha-\gamma'} \\ & \times [1+\xi(1-ze^{-x}) + \eta(1-te^{-x}) + \eta\xi(1-te^{-x}-ze^{-x})^{-\alpha}] \\ & \times {}_1F_1\left(\rho; \sigma; -p\left(2+\xi+\frac{1}{\xi}\right)^r\right) \\ & \times {}_1F_1\left(\rho'; \sigma'; -p\left(2+\eta+\frac{1}{\eta}\right)^r\right) d\xi d\eta dx. \end{aligned}$$

Corollary 3.6. *Taking $\mu = \sin^2 \theta$ and $\vartheta = \sin^2 \phi$ in (3.5), and after further simplification we arrive at the following result:*

$$\begin{aligned} (3.7) \quad & \phi *_{(\alpha,\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} (z, t, s, a; p) \\ & = \frac{4}{\Gamma(s)B(\beta, \gamma-\beta)B(\beta', \gamma'-\beta')} \\ & \times \int_0^\infty \int_0^{\pi/2} \int_0^{\pi/2} x^{s-1} e^{-ax} \sin^{2\beta-1} \theta \sin^{2\beta'-1} \phi \\ & \times \cos^{2\gamma-2\beta-1} \theta \cos^{2\gamma'-2\beta'-1} \phi (1-ze^{-x} \sin^2 \theta - te^{-x} \sin^2 \phi)^{-\alpha} \\ & \times {}_1F_1(\rho; \sigma; -p \operatorname{cosec}^{2r} \theta \sec^{2r} \theta) \\ & \times {}_1F_1(\rho'; \sigma'; -p \operatorname{cosec}^{2r} \phi \sec^{2r} \phi) d\phi d\theta dx. \end{aligned}$$

4. FEW TRANSFORMATION FORMULAS

Theorem 4.1. *Each of the following transformation formulas for $\phi *_{(\alpha,\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} (z, t, s, a; p)$ holds true:*

$$(4.1) \quad \begin{aligned} & \phi *_{(\alpha,\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} (z, t, s, a; p) \\ & = (1-te^{-x})^{-\alpha} \phi *_{(\alpha,\beta,\gamma'-\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} \left(\frac{z}{1-te^{-x}}, \frac{t}{te^{-x}-1}, s, a; p \right). \end{aligned}$$

$$(4.2) \quad \begin{aligned} & \phi *_{(\alpha,\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} (z, t, s, a; p) \\ & = (1-ze^{-x})^{-\alpha} \phi *_{(\alpha,\gamma-\beta,\beta';\gamma,\gamma')}^{(\rho,\sigma,\rho',\sigma';r)} \left(\frac{z}{ze^{-x}-1}, \frac{t}{1-ze^{-x}}, s, a; p \right). \end{aligned}$$

(4.3)

$$\begin{aligned} & \phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \sigma'; r)} (z, t, s, a; p) \\ &= (1 - ze^{-x}te^{-x})^{-\alpha} \phi *_{(\alpha, \gamma-\beta, \gamma'-\beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \sigma'; r)} \left(\frac{z}{ze^{-x} + te^{-x} - 1}, \frac{t}{ze^{-x} + te^{-x} - 1}, s, a; p \right). \end{aligned}$$

Proof. Setting $\xi = 1 - \vartheta$ and $\eta = 1 - \mu$ respectively in (3.5), after simplifications, this yields (4.1) and (4.2). On taking $\xi = 1 - \vartheta$ and $\eta = 1 - \mu$, (3.5) results into (4.3). \square

5. MELLIN TRANSFORMATION

Mellin Transformation of an integral $f(x)$ is defined as:

$$(5.1) \quad \mathcal{M}\{f(x) : x \rightarrow \alpha\} = \int_0^\infty x^{\alpha-1} f(x) dx.$$

where $0 < x < \infty$ and $\alpha > 0$ is a parameter.

Theorem 5.1. *The Mellin Transform of the function $\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)$ is given as:*

$$\begin{aligned} (5.2) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)) : p \rightarrow \lambda \\ &= \frac{\Gamma(\lambda)}{B(\beta, \gamma - \beta)B(\beta', \gamma' - \beta')} \sum_{m,n,N=0}^{\infty} \left[\frac{(-1)^N (\alpha)_{m+n} (\rho)_N (\lambda)_N z^m t^n}{(\sigma)_N (a+m+n)^s m! n! N!} \right. \\ & \quad \times B(\beta - Nr + m; \gamma - \beta - Nr) \\ & \quad \times B(\beta' + \lambda r + Nr + n; \gamma' - \beta' + \lambda r + Nr)], \end{aligned}$$

where $Re(r) \geq 0$, $Re(\lambda) \geq 0$, $Re(p) \geq 0$, $\min\{Re(\rho), Re(\sigma), Re(\beta+m), Re(\gamma-\beta)\} > 0$ and $\min\{Re(\beta'+n), Re(\gamma'-\beta')\} > 0$.

Proof. Taking the Mellin Transform for $\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)$, we get

$$\begin{aligned} & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)) : p \rightarrow \lambda \\ &= \int_0^\infty p^{\lambda-1} \phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p) dp. \end{aligned}$$

Using (2.1) and afterwards changing the order of summation and integration, this leads to:

$$\begin{aligned} (5.3) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)) : p \rightarrow \lambda \\ &= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} z^m t^n}{(a+m+n)^s m! n!} \\ & \quad \times \int_0^\infty p^{\lambda-1} \frac{B_p^{(\rho, \sigma; r)}(\beta + m, \gamma - \beta) B_p(\beta' + n, \gamma' - \beta')}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} dp. \end{aligned}$$

Using (1.7) and (1.9), the above equation (5.3) becomes:

$$\begin{aligned} (5.4) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p)) : p \rightarrow \lambda \\ &= \frac{1}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} z^m t^n}{(a+m+n)^s m! n!} \end{aligned}$$

$$\begin{aligned} & \times \int_0^1 \int_0^1 \mu^{\beta+m-1} (1-\mu)^{\gamma-\beta-1} \vartheta^{\beta'+n-1} (1-\vartheta)^{\gamma'-\beta'-1} \\ & \times \left(\int_0^\infty p^{\lambda-1} {}_1F_1 \left(\rho; \sigma; -\frac{p}{\mu^r(1-\mu)^r} \right) e^{\frac{-p}{\vartheta^r(1-\vartheta)^r}} dp \right) d\mu d\vartheta. \end{aligned}$$

Setting $\xi = \frac{p}{\mu^r(1-\mu)^r}$ and $c = \frac{\mu^r(1-\mu)^r}{\vartheta^r(1-\vartheta)^r}$ and then using (5.5), we get (5.6),

$$(5.5) \quad \int_0^\infty t^{s-1} e^{-ct} {}_1F_1(a; b; -t) dt = c^{-s} \Gamma(s) {}_2F_1 \left(a, s; b; -\frac{1}{c} \right).$$

$$\begin{aligned} (5.6) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p) : p \rightarrow \lambda) \\ & = \frac{\Gamma(\lambda)}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} z^m t^n}{(a + m + n)^s m! n!} \\ & \times \int_0^1 \int_0^1 [\mu^{\beta+m-1} (1-\mu)^{\gamma-\beta-1} \vartheta^{\beta'+n-1+\lambda r} (1-\vartheta)^{\gamma'-\beta'-1+\lambda r} \\ & \times {}_2F_1 \left(\rho, \lambda; \sigma; -\frac{\vartheta^r(1-\vartheta)^r}{\mu^r(1-\mu)^r} \right)] d\mu d\vartheta. \end{aligned}$$

$$\begin{aligned} (5.7) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \sigma, \rho', \rho'; r)} (z, t, s, a; p) : p \rightarrow \lambda) \\ & = \frac{\Gamma(\lambda)}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} z^m t^n}{(a + m + n)^s m! n!} \\ & \times \int_0^1 \int_0^1 [\mu^{\beta+m-1} (1-\mu)^{\gamma-\beta-1} \vartheta^{\beta'+n-1+\lambda r} (1-\vartheta)^{\gamma'-\beta'-1+\lambda r} \\ & \times \sum_{N=0}^{\infty} \frac{(\rho)_N (\lambda)_N (-1)^N}{(\sigma)_N} \left(\frac{\vartheta^r(1-\vartheta)^r}{\mu^r(1-\mu)^r} \right)^N] d\mu d\vartheta. \end{aligned}$$

After further simplifications and use of (1.10) in (5.7), we obtain the desired result. \square

Theorem 5.2. *The Mellin Transform of the function $\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \rho, \rho', \sigma'; r)} (z, t, s, a; p)$ is:*

$$\begin{aligned} (5.8) \quad & \mathcal{M}(\phi *_{(\alpha, \beta, \beta'; \gamma, \gamma')}^{(\rho, \rho, \rho', \sigma'; r)} (z, t, s, a; p) : p \rightarrow \lambda) \\ & = \frac{\Gamma(\lambda)}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \\ & \times \sum_{m,n,N=0}^{\infty} \left[\frac{(-1)^N (\alpha)_{m+n} (\rho')_N (\lambda)_N z^m t^n}{(\sigma')_N (a + m + n)^s m! n! N!} \right. \\ & \times B(\beta + \lambda r + Nr + m; \gamma - \beta + \lambda r + Nr) \\ & \times B(\beta' - Nr + n; \gamma' - \beta' - Nr)], \end{aligned}$$

where $Re(r) \geq 0$, $Re(\lambda) \geq 0$, $Re(p) \geq 0$, $\min\{Re(\beta + m), Re(\gamma - \beta)\} > 0$ and $\min\{Re(\rho'), Re(\sigma'), Re(\beta' + n), Re(\gamma' - \beta')\} > 0$.

Proof. Proof of Theorem 5.2 is similar to Theorem 5.1. \square

6. CONCLUSION

In this paper, we introduced a new generalization of Extended Hurtwitz–Lerch Zeta Function. In the light of techniques used by Gupta ([3] and [9]), this study can be further extended in the field of q -calculus. Furthermore, it is possible to extend q -calculus to post-quantum calculus.

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REFERENCES

- [1] P. Agarwal, J. Choi and S. Jain, Extended hypergeometric function of two and three variables, *Commun. Korean Math. Soc.* 30(4) (2015), 403-414.
- [2] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York (1976).
- [3] A. Aral, V. Gupta and R.P. Agarwal, *Applications of q Calculus in Operator Theory*, Springer-Verlag, New York, 2013.
- [4] S. Batra and P. Rai, A further extension of generalized Hurwitz Lerch Zeta function of two variables, *International Journal of Pure and Applied Mathematics* 119(18) (2018), 1551-1556.
- [5] M.A. Chaudhry, A. Qadir, H.M. Srivastava and R.B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.* 159(2) (2004), 589-602.
- [6] M.A. Chaudhry, A. Qadir, M. Rafique and S.M. Zubair, Extension of Euler beta function, *J. Comput. Appl. Math.* 78(1) (1997), 19-32.
- [7] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw Hill book Co., New York, Toronto and London (1953).
- [8] S.P. Goyal and R.K. Laddha, On the generalized Zeta function and generalized Lambert's function, *Ganita Sandesh* 11(1997), 99-108.
- [9] V. Gupta, T.M. Rassias, P.N. Agrawal and A. M Acu, *Recent Advances of Constructive Approximation Theory*, Springer Optimization and Its Applications, vol. 138(2018), Springer, Cham.
- [10] T. Kim, Euler numbers and polynomials associated with Zeta functions, *Abstr. Appl. Anal.* (Article ID 581582) (2008), 1-11.
- [11] D.M. Lee, A.K. Rathie, R.K. Parmar and Y.S. Kim, Generalization of extended Beta function, hypergeometric and confluent hypergeometric functions, *Honam Math. J.* 33(2) (2011), 187-206.
- [12] H. Liu, Some generating relations for extended Appell's and Lauricella's hypergeometric functions, *Rocky Mountain J. Math.* 44(6) (2014), 1987-2007.
- [13] E. Özergin, M.A. Özarslan and A. Altın, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.* 235(16) (2011), 4601-4610.
- [14] R.K. Parmar, J. Choi and S.D. Purohit, Further generalization of the extended Hurwitz Lerch Zeta functions, *Bol. Soc. Paran. Mat.* 37(1) (2019), 177-190.
- [15] M.A. Pathan and O.A. Daman, On generalization of Hurwitz zeta function, *Math. Sci. Res. J.* 16(10) (2012), 251-159.
- [16] H.M. Srivastava and J. Choi, *Zeta and q -Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York (2012).

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