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Operations over 3-Dimensional Extended Intuitionistic Fuzzy Index Matrices

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Abstract: In the current paper, an algorithm for index matrix ordering is discussed. The procedure can be applied by either rows or columns. Thereafter, the ordered 3-Dimensional Extended Intuitionistic Fuzzy Index Matrix is divided into sub-matrices. Reduction of the empty cells is proposed. In the end, the sub-matrices are united in the form of 3-Dimensional Extended Intuitionistic Fuzzy Index Matrix.

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1. Introduction of Three dimensional Extended Intuitionistic Fuzzy Index Matrix (3D-EIFIM)

In the current paper a 3D-EIFIM ordering procedure is presented (as a continuation from [1]). The operations for reduction of the empty cells are discussed [4, 8]. The concept of a Three-Dimensional Extended Intuitionistic Fuzzy Index Matrix (3D-EIFIM) is based on the concepts of an Intuitionistic Fuzzy Index Matrix (IFIM), an Extended Index Matrix (EIM) and a Three-Dimensional Index Matrix (IM) [6]. In the current section, we discuss the preliminary remarks for operations in the next sections.

Firstly, we introduce the Intuitionistic Fuzzy Pairs (IFP). IFS is an object in the form $\langle a, b\rangle$, where a, $b \in [0, 1]$ and $a + b < 1$, that is used as an estimation of some object or process and which components (a and b) are represented as degrees of membership and degrees of non-membership, degree of validity and degree of non-validity or degree of correctness and degree of non-correctness, etc [5, 7, 9].

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. Then, following [1], we define

$$x \leq y \text{ if and only if } a \leq c \text{ and } b \geq d$$

$$x \geq y \text{ if and only if } a \geq c \text{ and } b \leq d$$

$$x = y \text{ if and only if } a = c \text{ and } b = d$$

Second, the definition of a Three-Dimensional Extended Index Matrix (3D-EIFIM) will be presented:

$$A=\left[K^{*},L^{*},H^{*},\left\{\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle\right\}\right]$$

$$\equiv \begin{array}{c|ccccc}
h_g,\left\langle \alpha_g^h,\beta_g^h\right\rangle & l_1,\left\langle \alpha_1^l,\beta_1^l\right\rangle & \dots & l_j,\left\langle \alpha_j^l,\beta_j^l\right\rangle & \dots & l_n,\left\langle \alpha_n^l,\beta_n^l\right\rangle \\
\hline
k_1,\left\langle \alpha_1^k,\beta_1^k\right\rangle & \left\langle \mu_{k_1,l_1,h_g},\nu_{k_1,l_1,h_g}\right\rangle & \dots & \left\langle \mu_{k_1,l_j,h_g},\nu_{k_1,l_j,h_g}\right\rangle & \dots & \left\langle \mu_{k_1,l_n,h_g},\nu_{k_1,l_n,h_g}\right\rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_i,\left\langle \alpha_i^k,\beta_i^k\right\rangle & \left\langle \mu_{k_i,l_1,h_g},\nu_{k_i,l_1,h_g}\right\rangle & \dots & \left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle & \dots & \left\langle \mu_{k_i,l_n,h_g},\nu_{k_i,l_n,h_g}\right\rangle \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
k_m,\left\langle \alpha_m^k,\beta_m^k\right\rangle & \left\langle \mu_{k_m,l_1,h_g},\nu_{k_m,l_1,h_g}\right\rangle & \dots & \left\langle \mu_{k_m,l_j,h_g},\nu_{k_m,l_j,h_g}\right\rangle & \dots & \left\langle \mu_{k_m,l_n,h_g},\nu_{k_m,l_n,h_g}\right\rangle
\end{array}$$

where for every $1 \le i \le m$, $1 \le j \le n$, $1 \le g \le f$:

$$0 \le \mu_{k_m,l_1,h_g},\nu_{k_m,l_1,h_g},\mu_{k_m,l_1,h_g}+\nu_{k_m,l_1,h_g} \in [0,1],$$
$$\alpha_i^k,\beta_i^k,\alpha_i^k+\beta_i^k \in [0,1],$$
$$\alpha_j^l,\beta_j^l,\alpha_j^l+\beta_j^l \in [0,1],$$
$$\alpha_g^h,\beta_g^h,\alpha_g^h+\beta_g^h \in [0,1]$$

and

$$K^{*}=\left\{\left\langle k_i,\alpha_i^k,\beta_i^k\right\rangle \middle| k_i \in K\right\}=\left\langle k_i,\alpha_i^k,\beta_i^k\right\rangle|1 \le i \le m,$$

$$L^{*}=\left\{\left\langle l_j,\alpha_j^l,\beta_j^l\right\rangle \middle| l_j \in L\right\}=\left\langle l_j,\alpha_j^l,\beta_j^l\right\rangle|1 \le j \le n,$$

$$H^{*}=\left\{\left\langle h_g,\alpha_g^h,\beta_g^h\right\rangle \middle| h_g \in H\right\}=\left\langle h_g,\alpha_g^h,\beta_g^h\right\rangle|1 \le g \le f,$$

where $\{k_i|1 \le i \le m\} \cup \{l_j|1 \le j \le n\} \cup \{h_g|1 \le g \le f\} \subseteq \phi$ -an index set.

When the evaluation of the indices k_i,l_j and h_g are omitted we obtain an 3D-IFIM. When the IFP $\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle$ of a 3D-EFIM are changed with arbitrary other objects (numbers, propositions or predicates, etc.) we obtain 3D-EIM. In the partial case when these elements are numbers we obtain 3D-IM.

2. Relations over 3D-EIFIM

Let us have two 3D-EIFIMs

$$A=\left[K^*,L^*,H^*,\left\{\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle\right\}\right],\ B=\left[P^*,Q^*,R^*,\left\{\left\langle \mu_{p_r,q_s,r_e},\nu_{p_r,q_s,r_e}\right\rangle\right\}\right].$$

The relations for these 3D-EIFIMs will be the following:

- The strict relation "inclusion about dimension":

$$A\subset_d B\, iff\, \Big(\Big(\big(K^*\subset P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subset R^*\big)\Big)$$
$$\vee\Big(\big(K^*\subseteq P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subset R^*\big)\Big)\vee\Big(\big(K^*\subset P^*\big)\&\big(L^*\subseteq Q^*\big)\&\big(H^*\subset R^*\big)\Big)\Big)$$
$$\vee\Big(\big(K^*\subset P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subseteq R^*\big)\Big)\&\,(\forall k\in K)(\forall l\in L)(\forall h\in H)$$
$$\big(\langle a_{k,l,h},b_{k,l,h}\rangle=\langle c_{k,l,h},d_{k,l,h}\rangle\big)$$

- The non-strict relation "inclusion about dimension"

$$A\subset_d B\ iff\ \big(K^*\subseteq P^*\big)\&\big(L^*\subseteq Q^*\big)\&\big(H^*\subseteq R^*\big)$$
$$\&\big(\forall k\in K\big)\big(\forall l\in L\big)\big(\forall h\in H\big)\big(\langle a_{k,l,h},b_{k,l,h}\rangle=\langle c_{k,l,h},d_{k,l,h}\rangle\big)$$

- The strict relation "inclusion about value"

$$A\subset_v B\, iff\big(K^*=P^*\big)\&\big(L^*=Q^*\big)\&\big(H^*=R^*\big)$$
$$\&\big(\forall k\in K\big)\big(\forall l\in L\big)\big(\forall h\in H\big)\big(\langle a_{k,l,h},b_{k,l,h}\rangle<\langle c_{k,l,h},d_{k,l,h}\rangle\big)$$

- The non-strict relation "inclusion about value"

$$A\subset_v B\ iff\ \big(K^*=P^*\big)\&\big(L^*=Q^*\big)\&\big(H^*=R^*\big)$$
$$\&\big(\forall k\in K\big)\big(\forall l\in L\big)\big(\forall h\in H\big)\big(\langle a_{k,l,h},b_{k,l,h}\rangle\leq\langle c_{k,l,h},d_{k,l,h}\rangle\big)$$

- The strict relation "inclusion"

$$A\subset B\, iff\,\Big(\Big(\big(K^*\subset P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subset R^*\big)\Big)\vee\Big(\big(K^*\subseteq P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subset R^*\big)\Big)$$
$$\vee\Big(\big(K^*\subset P^*\big)\&\big(L^*\subseteq Q^*\big)\&\big(H^*\subset R^*\big)\Big)\vee\Big(\big(K^*\subset P^*\big)\&\big(L^*\subset Q^*\big)\&\big(H^*\subseteq R^*\big)\Big)\Big)$$
$$\&\,(\forall k\in K)(\forall l\in L)(\forall h\in H)\big(\langle a_{k,l,h},b_{k,l,h}\rangle<\langle c_{k,l,h},d_{k,l,h}\rangle\big)$$

- The strict relation "inclusion"

$$A \subset B \ \mathit{iff}\ \left(K^{*} \subseteq P^{*}\right) \& \left(L^{*} \subseteq Q^{*}\right) \& \left(H^{*} \subseteq R^{*}\right)$$
$$\& \left(\forall k \in K\right)\left(\forall l \in L\right)\left(\forall h \in H\right)\left(\left\langle a_{k,l,h}, b_{k,l,h}\right\rangle \leq \left\langle c_{k,l,h}, d_{k,l,h}\right\rangle\right)$$

- The relation “equality”

$$A \subset B \ \mathit{iff}\ \left(K^{*} = P^{*}\right) \& \left(L^{*} = Q^{*}\right) \& \left(H^{*} = R^{*}\right)$$
$$\& \left(\forall k \in K\right)\left(\forall l \in L\right)\left(\forall h \in H\right)\left(\left\langle a_{k,l,h}, b_{k,l,h}\right\rangle = \left\langle c_{k,l,h}, d_{k,l,h}\right\rangle\right)$$

3. Operations “Substitution” over 3D-EIFIM

For the following 3D-EIFIM:

$$A = \left[K^{*}, L^{*}, H^{*}, \left\{\left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}\right\rangle\right\}\right]$$

$$= \begin{array}{c|cccccc}
h_g, \left\langle \alpha_g^h, \beta_g^h \right\rangle & \cdots & l_j, \left\langle \alpha_j^l, \beta_j^l \right\rangle & \cdots & l_q, \left\langle \alpha_q^l, \beta_q^l \right\rangle & \cdots \\
\hline
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
k_i, \left\langle \alpha_i^k, \beta_i^k \right\rangle & \cdots & \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_i,l_q,h_g}, \nu_{k_i,l_q,h_g} \right\rangle & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
k_p, \left\langle \alpha_p^k, \beta_p^k \right\rangle & \cdots & \left\langle \mu_{k_p,l_j,h_g}, \nu_{k_p,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_p,l_q,h_g}, \nu_{k_p,l_q,h_g} \right\rangle & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

The operators [1] will be extended in the case of 3D-EIFIM:

- Substitution of the indices by dimension K

$$\left\{\frac{k_p}{k_i}\right\}_{\rho} A = \begin{array}{c|cccccc}
h_g, \left\langle \alpha_g^h, \beta_g^h \right\rangle & \cdots & l_j, \left\langle \alpha_j^l, \beta_j^l \right\rangle & \cdots & l_q, \left\langle \alpha_q^l, \beta_q^l \right\rangle & \cdots \\
\hline
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
k_p, \left\langle \alpha_p^k, \beta_p^k \right\rangle & \cdots & \left\langle \mu_{k_p,l_j,h_g}, \nu_{k_p,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_p,l_q,h_g}, \nu_{k_p,l_q,h_g} \right\rangle & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
k_i, \left\langle \alpha_i^k, \beta_i^k \right\rangle & \cdots & \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_i,l_q,h_g}, \nu_{k_i,l_q,h_g} \right\rangle & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

- Substitution of the indices by dimension L

$$\left\{\frac{l_q}{l_j}\right\}_{\sigma} A = \begin{array}{c|cccccc} h_g,\left\langle \alpha_g^h, \beta_g^h \right\rangle & \cdots & l_q,\left\langle \alpha_q^l, \beta_q^l \right\rangle & \cdots & l_j,\left\langle \alpha_j^l, \beta_j^l \right\rangle & \cdots \\ \hline \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ k_i,\left\langle \alpha_i^k, \beta_i^k \right\rangle & \cdots & \left\langle \mu_{k_i,l_q,h_g}, \nu_{k_p,l_q,h_g} \right\rangle & \cdots & \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ k_p,\left\langle \alpha_p^k, \beta_p^k \right\rangle & \cdots & \left\langle \mu_{k_p,l_q,h_g}, \nu_{k_i,l_q,h_g} \right\rangle & \cdots & \left\langle \mu_{k_p,l_j,h_g}, \nu_{k_p,l_j,h_g} \right\rangle & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{array}$$

- Substitution of the indices by dimension H

$$\left\{\frac{h_r}{h_g}\right\}_{\sigma} A = \begin{array}{c|cccccc} l_j,\left\langle \alpha_j^l, \beta_j^l \right\rangle & \cdots & h_r,\left\langle \alpha_r^h, \beta_r^h \right\rangle & \cdots & h_g,\left\langle \alpha_g^h, \beta_g^h \right\rangle & \cdots \\ \hline \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ k_i,\left\langle \alpha_i^k, \beta_i^k \right\rangle & \cdots & \left\langle \mu_{k_i,l_j,h_r}, \nu_{k_i,l_j,h_r} \right\rangle & \cdots & \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ k_p,\left\langle \alpha_p^k, \beta_p^k \right\rangle & \cdots & \left\langle \mu_{k_p,l_j,h_r}, \nu_{k_p,l_j,h_r} \right\rangle & \cdots & \left\langle \mu_{k_p,l_j,h_g}, \nu_{k_p,l_j,h_g} \right\rangle & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{array}$$

Thereafter the operation „substitutuion by values” will be extended in the case of 3D-EIFIM.

$$V\left(A; k_i, l_j, h_g \left\langle \varphi, \psi \right\rangle\right)$$

$$\equiv \begin{array}{c|ccccc} h_g,\left\langle \alpha_g^h, \beta_g^h \right\rangle & l_1,\left\langle \alpha_1^l, \beta_1^l \right\rangle & \cdots & l_j,\left\langle \alpha_j^l, \beta_j^l \right\rangle & \cdots & l_n,\left\langle \alpha_n^l, \beta_n^l \right\rangle \\ \hline k_1,\left\langle \alpha_1^k, \beta_1^k \right\rangle & \left\langle \mu_{k_1,l_1,h_g}, \nu_{k_1,l_1,h_g} \right\rangle & \cdots & \left\langle \mu_{k_1,l_j,h_g}, \nu_{k_1,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_1,l_n,h_g}, \nu_{k_1,l_n,h_g} \right\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i,\left\langle \alpha_i^k, \beta_i^k \right\rangle & \left\langle \mu_{k_i,l_1,h_g}, \nu_{k_i,l_1,h_g} \right\rangle & \cdots & \left\langle \varphi, \psi \right\rangle & \cdots & \left\langle \mu_{k_i,l_n,h_g}, \nu_{k_i,l_n,h_g} \right\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m,\left\langle \alpha_m^k, \beta_m^k \right\rangle & \left\langle \mu_{k_m,l_1,h_g}, \nu_{k_m,l_1,h_g} \right\rangle & \cdots & \left\langle \mu_{k_m,l_j,h_g}, \nu_{k_m,l_j,h_g} \right\rangle & \cdots & \left\langle \mu_{k_m,l_n,h_g}, \nu_{k_m,l_n,h_g} \right\rangle \end{array}$$

Obviously

$$V\left(V(A;k_i,l_j,h_g;\langle\varphi,\psi\rangle);k_i,l_j,h_g;\left\langle\mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle\right)=A$$

and

$$V\left(V(A;k_i,l_j,h_g;\langle\varphi,\psi\rangle);k_p,l_q,h_r;\langle\chi,\omega\rangle\right)=V\left(V\left(A;k_p,l_q,h_r;\langle\chi,\omega\rangle\right);k_i,l_j,h_g;\langle\varphi,\psi\rangle\right)$$

.

4. Operations "Reduction" over 3D-EIFIM

It holds that

$$A_{\left(\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right),\perp,\perp\right)}=\left[K-\left\{\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\right\},L,H,\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle\right]$$

where

$$\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle=\left\langle\mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle$$

for $t_u=k_i\in K-\left\{\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\right\}$, $v_w=l_j\in L$, $x_y=h_g\in H$,

$$A_{\left(\perp,\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right),\perp\right)}=\left[K,L-\left\{\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\right\},H,\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle\right]$$

where

$$\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle=\left\langle\mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle$$

for $t_u=k_i\in K,v_w=l_j\in L-\left\{\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\right\}$, $x_y=h_g\in H$,

$$A_{\left(\perp,\perp,\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right)}=\left[K,L,H-\left\{\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right\},\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle\right]$$

where

$$\left\langle\mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle=\left\langle\mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle$$

for $t_u=k_i\in K,v_w=l_j\in L,\ x_y=h_g\in H-\ \left\{\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right\}$.

Therefore,

$$A_{\left(\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right),\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right),\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right)}=\left(\left(A_{\left(\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right),\perp,\perp\right),}\right)_{\left(\perp,\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right),\perp\right)}\right)_{\left(\perp,\perp,\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right)}$$

$$A_{\left(\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right),\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right),\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right)}$$

$$=\left[K-\left\{\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\right\},L-\left\{\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\right\},H-\left\{\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right\},\left\langle \mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle\right]$$

where

$$\left\langle \mu_{t_u,v_w,x_y},\nu_{t_u,v_w,x_y}\right\rangle=\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle$$

for

$$t_u=k_i\in K-\left\{\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\right\},v_w=l_j\in L-\left\{\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\right\},x_y=h_g\in H-\left\{\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\right\}$$

We can write an example of operation „automatic reduction" for 3-dimensional extended intuitionistic fuzzy index matrix A.

$$@(A)=\left[P,Q,R,\left\{\left\langle \mu_{p_r,q_s,r_d},\nu_{p_r,q_s,r_d}\right\rangle\right\}\right]$$

where $P\subseteq K,Q\subseteq L,R\subseteq H$ are the index sets having the following property:

$$\left(\forall\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\in K-P\right)\left(\forall\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\in L\right)\left(\forall\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\in H\right)\left(\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle=\perp\right)$$

$$\&\left(\forall\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\in K\right)\left(\forall\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\in L-Q\right)\left(\forall\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\in H\right)\left(\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle=\perp\right)$$

$$\&\left(\forall\left(k_i,\left\langle a_i^k,b_i^k\right\rangle\right)\in K\right)\left(\forall\left(l_j,\left\langle a_j^l,b_j^l\right\rangle\right)\in L\right)\left(\forall\left(h_g,\left\langle a_g^h,b_g^h\right\rangle\right)\in H-R\right)\left(\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle=\perp\right)$$

$$\&\left(\forall\left(p_r,\left\langle a_r^p,b_r^p\right\rangle\right)=\left\langle \mu_i,\nu_i\right\rangle\in P\right)\left(\forall\left(q_s,\left\langle a_s^q,b_s^q\right\rangle\right)=\left\langle \mu_j,\nu_j\right\rangle\in Q\right)\left(\forall\left(r_d,\left\langle a_d^r,b_d^r\right\rangle\right)=\left\langle \mu_g,\nu_g\right\rangle\in H\right)$$

$$\left(\left\langle \mu_{p_r,q_s,r_d},\nu_{p_r,q_s,r_d}\right\rangle=\left\langle \mu_{k_i,l_j,h_g},\nu_{k_i,l_j,h_g}\right\rangle\right)$$

5. Extremal sets

We use the notation presented in [1] where a new operation is introduced with the aim to determine the indices of the *A*-element with an extremal value.

Let *ext*$\in$ {min, max}.

$$\varepsilon_{ext}A=\left\langle\left\{\left\langle k_{r_1},l_{s_1},h_{d_1}\right\rangle,\ldots,\left\langle k_{r_t},l_{s_t},h_{d_t}\right\rangle\right\};\langle\mu,\nu\rangle\right\rangle,$$

where $1 \leq t \leq \max(m, n)$, $\langle\mu,\nu\rangle=\left\langle\mu_{k_{r_1},l_{s_1},h_{d_1}},\nu_{k_{r_1},l_{s_1},h_{d_1}}\right\rangle=\ldots=\left\langle\mu_{k_{r_t},l_{s_t},h_{d_t}},\nu_{k_{r_t},l_{s_t},h_{d_t}}\right\rangle$ and the IFP $\langle\mu,\nu\rangle$ has an extremal value (maximum or minimum) among the *A*-elements. The *ext* can contain one or more elements. The IFP are compared using the relation $\leq$ discussed above. If *ext* contains more than one extremal value then we compare the IFP of indices represented as intuitionistic fuzzy estimations of the elements. We construct (*m*, *n*) extremal sets having the indices and values of the elements by rows or by columns.

The extremal sets by dimension *L* have the following form:

$$\varepsilon_{ext}\left(A,l_j\right)=\left\langle\left\{\left\langle k_{r_1},l_{s_j},h_{e_1}\right\rangle,\ldots,\left\langle k_{r_t},l_{s_j},h_{e_t}\right\rangle\right\};\langle\mu,\nu\rangle\right\rangle,$$

where $1 \leq t \leq \max(m, n, f)$ and $1 \leq j \leq n$,

$$\langle\mu,\nu\rangle=\left\langle\mu_{k_{r_1},l_{s_j},h_{e_1}},\nu_{k_{r_1},l_{s_j},h_{e_1}}\right\rangle=\ldots=\left\langle\mu_{k_{r_t},l_{s_j},h_{e_t}},\nu_{k_{r_t},l_{s_j},h_{e_t}}\right\rangle$$

The extremal sets by dimension *K* have the following form:

$$\varepsilon_{ext}\left(A,k_i\right)=\left\langle\left\{\left\langle k_{r_i},l_{s_1},h_{e_1}\right\rangle,\ldots,\left\langle k_{r_i},l_{s_t},h_{e_t}\right\rangle\right\};\langle\mu,\nu\rangle\right\rangle,$$

where $1 \leq t \leq \max(m, n, f)$ and $1 \leq i \leq m$,

$$\langle\mu,\nu\rangle=\left\langle\mu_{k_{r_i},l_{s_1},h_{e_1}},\nu_{k_{r_i},l_{s_1},h_{e_1}}\right\rangle=\ldots=\left\langle\mu_{k_{r_i},l_{s_t},h_{e_t}},\nu_{k_{r_i},l_{s_t},h_{e_t}}\right\rangle$$

The extremal sets by dimension *H* have the following form:

$$\varepsilon_{ext}\left(A,h_g\right)=\left\langle\left\{\left\langle k_{r_1},l_{s_1},h_{e_g}\right\rangle,\ldots,\left\langle k_{r_t},l_{s_t},h_{e_g}\right\rangle\right\};\langle\mu,\nu\rangle\right\rangle,$$

where $1 \leq t \leq \max(m, n, f)$ and $1 \leq g \leq f$,

$$\langle \mu,\nu \rangle = \left\langle \mu_{k_{r_1},l_{s_1},h_{e_g}}, \nu_{k_{r_1},l_{s_1},h_{e_g}} \right\rangle = \ldots = \left\langle \mu_{k_{r_t},l_{s_t},h_{e_g}}, \nu_{k_{r_t},l_{s_t},h_{e_g}} \right\rangle .$$

Obviously, the number of extremal sets in one dimension is equal to the number of columns/rows/indices in this dimension.

6. Procedure for 3D-EIFIM ordering and columns / rows empty cells removing

Step 1) Let we have 3D-EIFIM A. We will discuss the procedure of ordering the elements of 3D-EIFIM and subsequently task of deleting the empty cells of 3D-EIFIM. We present the case when the extremum is maximum. The similar procedure is possible to be applied when the extremum is minimum.

Step 2) We have the 3D-EIFIM $B = A$.

Step 3) We construct extremal sets for all different columns/rows indices in the dimensions of the 3D-EIFIM B.

Step 4) Construct the 3D-EIFIM C with dimension $m \times n \times f$, whose elements are $\langle 0, 1\rangle$. When the extremum is minimum, these elements will be $\langle 1, 0\rangle$.

Step 5.1) In the next step we order the elements using the extremal sets from 3D-EIFIM B. For example we can order the elements of dimension L column by column using the extremal sets. We compare the values using the relation ≤ for the sets with extremal elements $\varepsilon_{ext}\left(A,l_1\right),\ldots,\varepsilon_{ext}\left(A,l_n\right)$ in each column separately. If exists two or more values that are equal we compare the IFPs of the indices $\left\langle \left\{ \left\langle k_{r_1}, l_{s_j}, h_{e_1} \right\rangle, \ldots, \left\langle k_{r_t}, l_{s_j}, h_{e_t} \right\rangle \right\} ; \langle \mu,\nu \rangle \right\rangle$. Using local substitutions the maximum elements are substituted in the first, second and etc. cells of the columns in 3D-EIFIM C. We continue the comparison of maximum values. The empty cells will be moved to the end of the columns, i.e. the empty cells will be in the bottom of the 3D-EIFIM C. The same procedure can be repeated for dimension H and dimension K – comparison by rows.

Step 5.2) Using substitution V, we change the values of B-elements from sets $E_{ext}B$ with value $\langle 0,1\rangle$. If all elements of B are already equal to $\langle 0,1\rangle$, the process stops, else, we return to point 3.

Step 6) When we have an ordered 3D-EIFIM C we separate it on the n 3D-EIFIM E_1, …, E_n. The purpose of the step is to divide the matrix to the several 3D-EIFIMs on the base of empty cells. We construct the 3D-EIFIMs E_1, …, E_n using operations "projection" and „structural subtraction" over 3D-EIFIM C. The 3D-EIFIM C is divided on n non-overlapping 3D-EIFIMs E_1, …, E_n (these 3D-EIFIMs do not have the common indices by rows and columns together/simultaneously).

Let $P \subseteq K$, $Q \subseteq L$, $R \subseteq H$. Then

$$pr_{P,Q,R}C = \left[P, Q, R, \left\{ \left\langle \phi_{k_i,l_j,h_g}, \psi_{k_i,l_j,h_g} \right\rangle \right\} \right],$$

where

$$(\forall k_i \in P)(\forall l_j \in Q)(\forall l_j \in Q)\left(\left\langle \phi_{k_i,l_j,h_g}, \psi_{k_i,l_j,h_g} \right\rangle = \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle \right).$$

This procedure will perform a segmentation of the 3D-EIFIMs. For example we can perform multiple projections of $A = \left[K^*, L^*, H^*, \left\{ \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle \right\} \right]$ having the condition for non-intersecting elements. The first two sub-matrices will have the form $pr_{[p_1,p_2,p_3],[q_1,q_2,q_3],[r_1,r_2,r_3]}C = \left[P, Q, R, \left\{ \left\langle \phi_{k_i,l_j,h_g}, \psi_{k_i,l_j,h_g} \right\rangle \right\} \right]$, $pr_{[p_4,p_5,p_6],[q_1,q_2,q_3],[r_1,r_2,r_3]}C = \left[P, Q, R, \left\{ \left\langle \phi_{k_i,l_j,h_g}, \psi_{k_i,l_j,h_g} \right\rangle \right\} \right]$ and etc. where $\{ p_1, p_2, p_3 \} \in P^1$, $\{ p_4, p_5, p_6 \} \in P^2$, $\{P^1, P^2,.., P^r\} \in P$ and $\{q_1, q_2, q_3 \} \in Q$, $\{Q^1, Q^2,.., Q^s\} \in Q$ and $\{r_1, r_2, r_3 \} \in R$, $\{R^1, R^2,.., R^d \in R$. The sub-dimensions have the property $D^x \cap D^y = \varnothing$, where $x \neq y$.

Thereafter we subtract the projected sub-matrix E_1 from 3D-EIFIM C using operation "Structural subtraction" (similar to "InterCube Difference" operation in OLAP). The operation "Structural subtraction" has the following form:

Let we have A = [K, L, $H, \left\{ \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle \right\}$]; B = [P, Q, R, $\left\{ \left\langle \rho_{p_r,q_s,r_d}, \sigma_{p_r,q_s,r_d} \right\rangle \right\}$]:

$$A - B = K - P, L - Q, H - R, \left\{ \left\langle \varphi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y} \right\rangle \right\}$$

where

$$c_{t_u,v_w,x_y} = a_{k_i,l_j,h_g} \text{ for } t_u = k_i \in K - P,\ v_w = l_j \in L–Q \text{ and } x_y = h_g \in H–R.$$

Thereafter, we receive the rest of 3D-EIFIM C: $C = C - pr_{P,Q,R}C$. Then, we can continue with "projection" operation. The procedure is repeated until $C = \varnothing$.

Step 7) We apply operations reduction/automatic reduction over n 3D-EIFIM E_1, …, E_n.

Step 8) We construct 3D-EIFIM B from 3D-EIFIM E_1,…, E_n using "addition" operation.

$$B = E_1 \oplus E_2 \oplus \ldots \oplus E_n$$

Note that we do not have overlapping elements or indices (by rows, by columns). In this case we have a reduced case of "addition" operation.

Obviously, the operation will present the union of two matrices with non-intersecting elements.

$$A \oplus_{(\circ,*)} B = \left[T^*, V^*, X^*, \left\{\left\langle \varphi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y} \right\rangle\right\}\right],$$

where

$$T^* = K^* \cup P^* = \left\{\left\langle t_u, a_u^t, b_u^t \right\rangle \middle| t_u \in K \cup P\right\},$$

$$V^* = L^* \cup Q^* = \left\{\left\langle v_w, a_w^v, b_w^v \right\rangle \middle| v_w \in L \cup Q\right\},$$

$$X^* = H^* \cup R^* = \left\{\left\langle x_y, a_y^x, b_y^x \right\rangle \middle| x_y \in H \cup R\right\},$$

$$\alpha^{t_u} = \begin{cases} \alpha^{k_i} \ \ if \ t_u \in K - P \\ \alpha^{p_r} \ \ if \ t_u \in P - K \end{cases}$$

$$\beta^{v_w} = \begin{cases} \beta^{l_j} \ \ if \ v_w \in L - Q \\ \beta^{q_s} \ \ if \ v_w \in Q - L \\ \beta^{h_g} \ \ if \ v_w \in H - R \\ \beta^{r_d} \ \ if \ v_w \in R - H \end{cases}$$

and

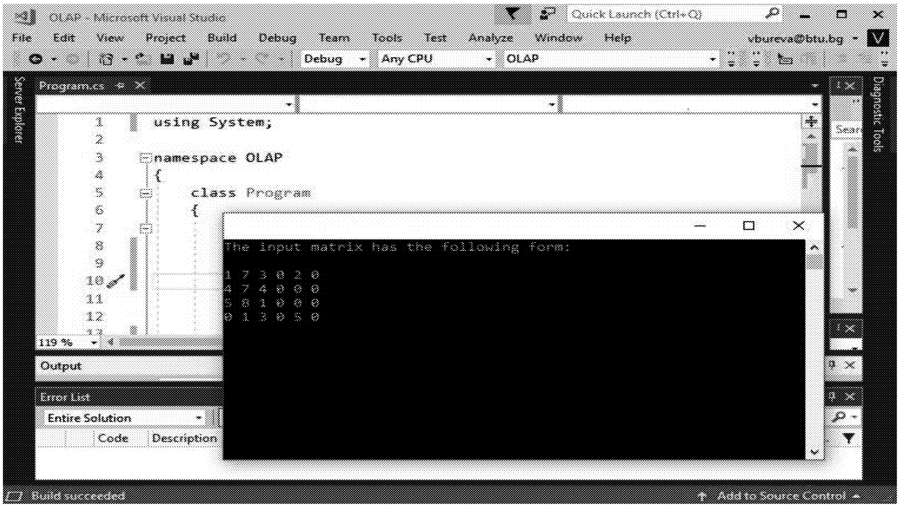
$$\left\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y} \right\rangle = \begin{cases} \left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle, & if \ t_u = k_i \in K, \ v_w = l_j \in L \ and \ x_y = h_g \in H - R \\ & or \ t_u = k_i \in K, \ v_w = l_j \in L - Q \ and \ x_y = h_g \in H \\ & or \ t_u = k_i \in K - P, \ v_w = l_j \in L \ and \ x_y = h_g \in H \\ \left\langle \rho_{p_r,q_s,r_d}, \sigma_{p_r,q_s,r_d} \right\rangle, & if \ t_u = p_r \in P, \ v_w = q_s \in Q \ and \ x_y = r_d \in R - H \\ & or \ t_u = p_r \in P, \ v_w = q_s \in Q - L \ and \ x_y = r_d \in R \\ & or \ t_u = p_r \in P - K, \ v_w = q_s \in Q \ and \ x_y = r_d \in R \\ \left\langle \circ\left\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g} \right\rangle, \right. & if \ t_u = k_i = p_r \in K \cap P, \ v_w = l_j = q_s \in L \cap Q \\ \left. *\left\langle \rho_{p_r,q_s,r_d}, \sigma_{p_r,q_s,r_d} \right\rangle \right\rangle & and \ x_y = h_g = r_d \in H \cap R \\ \langle 0,1 \rangle & otherwise \end{cases}$$

where $\langle \circ, * \rangle \in \{\langle \max, \min \rangle, \langle \min, \max \rangle\}$.

The theory of index matrices is applied in applications of different scientific areas: neural networks [2, 3], databases [10], solving problems [11] and the topic of the OLAP cubes [12–14, 17–19].

7. Algorithm explanation

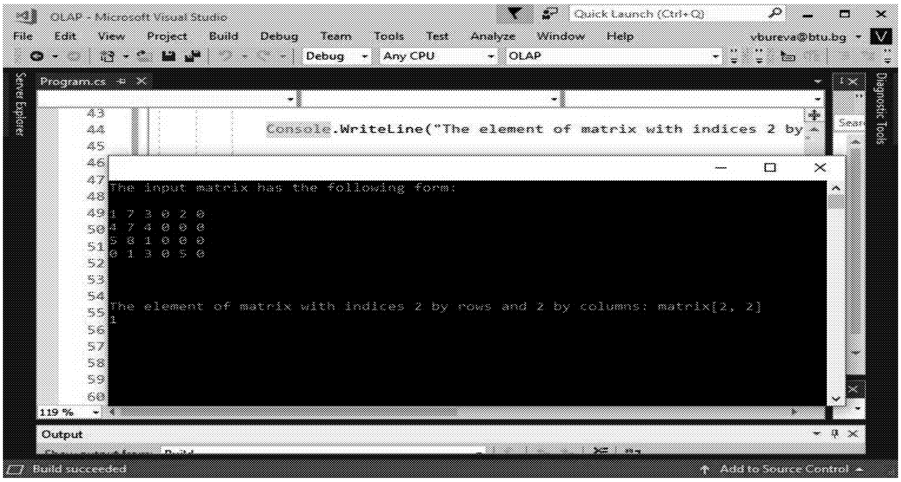
The algorithm will be explained using an example of arrays for representation of index matrices. Each array has indices by rows and by columns beginning with the value 0. Let us have a 2D slice of OLAP cube. We have measure values represented in the



```
using System;

namespace OLAP
{
    class Program
    {
```

```
The input matrix has the following form:

1 7 3 0 2 0
4 7 4 0 0 0
5 8 1 0 0 0
0 1 3 0 5 0
```

Fig.1 The input slice/matrix

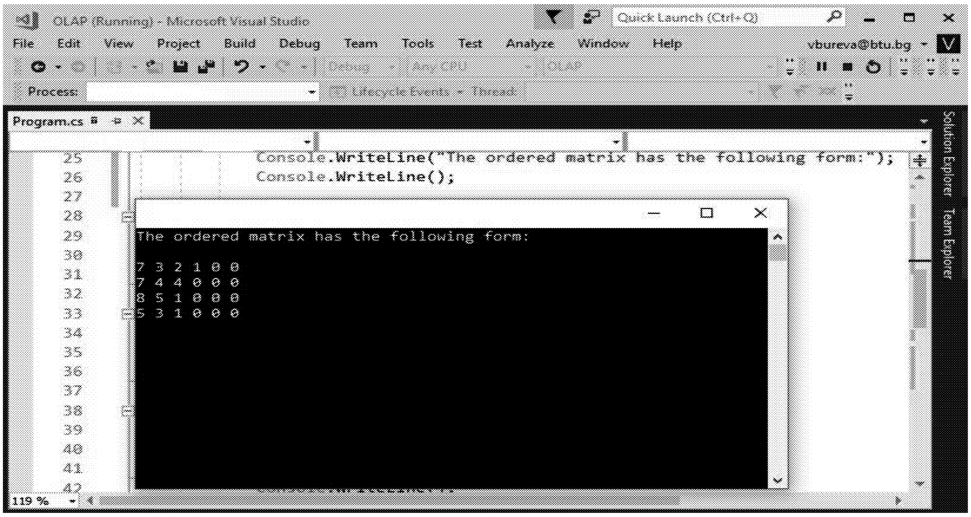


```
Console.WriteLine("The element of matrix with indices 2 by
```

```
The input matrix has the following form:

1 7 3 0 2 0
4 7 4 0 0 0
5 8 1 0 0 0
0 1 3 0 5 0



The element of matrix with indices 2 by rows and 2 by columns: matrix[2, 2]
1
```

Fig.2 Selected element of the matrix

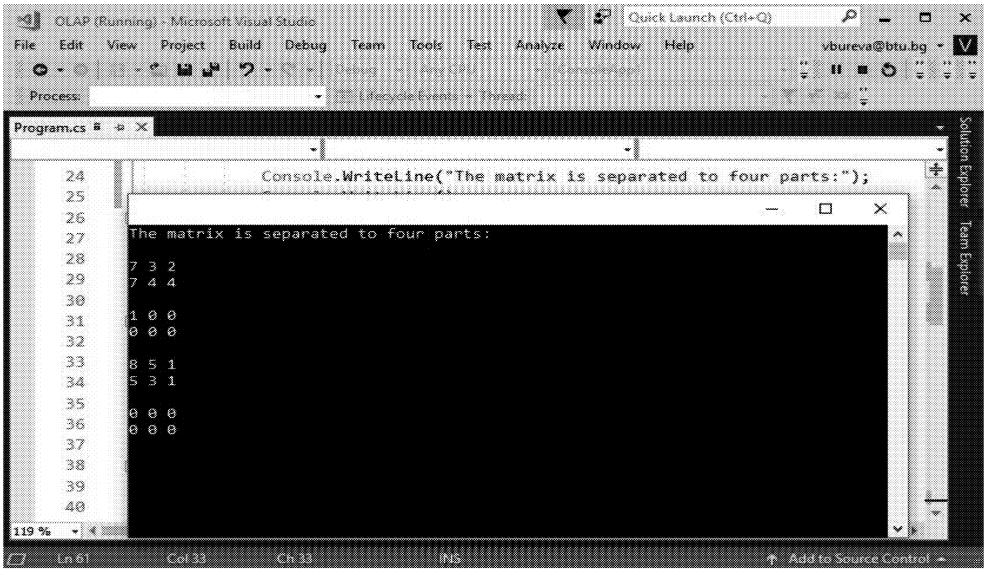
form of an index matrix (Fig.1). The indices of the matrices in the form of arrays are not displayed but we can address them by code. In this way we can display the value saved in the element of matrix placed on the row 3 and on the column 3 writing the code Console.WriteLine(matrix[2,2]) using a standard array in C#. We will obtain the number 1 because the numbering of rows and numbering of columns begins by 0 in standard arrays. The associative array/dictionary works in a different way and that will not be discussed in this example.



```
Console.WriteLine("The ordered matrix has the following form:");
Console.WriteLine();
```

```
The ordered matrix has the following form:

7 3 2 1 0 0
7 4 4 0 0 0
8 5 1 0 0 0
5 3 1 0 0 0
```

Fig.3 Sorted slice/matrix

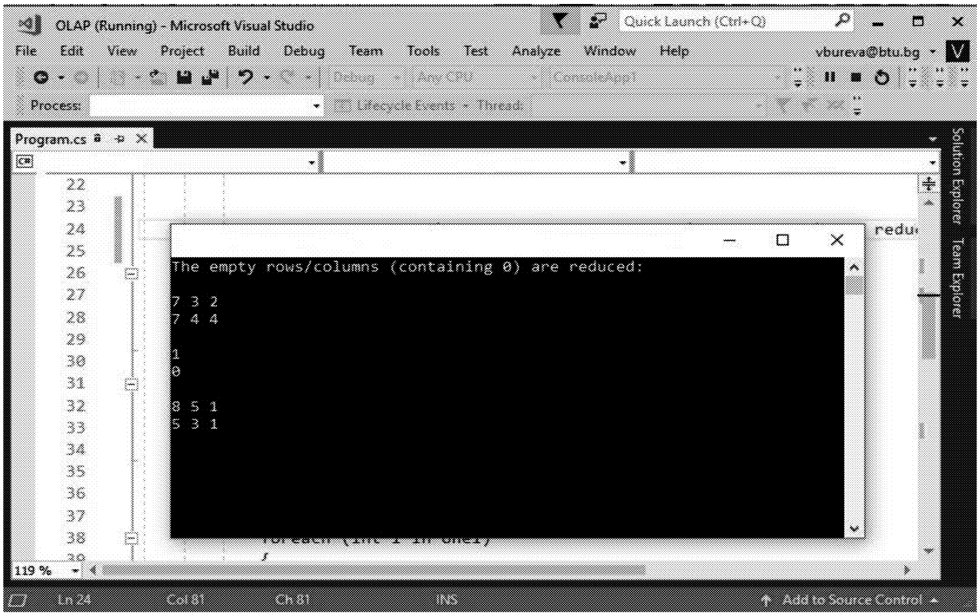


```
Console.WriteLine("The matrix is separated to four parts:");
```

```
The matrix is separated to four parts:

7 3 2
7 4 4

1 0 0
0 0 0

8 5 1
5 3 1

0 0 0
0 0 0
```

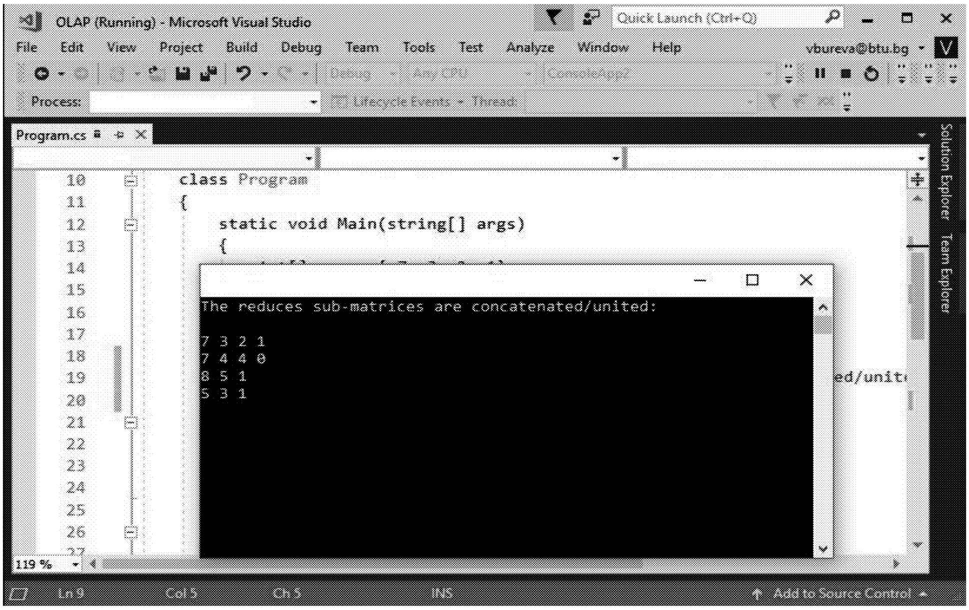
Fig.4 Separated slice/matrix

The input slice / matrix is ordered in a descending order by rows. The maximal elements are placed to the left, the minimal elements are placed to the right (Fig.3). The ordered slice / matrix is separated in four equal parts (Fig.4). Thereafter the reducing operation by the columns of the matrices is applied (Fig.5).



```
The empty rows/columns (containing 0) are reduced:

7 3 2
7 4 4

1
0

8 5 1
5 3 1
```

Fig.5 Empty cells reduction



```
class Program
{
    static void Main(string[] args)
    {
```

```
The reduces sub-matrices are concatenated/united:

7 3 2 1
7 4 4 0
8 5 1
5 3 1
```

Fig.6 Resulting slice/matrix

At the last step, the reduced slices / matrices are joined together again. The received structure can be explained as a jagged array (Fig.6).

8. Presentation of the OLAP cube operations for ordering and deleting

In the current section, OLAP operations for the data ordering and deleting empty cells will be presented. The operations are performed over the processed "Bookshops" OLAP cube in the IDE (Integrated development environment) Visual Studio using SSDT (SQL Server Data Tools). The "Bookshops" OLAP cube is constructed using a data source of the relational database "Bookshops Database". The "Bookshops" OLAP cube has five dimensions: Books, Bookshops, Authors, Time, Location and one fact table: Sales. The attributes of Books dimension are presented in the Books hierarchy (up-down): Publisher / Genre / Title. Bookshops hierarchy has the form: Owner / Regional Manager / Bookshop Name. The Authors hierarchy is presented as: Type / Author Name. The attributes of Location dimension are organized in the following hierarchy: Country / Town. Time hierarchy is constructed by the attributes: Year / Month / Date. The measure of the Bookshop OLAP cube presents the sold books [15, 16]. The attributes have assigned members. The data source view for Bookshop OLAP cube is presented (Fig.7).

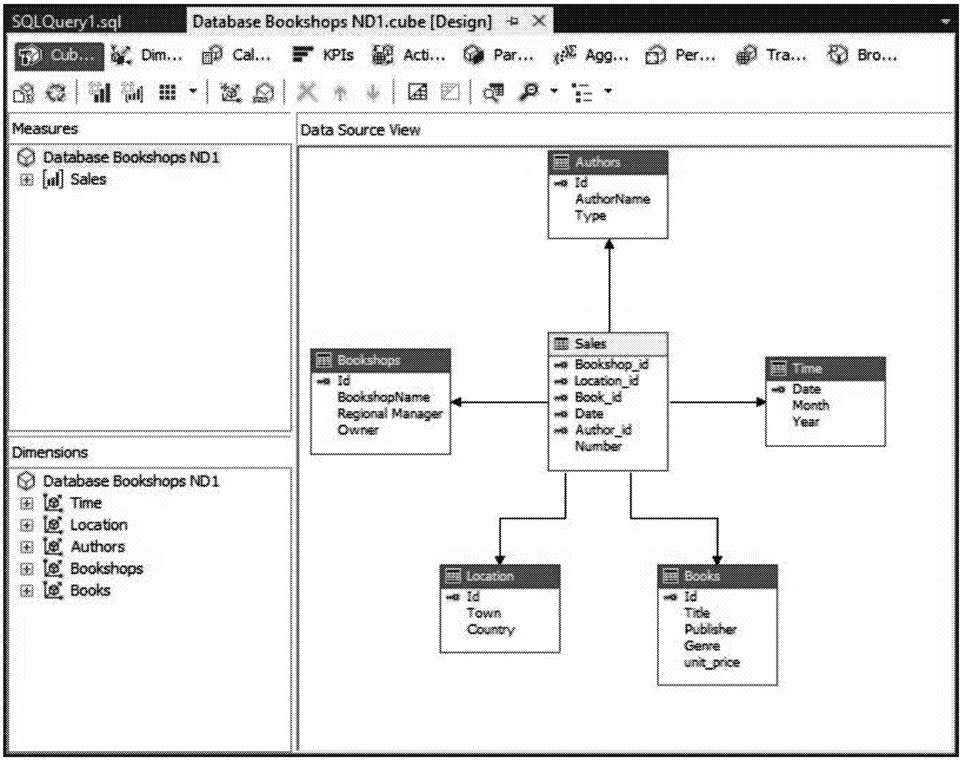
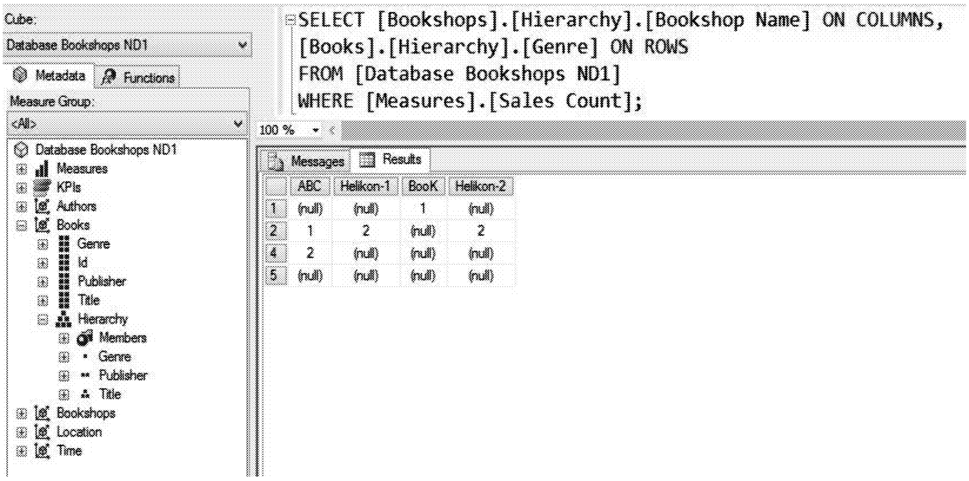


Fig.7 The data source view of the Bookshops OLAP cube in Visual Studio IDE

The selected language for request performing is Multidimensional Expressions (MDX). A simple query is presented in the Fig.8. The result presents sold books by genre. Intentionally the OLAP cube contains empty cells: these are used for deleting operations.

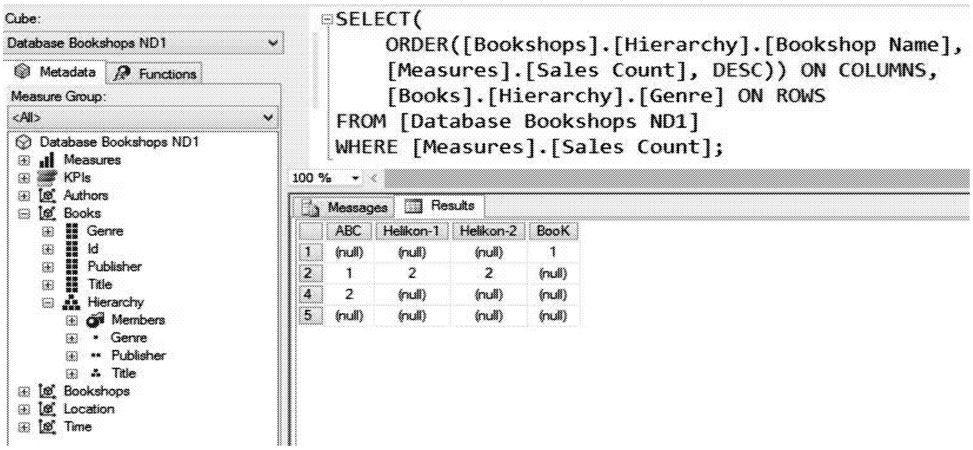
The function ORDER() is applied onto the next query to sort the sold books by genre. Obviously, the ordering is performed with respect to the column with maximal values according to the selected genres. The next columns are sorted by the selected order of the attributes of the first column (Fig.9). Here the notation for local reduction with the property of saving indices and intuitionistic fuzzy estimations is not included.



```
SELECT [Bookshops].[Hierarchy].[Bookshop Name] ON COLUMNS,
[Books].[Hierarchy].[Genre] ON ROWS
FROM [Database Bookshops ND1]
WHERE [Measures].[Sales Count];
```

	ABC	Helikon-1	BooK	Helikon-2
1	(null)	(null)	1	(null)
2	1	2	(null)	2
4	2	(null)	(null)	(null)
5	(null)	(null)	(null)	(null)

Fig.8 MDX query

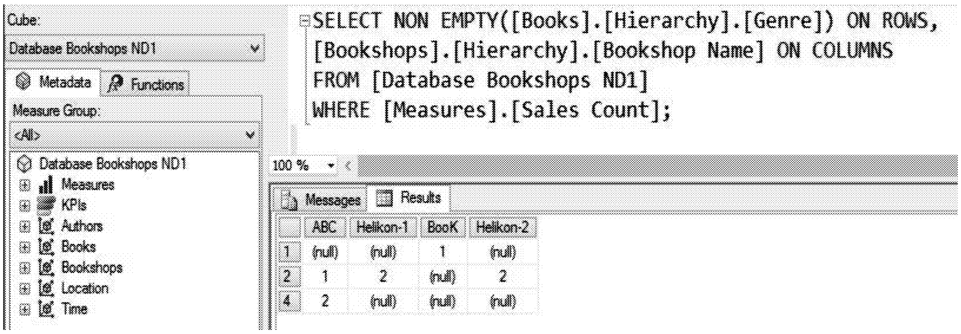


```
SELECT(
    ORDER([Bookshops].[Hierarchy].[Bookshop Name],
    [Measures].[Sales Count], DESC)) ON COLUMNS,
    [Books].[Hierarchy].[Genre] ON ROWS
FROM [Database Bookshops ND1]
WHERE [Measures].[Sales Count];
```

	ABC	Helikon-1	Helikon-2	BooK
1	(null)	(null)	(null)	1
2	1	2	2	(null)
4	2	(null)	(null)	(null)
5	(null)	(null)	(null)	(null)

Fig.9 OLAP function ORDER()

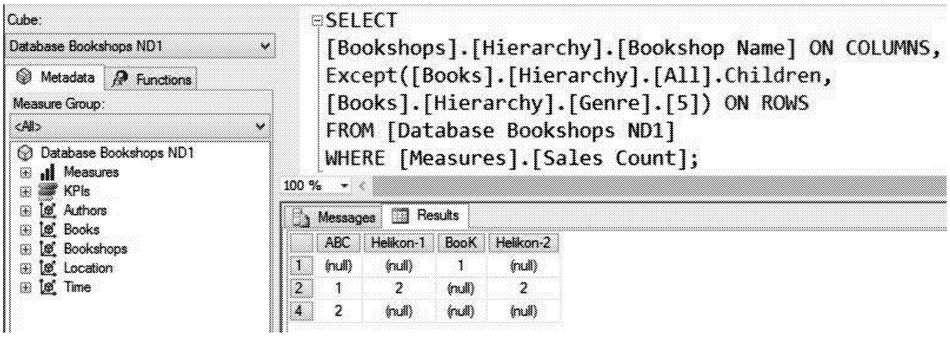
An alternative of temporary hiding of the rows/columns with null values in OLAP is to use the functions NONEMPTY() or NON EMPTY(). This method does not delete the values of row with empty cells as the row for genre number 5 (Fig.10)



```
SELECT NON EMPTY([Books].[Hierarchy].[Genre]) ON ROWS,
[Bookshops].[Hierarchy].[Bookshop Name] ON COLUMNS
FROM [Database Bookshops ND1]
WHERE [Measures].[Sales Count];
```

	ABC	Helikon-1	BooK	Helikon-2
1	(null)	(null)	1	(null)
2	1	2	(null)	2
4	2	(null)	(null)	(null)

Fig.10 OLAP function NON EMPTY()

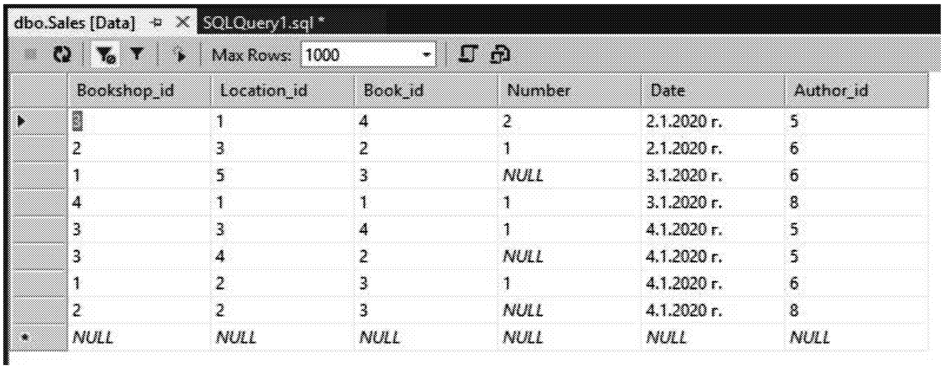


```
SELECT
[Bookshops].[Hierarchy].[Bookshop Name] ON COLUMNS,
Except([Books].[Hierarchy].[All].Children,
[Books].[Hierarchy].[Genre].[5]) ON ROWS
FROM [Database Bookshops ND1]
WHERE [Measures].[Sales Count];
```

	ABC	Helikon-1	BooK	Helikon-2
1	(null)	(null)	1	(null)
2	1	2	(null)	2
4	2	(null)	(null)	(null)

Fig.11 OLAP function EXCEPT()

The same situation is presented using the EXCEPT() function. It temporary hides the selected member 5 of attribute Genre in dimension Books (Fig.11). The information deleting has to be performed in the database. The database table Sales is presented (Fig.12). Obviously, there are three sales without number.

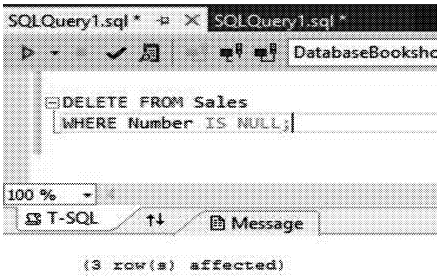


	Bookshop_id	Location_id	Book_id	Number	Date	Author_id
▶	[illegible]	1	4	2	2.1.2020 г.	5
	2	3	2	1	2.1.2020 г.	6
	1	5	3	NULL	3.1.2020 г.	6
	4	1	1	1	3.1.2020 г.	8
	3	3	4	1	4.1.2020 г.	5
	3	4	2	NULL	4.1.2020 г.	5
	1	2	3	1	4.1.2020 г.	6
	2	2	3	NULL	4.1.2020 г.	8
*	NULL	NULL	NULL	NULL	NULL	NULL

Fig.12 Table Sales

We execute an SQL Delete query to remove the rows without assigned values for column

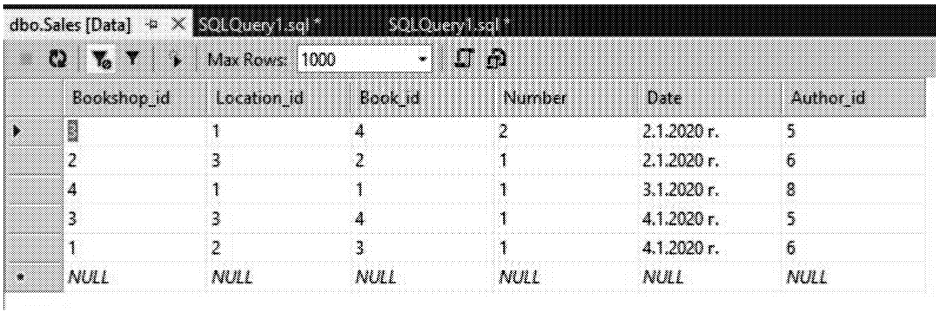
Number (Fig.13). Three rows from table Sales are deleted (Fig.14). Thereafter we can synchronize the information of the data source with the data in OLAP cube. The Bookshops OLAP cube has to be processed again to upload the corrected data (Fig. 15). Two ways are presented here: to process the OLAP cube directly using Visual Studio and SQL Server Analysis Service (Fig.9), process the Cube using SQL Server Integration Service (Fig.10). Additionally processing can be applied in a different scenario: Process Full, Process Update. The Process Update processing add, remove or update the information of OLAP cube without deleting and uploading the whole data from the database to the OLAP cube (as ETL task). The Process Full option deletes all the information from the OLAP cube and uploads again the information from the database. It is important to pay attention that the OLAP (Online Analytical Processing) is used for analyzing the data while OLTP (Online Transactional Processing) is used for transaction-oriented tasks (inserting, updating, and/or deleting small amounts of data in the database).



```
DELETE FROM Sales
WHERE Number IS NULL;
```

```
(3 row(s) affected)
```

Fig.13 SQL DELETE query



Bookshop_id	Location_id	Book_id	Number	Date	Author_id
[illegible]	1	4	2	2.1.2020 г.	5
2	3	2	1	2.1.2020 г.	6
4	1	1	1	3.1.2020 г.	8
3	3	4	1	4.1.2020 г.	5
1	2	3	1	4.1.2020 г.	6
NULL	NULL	NULL	NULL	NULL	NULL

Fig.14 Table Books after Delete query

The OLAP cube can be updated from the database using the SSIS (SQL Server Integration Service) package. We add the Analysis Services Processing Task (Fig.16) in the Control Flow tab. We have to edit the Analysis Services Processing Task. We have to update the measure of the OLAP cube. We select full processing of the Measure Group of the OLAP cube. In the case of big OLAP structures, the Process Full operation is not desired: it can be completed for hours or days. This task of OLAP cube processing will be presented step by step (Fig.17).

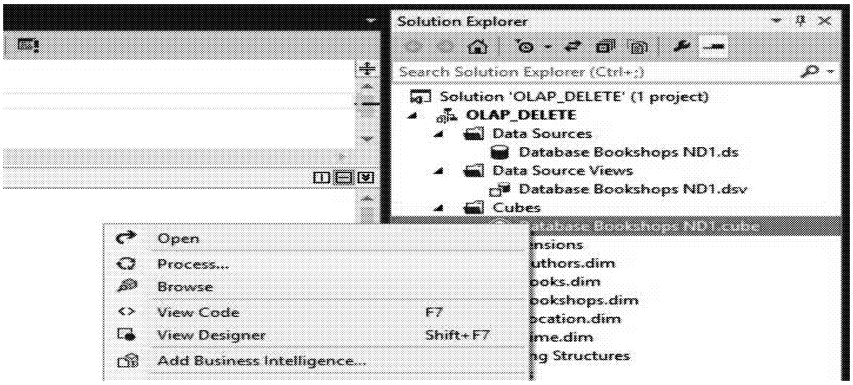


Fig.15 Process OLAP cube in Visual Studio

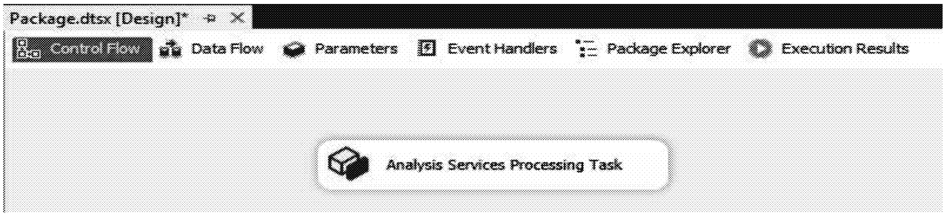
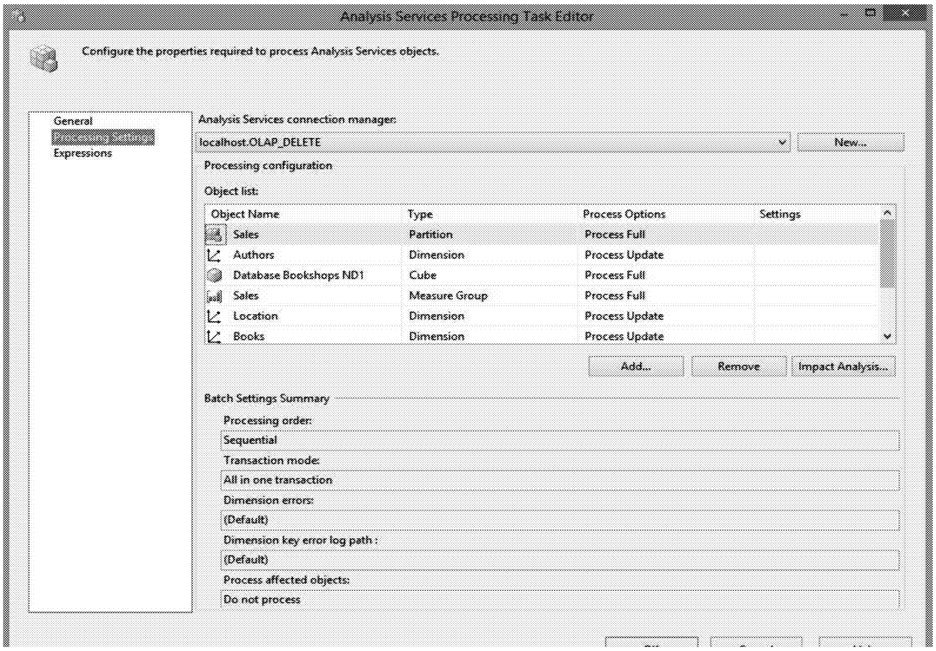


Fig.16 SQL Server Integration Services (SSIS) Project



Configure the properties required to process Analysis Services objects.

Analysis Services connection manager: localhost.OLAP_DELETE

Processing configuration

Object list:

Object Name	Type	Process Options	Settings
Sales	Partition	Process Full	
Authors	Dimension	Process Update	
Database Bookshops ND1	Cube	Process Full	
Sales	Measure Group	Process Full	
Location	Dimension	Process Update	
Books	Dimension	Process Update	

Batch Settings Summary

Processing order: Sequential

Transaction mode: All in one transaction

Dimension errors: (Default)

Dimension key error log path : (Default)

Process affected objects: Do not process

Fig.17 Edit Analysis Services Processing Task

The connection to the Bookshops OLAP cube is completed. The type of processing can be selected. We use the default options for our presentation. Thereafter we can to execute the Analysis Services Processing Task (Fig.18). The task is successfully executed (Fig.19).

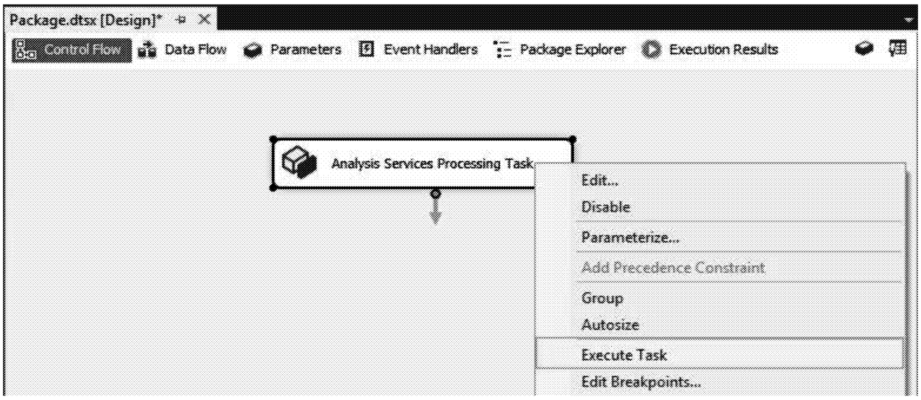


Fig.18 Analysis Services Processing Task Execution

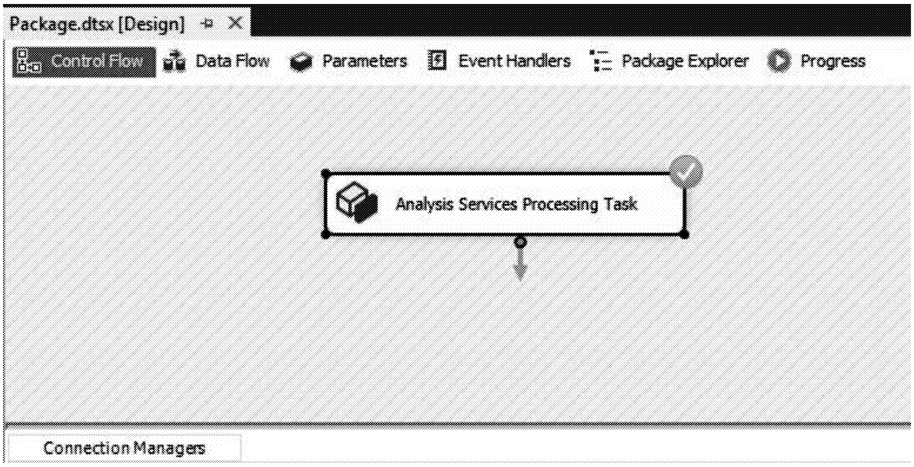


Fig.19 Succesfully executed task

Obviously, the OLAP application presenting the MDX possibilities for empty cells reduction does not have the same execution plan as the proposed index matrix operations. The ordering procedure in OLAP cube is according to the dimension attributes. Therefore for one level of aggregation the proposed index matrix operation is not possible. In the case of the new ordering procedure, the indices can change their hierarchy. The possible partial decision for ordering the operation construction in OLAP is to be performed "one to many" relationships between the attributes in the dimensions. Thereafter the history of the elements indices has to be saved.

9. Conclusion

In the current research paper a new algorithm of index matrix ordering is discussed. The operations for substitution and reduction of 3D-EIFIM are presented. The comparison of the proposed operations and the standard OLAP operations for ordering and empty cells reduction is made. An example of the algorithm is discussed.

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References

[1] Atanassov K., Bureva V., *Four operations over extended intuitionistic fuzzy index matrices and some of their applications*, Studies in Computational Intelligence, Springer series, High Performance Computing 2019 Conference, September 2-6, 2019, Borovets, Bulgaria (in press)

[2] Atanassov K., Sotirov S., *Index matrix interpretation of the Multilayer perceptron*. 2013 IEEE INISTA, Albena, 2013, pp. 1-3, doi: 10.1109/INISTA.2013.6577637.

[3] Atanassov K., Sotirov S., *Index matrix interpretation of one type of extended neural networks*, International Journal of Reasoning-based Intelligent Systems, Vol.6, No.3/4, 2014

[4] Atanassov, K. *On extended intuitionistic fuzzy index matrices*, Notes on Intuitionistic Fuzzy Sets, Vol. 19, 2013, No. 4, 27-41.

[5] Atanassov, K., E. Szmidt, J. Kacprzyk, *On intuitionistic fuzzy pairs*, Notes on Intuitionistic Fuzzy Sets, Vol. 19, 2013, No. 3, 1-13.

[6] Atanassov, K., *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham, 2014.

[7] Atanassov, K., *Intuitionistic Fuzzy Sets*, Springer, Heidelberg, 1999.

[8] Atanassov, K., *n-Dimensional extended index matrices*. Advanced Studies in Contemporary Mathematics,, Vol. 28 (2018), No. 2, 245-259.

[9] Atanassov, K., *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[10] Bureva V., *Application of the Index Matrix Operations Over Relational Databases*, Advanced Studies in Contemporary Mathematics, 2016, No 4, p. 669-679.

[11] Bureva V., Sotirova E., Moskova M., Szmidt E., *Solving the Problem of appointments using index matrices*, Proceedings of the 13th International Workshop on Generalized Nets 29, 2012, p.43-48

[12] Bureva V., Traneva V., Sotirova E., Atanassov K., *Index Matrices and OLAP-Cube Part 4: A presentation of the OLAP “Drill-Across” operation by index matrices*, Advanced Studies in Contemporary Mathematics, 29,1, 2019, p.109-123

[13] Bureva V., Traneva V., Sotirova E., Atanassov K., *Index matrices and OLAP-Cube, Part 5: Index matrix operations over OLAP-cube*, Adv. Stud. Contemp. Math. (Kyungshang) Vol. 30 (2020), No. 1, 69-88

[14] Bureva V., V. Traneva, E. Sotirova, K. Atanassov, *Index Matrices And OLAP-Cube Part 2: An presentation of the OLAP-analysis by Index Matrices*, Advanced Studies in Contemporary Mathematics, Vol.27, 2017, No 4, p. 647-672.

[15] Inmon W., *Building the Data Warehouse*, 4th Edition, Wiley, 2005

[16] Kimball R., Ross M., *The Data Warehouse Toolkit: The Definitive Guide to Dimensional Modeling*, 3rd Edition, Wiley, 2013

[17] Traneva V., Sotirova E., Bureva V., Atanassov K., *Aggregation operations over 3-dimensional extended index matrices*, Advanced Studies in Contemporary Mathematics, 2015 (July), vol.25, No 3, p.407-416.

[18] Traneva V., V. Bureva, E. Sotirova, K. Atanassov, *Index Matrices And OLAP-Cube Part 1: Application of the Index Matrices to presentation of operations in OLAP-Cube*, Advanced Studies in Contemporary Mathematics, Vol.27, 2017, No 2, p. 253-278.

[19] Traneva, V., V. Bureva, E. Sotirova, K. Atanassov, *Index matrices and OLAP-cube part 3: A presentation of the OLAP "InterCube Set" and "Data cube" operations by index matrices*, Advanced Studies in Contemporary Mathematics (Kyungshang), 2018.