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Hub number of some wheel related graphs

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Abstract: A *hub set* in a graph G is a set $S \subseteq V(G)$ such that any two vertices outside S are connected by a path all of whose internal vertices are members of S. The minimum cardinality of a hub set is called *hub number*. In this paper, we give results for the hub number of some wheel related graphs.

1 Introduction

All graphs considered in this paper are non-trivial, connected, simple and undirected. Let $G = (V(G), E(G)) = (V, E)$ be a graph with vertex set V and edge set E. $|V| = n$ and $|E| = m$ are respectively called the *order* and *size* of G. The *girth* of a graph G denoted by $g(G)$ is defined as the length of the shortest cycle in a graph G. The *subdivision graph* $S(G)$ of a graph G is the graph obtained by inserting a new vertex onto each edge of G. A path, a cycle and a complete graph of order n are denoted by P_n, C_n and K_n, respectively. For unexplained graph terminology and notations, refer to [3].

Suppose $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices lie in S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus S$, there is an S-path in G between x and y. The minimum cardinality of a hub set is called *hub number* and is denoted by $h(G)$.

In 2006, Walsh have defined the hub number of a graph to study a network related problem, [6]. He also gave hub number of several classes of graphs and shown that the

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hub number of any cyclic graph G is at least $g(G) - 3$. Continuing this work, Grauman et. al. gave the relationship between hub number, connected hub number and connected domination number of a graph, [2]. Further, Cauresma Jr. and Paluga obtained the hub numbers of join, corona and cartesian product of two connected graphs in [4]. Inspired by this, we obtain the hub number of some wheel related graphs in this paper.

2 Preliminaries

Let G_1 and G_2 be two graphs of order n_1, n_2 and size m_1, m_2, respectively. The *union* of G_1 and G_2 is a graph denoted by $G_1 \cup G_2$ having vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, [3]. It is obvious that $|V(G_1 \cup G_2)| = n_1 + n_2$ and $|E(G_1 \cup G_2)| = m_1 + m_2$. The *join* $G_1 + G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2 by an edge, [3].

The *fan* graph F_n for $n \geq 3$ is defined as the join of K_1 and P_{n-1}. The *wheel* $W_n = K_1 + C_{n-1}$ is a graph with $n+1$ vertices and $2n-2$ edges, where the vertex of degree $n-1$ is called the *central vertex* and all other vertices on the cycle C_{n-1} are called *rim vertices*.

The *gear graph* G_n is a wheel graph with a vertex added between each pair of adjacent vertices on the outer circle (rim). The *helm* H_n is a graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex of W_n. A *closed helm* CH_n is the graph obtained from a helm by joining pendant vertices to form a cycle. The *flower* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex c of the helm. The *sunflower graph* SF_n is a graph obtained from a wheel with central vertex c and an $n-1$-cycle $v_0, v_1, \cdots, v_{n-2}$ by adding n new vertices $w_0, w_1, \cdots, w_{n-2}$ such that w_i is joined by two end vertices v_i and v_{i+1} for $i = 0, 1, \cdots, n-2$ where i is taken modulo $n-1$.

The *friendship graph* f_n is a collection of n triangles all having a common vertex. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternating edges of the cycle. That is $f_n = K_1 + nK_2$. A *web graph* $W(2, n)$ is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t, n)$ is the generalized web with t cycles each of order n. The *crown* (or sun) graph CW_n is the corona product $C_n \circ K_1$ where $n \geq 3$. That is, a crown graph is a helm without central vertex.

The *duplication of an edge* $e = uv$ by a new vertex v' in a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = \{u, v\}$. Consider a wheel $W_n = C_{n-1} + K_1$ with $v_1, v_2, \cdots, v_{n-1}$ as its rim vertices and x as its central vertex. Let $e_1, e_2, \cdots, e_{n-1}$ be the rim edges of W_n which are duplicated by new vertices $w_1, w_2, \cdots, w_{n-1}$, respectively. Let $y_1, y_2, \cdots, y_{n-1}$ be the spoke edges of W_n which are duplicated by the vertices $u_1, u_2, \cdots, u_{n-1}$, respectively. The resultant graph is called *duplication of the wheel* denoted by DuW_n, [5]. In many sources, duplication of the wheel graph is also called the semi-total point graph of wheel. The definitions of these wheel related graphs and more can be found in [1].

In the next section, we obtain the hub number of these wheel related graphs. The following theorem given by Cuaresma et. al. is used in proving our results, [4].

Theorem 2.1. *[4] For any connected graphs G and H,*

$$h(G+H)=\begin{cases}0 & \textit{if } G \textit{ and } H \textit{ are complete,}\\ 1 & \textit{if } G \textit{ is complete and } H \textit{ is non-complete,}\\ min\{h(G),h(H),2\} & \textit{if } G \textit{ and } H \textit{ both are non-complete.}\end{cases}$$

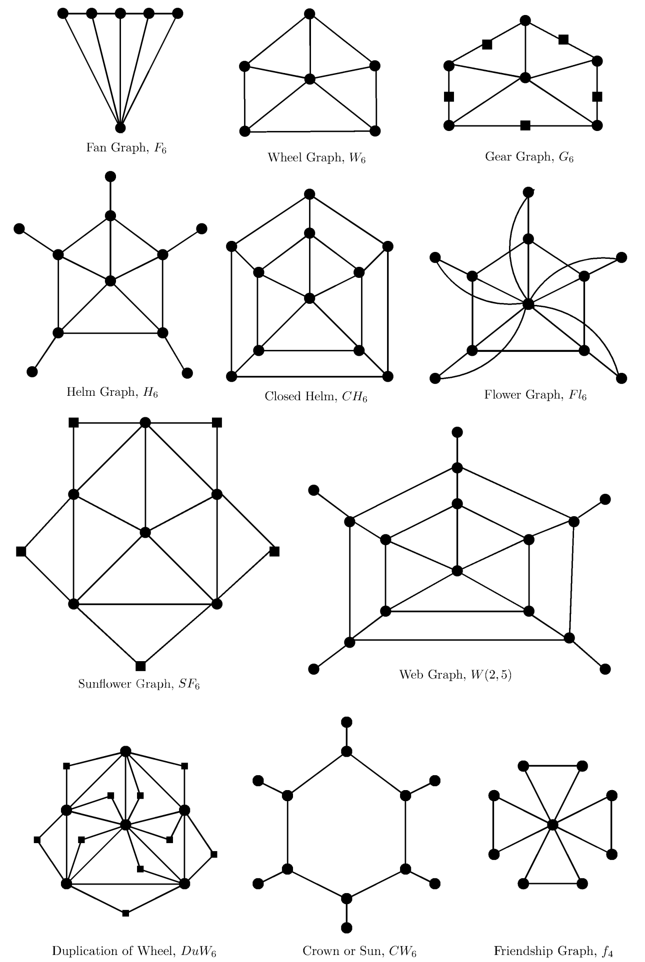


Figure 1: Examples of some wheel related graphs

3 Hub number of some wheel related graphs

Theorem 3.1. *Let F_n and W_n be fan graph and wheel graph of order n respectively. Then*

$$h(W_n) = 1 = h(F_n).$$

Proof. The fan graph is $K_1 + P_{n-1}$ and the wheel graph is $K_1 + C_{n-1}$. Both are join of one complete graph and another non-complete graph. Thus, by case 2 of Theorem 2.1, we get the desired results. □

Theorem 3.2. *Let G_n be the gear graph of order $2n - 1$. Then*

$$h(G_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proof. Let $x_1, x_2, x_3, \cdots, x_{n-1}$ be the vertices on the rim of wheel graph with center vertex c. Gear graph is obtained by adding a new vertex on each edge on the rim. Let $S \subset V(G_n)$. We discuss the three cases of choosing the set S as shown in Fig. 2.

Case 1. If we choose the set $S = \{c, x_1, x_2, x_3, \cdots, x_{n-1}\}$ as shown in Fig. 2 (a), then for any two vertices $x, y \in V(G_n) \setminus S$, there is an $S-$path from x to y whose all intermediate vertices are in S. Thus S is a hub set. This gives $|S| = n$.

Case 2. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 2 (b), then for two vertices $x, y \in V(G_n) \setminus S$ as marked in Fig. 2 (b), there is no $S-$path from x to y. Thus S is not a hub set.

Case 3. If we choose the set $S = \{c, x_1, x_2, x_3, \cdots, x_{\lfloor \frac{n}{2} \rfloor}\}$ as shown in Fig. 2 (c), then for any two vertices $x, y \in V(G_n) \setminus S$, there is an $S-$path from x to y whose all intermediate vertices are in S. Thus S is a hub set. In this case we have $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1$. □

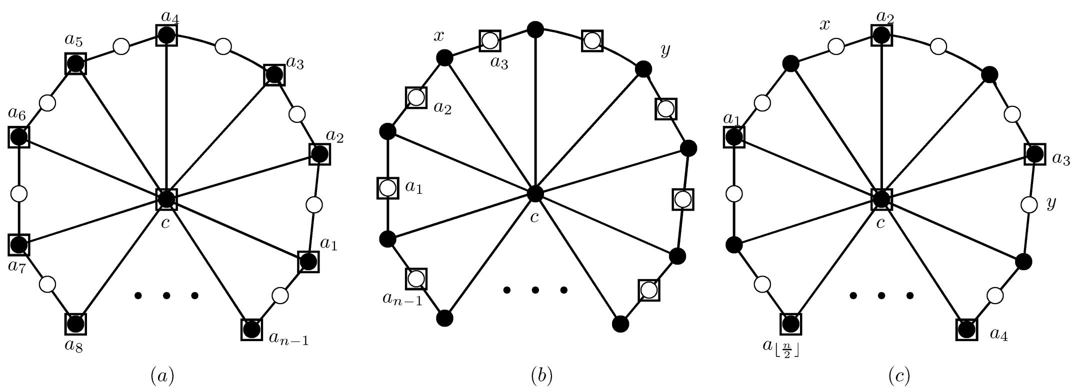


Figure 2: Choosing minimum hub set in a gear graph

Theorem 3.3. *Let H_n be the helm graph of order $2n-1$. Then*

$$h(H_n) = n-1.$$

Proof. Let $x_1, x_2, x_3, \cdots, x_{n-1}$ be the vertices on the rim of wheel graph with center vertex c. Helm graph is obtained by attaching a pendant edge to each rim vertex of wheel graph. Let $S \subset V(H_n)$. We show three cases to choose the set S (as shown in Fig. 3) .

Case 1. If we choose the set $S = \{x_1, x_2, x_3, \cdots, x_{n-1}\}$ as shown in Fig. 3 (a), then for any two vertices $x, y \in V(H_n) \setminus S$, there is an S-path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. In this case we have $|S| = n-1$.

Case 2. If we choose the set $S = \{c, x_1, x_2, x_3, \cdots, x_{n-1}\}$ as shown in Fig. 3 (b), then for two vertices $x, y \in V(H_n) \setminus S$ in Fig. 3 (b), there is an S-path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. Here we have $|S| = n$.

Case 3. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 3 (c), then for any two vertices $x, y \in V(H_n) \setminus S$, there is no S-path from x to y. Thus S is not a hub set. □

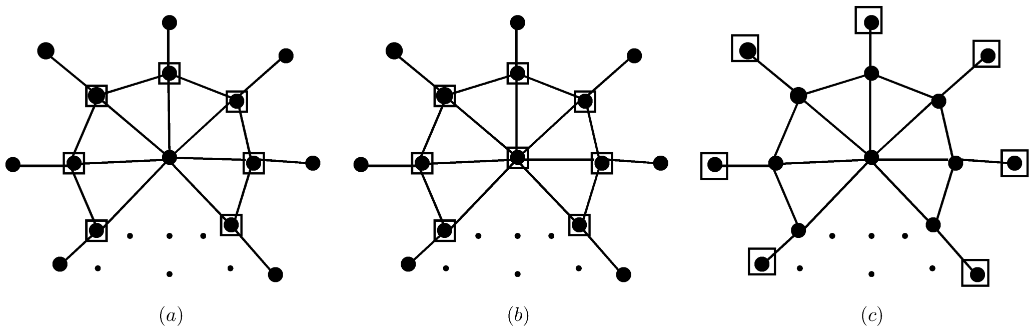


Figure 3: Choosing minimum hub set in helm graph.

Theorem 3.4. *Let CH_n be the closed helm graph of order $2n-1$. Then*

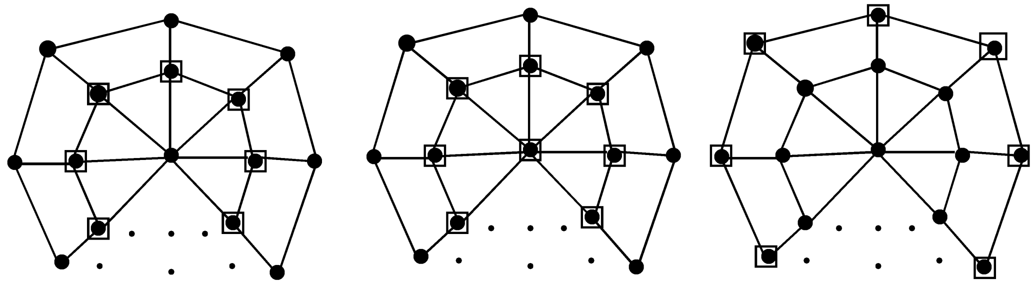
$$h(CH_n) = n-1.$$

Proof. Let $x_1, x_2, x_3, \cdots, x_{n-1}$ be the vertices on the rim of wheel graph with center vertex c. A *closed helm* CH_n is the graph with central vertex c, obtained from a helm by joining pendant vertices to form a cycle. Let $S \subset V(CH_n)$. We discuss three cases of choosing the set S (as shown in Fig. 4) .

Case 1. If we choose the set $S = \{x_1, x_2, x_3, \cdots, x_{n-1}\}$ as shown in Fig. 4 (a) such that vertices of S form inner cycle. Then for any two vertices $x, y \in V(CH_n) \setminus S$, there is an S-path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. This gives $|S| = n-1$.

Case 2. If we choose the set $S = \{c, x_1, x_2, x_3, \cdots, x_{n-1}\}$ as shown in Fig. 4 (b), then for two vertices $x, y \in V(CH_n) \setminus S$ as in Fig. 4 (b), there is an $S-$path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. This gives $|S| = n$.

Case 3. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 4 (c), then for any two vertices $x, y \in V(CH_n) \setminus S$, there is a S-path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. Thus, $|S| = n - 1$. Therefore, this way of choosing the set S gives us the minimum hub set. □



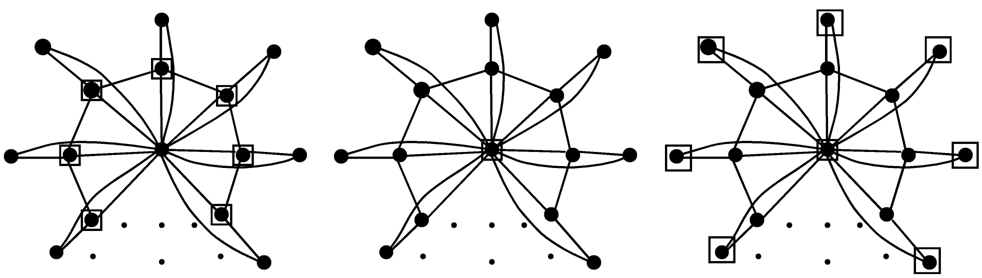
(a) (b) (c)

Figure 4: Choosing minimum hub set in closed helm graph.

Theorem 3.5. *Let Fl_n be the flower graph of order $2n - 1$. Then*

$$h(Fl_n) = 1.$$

Proof. Since the central vertex of a flower graph is adjacent to every other vertex, consider $S = \{c\}$. Then for any two vertices $x, y \in V(Fl_n) \setminus S$, there is an $S-$path from x to y whose intermediate vertices are in S. Therefore, this way of choosing S gives us the minimum hub set. Hence, $|S| = 1$. Therefore $h(Fl_n) = 1$. □



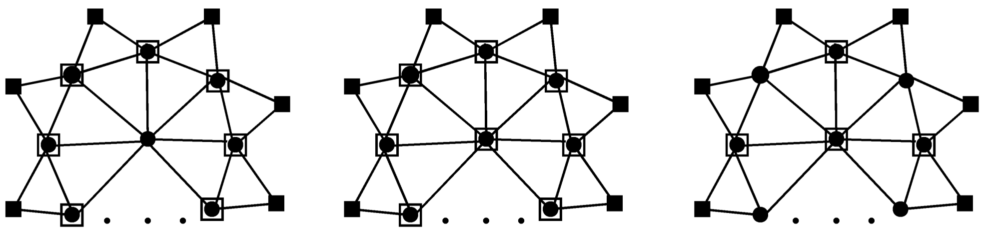
(a) (b) (c)

Figure 5: Choosing minimum hub set in flower graph.

Theorem 3.6. *Let SF_n be the sunflower graph of order $2n - 1$. Then,*

$$h(SF_n) = \left\lceil \frac{n-1}{2} \right\rceil + 1.$$

Proof. The *sunflower graph* SF_n is a graph obtained from a wheel with central vertex c, an $n-1$-cycle $\{v_0, v_1, \cdots, v_{n-2}\}$ and $n-1$ additional vertices $w_0, w_1, \cdots, w_{n-2}$ where w_i is joined to v_i and v_{i+1} for $i = 0, 1, \cdots, n-2$ where i is taken modulo $n-1$. Let $S \subset V(SF_n)$. We discuss the three cases of choosing the set S as shown in Fig. ??:
Case 1. If we choose the set $S = \{v_0, v_1, \cdots, v_{n-2}\}$ as shown in Fig. 6 (a), then for any two vertices $x, y \in V(SF_n) \setminus S$, there is an $S-$path from x to y whose all intermediate vertices are in S. Thus, S is a hub set. This gives $|S| = n-1$.
Case 2. If we choose the set $S = \{c, v_0, v_1, \cdots, v_{n-2}\}$ as shown in Fig. ?? (b), then for two vertices $x, y \in V(SF_n) \setminus S$ marked in Fig. 6 (b), there is an $S-$path from x to y. Thus S is a hub set and $|S| = n$.
Case 3. If we choose the set $S = \{c, v_0, v_1, \cdots, v_{\lceil \frac{n-1}{2} \rceil}\}$ as shown in Fig. ?? (c), then for any two vertices $x, y \in V(SF_n) \setminus S$, there is an S-path from x to y whose all intermediate vertices are in S. Thus S is a hub set. This gives $|S| = \lceil \frac{n-1}{2} \rceil + 1$. □



(a) (b) (c)

Figure 6: Choosing minimum hub set in sunflower graph.

Theorem 3.7. *Let $W(2, n)$ be the web graph of order $3n-2$. Then*

$$h(W(2, n)) = n.$$

Proof. A *web graph* is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t, n)$ is the generalized web with t cycles each of order $n-1$. Let $S \subset V(W(2, n))$. We study three cases to choose the set S as shown in Fig. 7.
Case 1. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 7 (a), then for two vertices $x, y \in V(W(2, n)) \setminus S$, marked in 7 (a), there is no $S-$path from x to y whose all intermediate vertices are in S. Thus S is not a hub set.
Case 2. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 7 (b), then for two vertices $x, y \in V(W(2, n)) \setminus S$ shown in 7 (b), there is no $S-$path from x to y. Thus S is not a hub set.
Case 3. If we choose the set $S = \{c, a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 7 (c), then for any two vertices $x, y \in V(W(2, n)) \setminus S$, there is an S-path from x to y whose all intermediate vertices are in S. Thus S is a hub set. This gives $|S| = n$. □

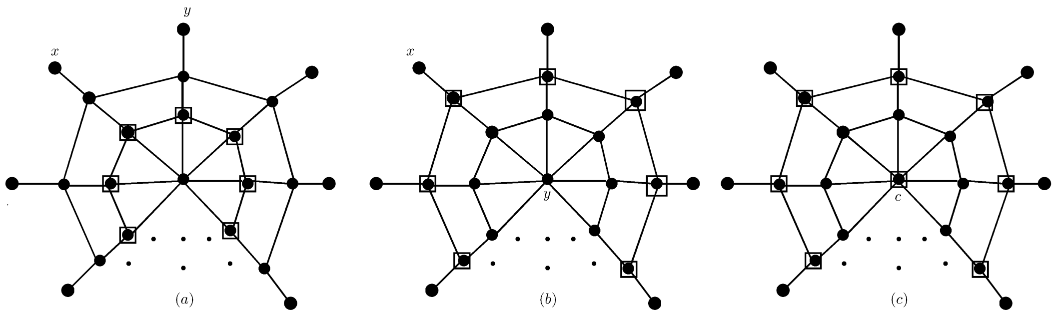


Figure 7: Choosing minimum hub set in web graph.

Theorem 3.8. *Let DuW_n be the duplication of the wheel graph of order $3n-2$. Then*

$$h(DuW_n) = n - 1.$$

Proof. Let $S \subset V(DuW_n)$. We show three cases to choose the set S (as shown in Fig. 8).
Case 1. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 8 (a), then for two vertices $x, y \in V(DuW_n) \setminus S$, there is a $S-$path from x to y whose all intermediate vertices are in S. Thus, S is a a hub set. Therefore $h(DuW_n) = n - 1$.
Case 2. If we choose the set $S = \{c, a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 8 (b), then for any two vertices $x, y \in V(DuW_n) \setminus S$ there is a $S-$path from x to y. Thus S is a hub set. Therefore, $h(DuW_n) = n$.
Case 3. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 8 (c), then for two vertices $x, y \in V(DuW_n) \setminus S$ marked in Fig. 8 (c), there is no $S-$path from x to y whose all intermediate vertices are in S. Thus S is not a hub set. □

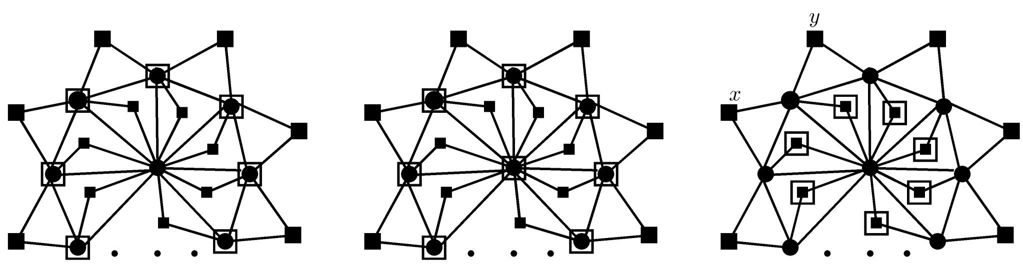


Figure 8: Choosing minimum hub set in duplication of the wheel graph.

Theorem 3.9. *Let CW_n be the crown graph of order n. Then*

$$h(CW_n) = n - 1.$$

Proof. Let $S \subset V(CW_n)$. We now consider three cases of choosing the set S as shown in Fig. 9.

Case 1. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 9 (a), then for any two vertices $x, y \in V(CW_n) \setminus S$, there is an $S-$path from x to y whose all intermediate vertices are in S. Thus S is a hub set which gives $|S| = n - 1$.

Case 2. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{\lfloor \frac{n-1}{2} \rfloor}\}$ as shown in Fig. 9 (b), then for two vertices $x, y \in V(CW_n) \setminus S$ marked in Fig. 9 (b), there is no $S-$path from x to y. Thus S is not a hub set.

Case 3. If we choose the set $S = \{a_1, a_2, a_3, \cdots, a_{n-1}\}$ as shown in Fig. 9 (c), then for two vertices $x, y \in V(CW_n) \setminus S$, marked in Fig. 9 (c), there is no $S-$path from x to y whose all intermediate vertices are in S. Thus S is not a hub set. $\square$

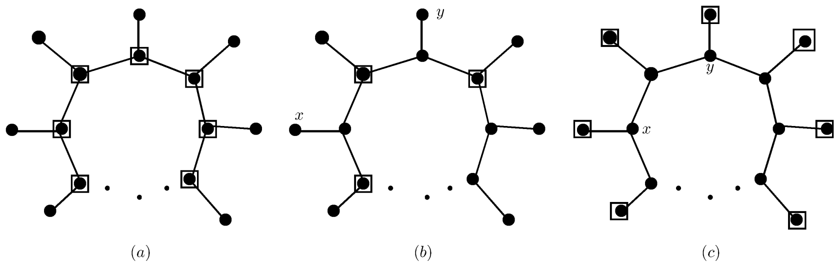


Figure 9: Choosing minimum hub set in crown graph

Theorem 3.10. *Let f_n be the friendship graph of order $2n + 1$. Then*

$$h(f_n) = 1.$$

Proof. The proof follows from the fact that in any friendship graph, the central vertex is adjacent to all other vertices. Therefore choosing $S = \{c\}$ gives the desired result. $\square$

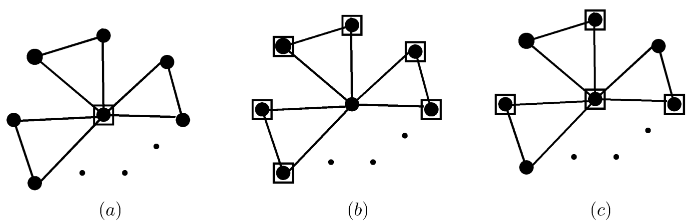


Figure 10: Choosing minimum hub set in friendship graph.

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