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SOME THEOREMS ON GENERATING FUNCTIONS AND THEIR APPLICATIONS

KHADIDJA BOUBELLOUTA, ALI BOUSSAYOUD, SERKAN ARACI, AND MOHAMED KERADA

Abstract. In this paper, we first consider some operators including symmetric functions. From those operators, we obtain some new generating functions of k-Fibonacci numbers and k-Pell numbers of third order and Chebyshev polynomials of the first and the second kind.

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1. Introduction and Preliminaries

In the last years, there is huge interest of natural science in the applications of Fibonacci and Lucas numbers. The sequences of the classical Fibonacci F_n, Lucas L_n and generalized Fibonacci G_n are defined for $n \geq 2$ by the recurrence relations [38]

$$F_n = F_{n-1} + F_{n-2},\ L_n = L_{n-1} + L_{n-2} \quad and \quad G_n = G_{n-1} + G_{n-2},$$

with inital conditions, respectively,

$$F_0 = 1,\ F_1 = 1;\ L_0 = 2,\ L_1 = 1;\ G_0 = a,\ G_1 = b, \qquad (a \in \mathbb{R},\ b \in \mathbb{R}).$$

For the generalizations of Fibonacci and Lucas sequences, one can look at [14]. Also, for useful applications of these numbers in science and nature, we refer the interested readers to see the references [19, 20, 21, 30].

From the relation

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2},$$

it appears in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. In [17], Gulec and Taskara derived some new properties of Fibonacci and Lucas numbers with binomial coefficients.

As a generalization of the Fibonacci sequences, Falcon and Plaza, in [14, 15], introduced k-Fibonacci sequence, denoted by $\{F_{k,n}\}_{n=0}^{\infty}$. They also studied not only new but also interesting properties of these numbers. Also, from a geometric point of view, they gave 3-dimensional k-Fibonacci spirals.

Definition 1.1. [38] *Let k be a positive real number. Then, the recurrence relation of generalized k-Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$ is defined by*

$$G_{k,n+1} = kG_{k,n} + G_{k,n-1},\ n \geq 1,$$

with initial conditions $G_{k,0} = a,\ G_{k,1} = b,\ (a, b \in \mathbb{R})$.

Generalized k-Fibonacci number is called to each element of Generalized k-Fibonacci sequence. Taking $a = 1$, $b = 1$ and $a = 2$, $b = 1$ gives k-Fibonacci sequence and k-Lucas sequence, respectively. A few special values for Generalized k-Fibonacci sequence$\{G_{k,n}\}_{n\in\mathbb{N}}$ are listed below:

i. If $k = 1$, then we have generalized Fibonacci sequence

$$\{G_{1,n}\} = \{a,\ b,\ a + b,\ a + 2b,\ 2a + 3b, ...\}.$$

- Putting $a = 1$, $b = 1$ the generalized Fibonacci sequence reduces to Fibonacci sequence known as $F_n = \{1, 1, 2, 3, 5, ...\}$.
- Substituting $a = 2$, $b = 1$ yields Lucas sequence given by $L_n = \{2, 1, 3, 4, 7, 11, ...\}$.

ii. If $k = 2$, then we have generalized Pell sequence [27, 38]

$$\{G_{2,n}\} = \{a,\ b,\ a + 2b,\ 2a + 5b,\ 5a + 12b,\ 12a + 29b, ...\}.$$

- Taking $a = 0, b = 1$ gives Pell sequence given by $P_n = \{0, 1, 2, 5, 12, 29, ...\}$.
- In the case when $a = 2$ and $b = 2$, it reduces to Pell-Lucas sequence known as $P_n = \{2, 2, 6, 14, 34, 82, ...\}$.

Definition 1.2. [22] *Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables. Then, the k-th elementary symmetric function $e_k(a_1, a_2, ..., a_n)$ is defined by*

$$e_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \qquad (0 \le k \le n),$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Definition 1.3. [22] *Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables. Then, the k-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by*

$$h_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \qquad (0 \le k \le n),$$

with $i_1, i_2, ..., i_n \ge 0$.

Remark 1.4. *Set $e_0(a_1, a_2, ..., a_n) = 1$ and $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$, we set $e_k(a_1, a_2, ..., a_n) = 0$ and $h_k(a_1, a_2, ..., a_n) = 0$.*

Definition 1.5. [2] *Let A and B be any two alphabets. We define $S_n(A - B)$ by the following form*

$$E(-t)\,H(t) = \sum_{n=0}^{\infty} S_n(A - B)\,t^n,$$

with $H(t) = \prod_{a\in A} (1 - at)^{-1}$, $E(-t) = \prod_{b\in B} (1 - bt)$.

Remark 1.6. $S_n(A - B) = 0$ *for* $n < 0$.

Remark 1.7. *Let $A = \{a_1, a_2, ..., a_n\}$ an alphabet, we have*

$$h_k(a_1, a_2, ..., a_n) = S_k(a_1 + a_3 + ... + a_n).$$

Corollary 1.8. *Given an alphabet $A = \{a_1, a_2, ..., a_n\}$, we have*

$$\sum_{a\in A} h_k(a_1,a_2,...,a_n)t^n = \frac{1}{\prod\limits_{a\in A}(1-at)}.$$

Definition 1.9. [5] *Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,...,x_i,x_{i+1},...x_n) - g(x_1,...,x_{i-1},x_{i+1},x_{i,}x_{i+2}...x_n)}{x_i - x_{i+1}}.$$

Definition 1.10. [7] *The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by*

$$\delta^k_{a_1a_2}(g) = \frac{a_1^k g(a_1) - a_2^k g(a_2)}{a_1 - a_2}, \quad (k\in\mathbb{N}).$$

Example 1.11. *For $g(a_1) = a_1$ we have*

$$\delta^k_{a_1a_2}(g) = h_k(a_1,a_2), \qquad (k\in\mathbb{N}).$$

In this paper, we derive new generating functions of the products of some well-known numbers and polynomials. In Sect 1, we give some more useful definitions from the literature which are used in the subsequent sections. In Sect 2, we provide some new theorems by making use of the symmetrizing operator defined in this paper. In Sect 3, we give some new generating functions of k-Fibonacci numbers of third order, k-Pell numbers of third order and Chebyshev polynomials of the first and the second kinds which were studied in great details by Kim *et al.* [31, 32, 33, 34, 35].

2. Main Results

In this section, we provide some new theorems by using the symmetrizing operator. We now begin with the following theorem.

Theorem 2.1. *Let the alphabets $A = \{a_1,a_2\}, B = \{b_1,b_2\}$ and $C = \{c_1,c_2\}$ be given, we have*

$$(1)\qquad \sum_{n=0}^{\infty} h_n(a_1,a_2)\,h_n(b_1,b_2)\,h_n(c_1,c_2)\,t^n$$

$$= \frac{b_1b_2}{(c_1-c_2)} \times \frac{\left(\begin{array}{c} \prod\limits_{a\in A}(1-ab_1c_1t)\prod\limits_{a\in A}(1-ab_2c_1t) \\ \times\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,h_{n-2}(b_1,b_2)\,c_2^{n+1}t^n \\ -\prod\limits_{a\in A}(1-ab_1c_2t)\prod\limits_{a\in A}(1-ab_2c_2t) \\ \times\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,h_{n-2}(b_1,b_2)\,c_1^{n+1}t^n \end{array}\right)}{\begin{array}{c}\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,b_2^n c_1^n t^n\right) \\ \times\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,b_2^n c_2^n t^n\right)\end{array}}.$$

Proof. Let $\sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n$ and $\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n$ be two sequences as $\sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n \times \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n = 1$. On one hand, since $g(b_1,c_1) = \sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n$, we have

$$\begin{aligned}
& \delta_{c_1c_2}\delta_{b_1b_2} g(b_1,c_1) \\
= \;& \delta_{c_1c_2}\left(\frac{\sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^{n+1} c_1^n t^n - \sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, b_2^{n+1} c_1^n t^n}{b_1-b_2}\right) \\
= \;& \delta_{c_1c_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2)\,\frac{b_1^{n+1}-b_2^{n+1}}{b_1-b_2} c_1^n t^n\right) \\
= \;& \delta_{c_1c_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_1^n t^n\right) \\
= \;& \frac{\sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_1^{n+1} t^n - \sum\limits_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_2^{n+1} t^n}{c_1-c_2} \\
= \;& \sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, h_n(c_1,c_2) t^n,
\end{aligned}$$

which is the left hand side of (1). On the other hand, since

$$g(b_1,c_1) = \frac{1}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n}$$

$$\delta_{c_1c_2}\delta_{b_1b_2} g(b_1,c_1) = \delta_{c_1c_2}\delta_{b_1b_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n}\right)$$

$$\begin{aligned}
= \;& \delta_{c_1c_2}\left(\frac{b_1b_2\left(\begin{array}{c}\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_2^{n-1} c_1^n t^n \\ -\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^{n-1} c_1^n t^n\end{array}\right)}{\begin{array}{c}(b_1-b_2)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n\right) \\ \times\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_2^n c_1^n t^n\right)\end{array}}\right) \\
= \;& \delta_{c_1c_2}\left(\frac{-b_1b_2\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, h_{n-2}(b_1,b_2)\, c_1^n t^n}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2)\, b_2^n c_1^n t^n\right)}\right)
\end{aligned}$$

$$= \frac{b_1 b_2}{c_1 - c_2} \left(\frac{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-2}(b_1,b_2) c_2^{n+1} t^n}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_2^n t^n\right)} - \frac{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-2}(b_1,b_2) c_1^{n+1} t^n}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_1^n t^n\right)} \right).$$

Using the fact that $\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_1^n t^n = \prod\limits_{a\in A} (1 - ab_1c_1t)$, then

$$\delta_{c_1c_2}\delta_{b_1b_2} g(b_1,c_1)$$

$$= \frac{b_1b_2}{c_1-c_2} \times \frac{\left(\begin{array}{c} \prod\limits_{a\in A}(1-ab_1c_1t)\prod\limits_{a\in A}(1-ab_2c_1t) \\ \times \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-2}(b_1,b_2) c_2^{n+1} t^n \\ -\prod\limits_{a\in A}(1-ab_1c_2t)\prod\limits_{a\in A}(1-ab_2c_2t) \\ \times \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-2}(b_1,b_2) c_1^{n+1} t^n \end{array}\right)}{\begin{array}{c}\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_1^n t^n\right) \\ \times \left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_2^n t^n\right)\end{array}}.$$

This completes the proof. □

Theorem 2.2. *Given three alphabets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ we have*

(2)
$$\sum_{n=0}^{\infty} h_n(a_1,a_2) h_{n-1}(b_1,b_2) h_{n-1}(c_1,c_2) t^n$$

$$= \frac{\left(\begin{array}{c} \prod\limits_{a\in A}(1-ab_1c_1t)\prod\limits_{a\in A}(1-ab_2c_1t) \\ \times \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-1}(b_1,b_2) c_2^n t^n \\ -\prod\limits_{a\in A}(1-ab_1c_2t)\prod\limits_{a\in A}(1-ab_2c_2t) \\ \times \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) h_{n-1}(b_1,b_2) c_1^n t^n \end{array}\right)}{\begin{array}{c}(c_1-c_2)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_1^n t^n\right) \\ \times \left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1,a_2) b_2^n c_2^n t^n\right)\end{array}}.$$

Proof. The proof is similar to the proof of theorem (2.1), but now using the divided difference operator $\partial_{c_1,c_2}\,\partial_{b_1,b_2}$. □

Theorem 2.3. *Given three alphabets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, h_{n-1}(c_1,c_2)\, t^n \tag{3}$$

$$= \frac{b_1 b_2}{(c_1-c_2)} \times \frac{\begin{pmatrix} \prod_{a\in A}(1-ab_1c_1t)\prod_{a\in A}(1-ab_2c_1t) \\ \times \sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, h_{n-2}(b_1,b_2)\, c_2^n t^n \\ -\prod_{a\in A}(1-ab_1c_2t)\prod_{a\in A}(1-ab_2c_2t) \\ \times \sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, h_{n-2}(b_1,b_2)\, c_1^n t^n \end{pmatrix}}{\begin{array}{c}\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_2^n c_1^n t^n\right) \\ \times\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_2^n t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_2^n c_2^n t^n\right)\end{array}}.$$

Proof. Let $\sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n$ and $\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n$ be two sequences as $\sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n \times \sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n = 1$. On one hand, since $g(b_1,c_1) = \sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n$, we have

$$\begin{aligned}
& \partial_{c_1c_2}\delta_{b_1b_2} g(b_1,c_1) \\
= \ & \partial_{c_1c_2}\delta_{b_1b_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^n c_1^n t^n\right) \\
= \ & \partial_{c_1c_2}\left(\frac{\sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_1^{n+1} c_1^n t^n - \sum_{n=0}^{\infty} h_n(a_1,a_2)\, b_2^{n+1} c_1^n t^n}{b_1-b_2}\right) \\
= \ & \partial_{c_1c_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2)\, \frac{b_1^{n+1}-b_2^{n+1}}{b_1-b_2} c_1^n t^n\right) \\
= \ & \partial_{c_1c_2}\left(\sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_1^n t^n\right) \\
= \ & \frac{\sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_1^n t^n - \sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, c_2^n t^n}{c_1-c_2} \\
= \ & \sum_{n=0}^{\infty} h_n(a_1,a_2)\, h_n(b_1,b_2)\, h_{n-1}(c_1,c_2) t^n,
\end{aligned}$$

which is the left hand side of (3). On the other hand, since

$$g(b_1,c_1) = \frac{1}{\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n}$$

$$\partial_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)=\partial_{c_1c_2}\delta_{b_1b_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n}\right)$$

$$\begin{aligned}
&= \partial_{c_1c_2}\left(\frac{b_1b_2\left(\begin{array}{c}\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^{n-1}c_1^n t^n\\ -\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^{n-1}c_1^n t^n\end{array}\right)}{\begin{array}{c}\left(b_1-b_2\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n\right)\\ \times\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_1^n t^n\right)\end{array}}\right)\\
&= \partial_{c_1c_2}\left(\frac{-b_1b_2\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_1^n t^n}{\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_1^n t^n\right)}\right)\\
&= \frac{b_1b_2}{c_1-c_2}\left(\frac{\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_2^n t^n}{\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_2^n t^n\right)}\right.\\
&\qquad\left.-\frac{\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_1^n t^n}{\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_1^n t^n\right)}\right).
\end{aligned}$$

Using the fact that $\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n=\prod\limits_{a\in A}\left(1-ab_1c_1t\right)$, then

$$\begin{aligned}
&\partial_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)\\
&= \frac{b_1b_2}{\left(c_1-c_2\right)}\times\frac{\left(\begin{array}{c}\prod\limits_{a\in A}\left(1-ab_1c_1t\right)\prod\limits_{a\in A}\left(1-ab_2c_1t\right)\\ \times\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_2^n t^n\\ -\prod\limits_{a\in A}\left(1-ab_1c_2t\right)\prod\limits_{a\in A}\left(1-ab_2c_2t\right)\\ \times\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_1^n t^n\end{array}\right)}{\begin{array}{c}\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_1^n t^n\right)\\ \times\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^n c_2^n t^n\right)\left(\sum\limits_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^n c_2^n t^n\right)\end{array}}.
\end{aligned}$$

This completes the proof. □

3. Application of Theorems

We now consider the previous theorems in order to derive a new generating functions of the k-Fibonacci numbers of third order, the k-Pell numbers of third order and Chebyshev polynomials of the first and the second kind.

Theorem 3.1. *Given three alphabets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ we have*

$$(4)\qquad \sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)\, h_{n-1}(b_1,b_2)\, h_{n-1}(c_1,c_2)\, t^n$$

$$= \frac{b_1 b_2}{c_1 - c_2} \times \frac{\begin{pmatrix} \prod_{a\in A}(1-ab_1c_1t)\prod_{a\in A}(1-ab_2c_1t) \\ \times \sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, h_{n-2}(b_1,b_2)\, c_2^{n+1}t^{n+1} \\ -\prod_{a\in A}(1-ab_1c_2t)\prod_{a\in A}(1-ab_2c_2t) \\ \times \sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, h_{n-2}(b_1,b_2)\, c_1^{n+1}t^{n+1} \end{pmatrix}}{\begin{array}{c}\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_1^n t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_2^n c_1^n t^n\right) \\ \times\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_1^n c_2^n t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\, b_2^n c_2^n t^n\right)\end{array}}.$$

Case 3.2. *By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2 = b_1b_2 = c_1c_2 = 1$ and $a_1 - a_2 = b_1 - b_2 = c_1 - c_2 = k$ in theorem 2.1, we have the following theorem.*

Theorem 3.3. *We have the following a new generating function for the cubes of k-Fibonacci numbers as :*

$$(5)\qquad \sum_{n=0}^{\infty} F_{k,n}^3 t^n = \frac{N_{F_{k,n}F_{k,n}F_{k,n}}}{D_{F_{k,n}F_{k,n}F_{k,n}}},$$

with

$$\begin{aligned} N_{F_{k,n}F_{k,n}F_{k,n}} &= 1 - \left(3k^2+3\right)t^2 - 2k^3t^3 + (3k^2+3)t^4 - t^6. \\ D_{F_{k,n}F_{k,n}F_{k,n}} &= 1 - k^3t - (k^4 + (2+k^2)(2k^2+2))t^2 - k^3(3k^2+5)t^3 \\ &\quad + (3k^4 + 12k^2 - k^6 + 6)t^4 + k^3(3k^2+5)t^5 \\ &\quad - (k^4 + (2+k^2)(2k^2+2))t^6 + k^3t^7 + t^8. \end{aligned}$$

By taking $k = 1$ in the identity (5) we obtain the generating function of the cubes of Fibonacci numbers as:

$$\sum_{n=0}^{\infty} F_n^3 t^n = \frac{1 - 2t - t^2}{1 - 3t - 6t^2 + 3t^3 + t^4}.$$

Foata in [13] give the formula (6) which is obtained by replacing a_1 by $2a_1$, a_2 by $(-2a_2)$, b_1 by $2b_1$, b_2 by $(-2b_2)$, c_1 by $2c_1$ and c_2 by $(-2c_2)$ and $4a_2a_1 = 4b_1b_2 = 4c_1c_2 = -1$ in the theorem 2.1

$$(6)\qquad \sum_{n=0}^{\infty} U_n(a_1-a_2)\, U_n(b_1-b_2)\, U_n(c_1-c_2)\, t^n = \frac{N_{U_nU_nU_n}}{D_{U_nU_nU_n}},$$

with

$$\begin{aligned} N_{U_nU_nU_n} &= 1-(4\,(a_1-a_2)^2+4\,(b_1-b_2)^2+4\,(c_1-c_2)^2-3)t^2 \\ &\quad +16\,(a_1-a_2)\,(b_1-b_2)\,(c_1-c_2)\,t^3-(4\,(a_1-a_2)^2 \\ &\quad +4\,(b_1-b_2)^2+4\,(c_1-c_2)^2-3)t^4+t^6. \\ D_{U_nU_nU_n} &= 1-8\,(a_1-a_2)\,(b_1-b_2)\,(c_1-c_2)\,t+(16\,(a_1-a_2)^2 \\ &\quad (c_1-c_2)^2+16\,(a_1-a_2)^2\,(b_1-b_2)^2+16\,(b_1-b_2)^2 \\ &\quad (c_1-c_2)^2-8\,(a_1-a_2)^2-8\,(b_1-b_2)^2-8\,(c_1-c_2)^2+4)t^2 \\ &\quad -(32\,(a_1-a_2)^3\,(b_1-b_2)\,(c_1-c_2)+32\,(b_1-b_2)^3\,(a_1-a_2) \\ &\quad (c_1-c_2)+32\,(c_1-c_2)^3\,(a_1-a_2)\,(b_1-b_2)-40\,(a_1-a_2) \\ &\quad (b_1-b_2)\,(c_1-c_2))t^3+(16\,(a_1-a_2)^4+16\,(b_1-b_2)^4 \\ &\quad +16\,(c_1-c_2)^4-16\,(a_1-a_2)^2-16\,(b_1-b_2)^2-16\,(c_1-c_2)^2 \\ &\quad +64\,(a_1-a_2)^2\,(b_1-b_2)^2\,(c_1-c_2)^2+6)t^4-(32\,(a_1-a_2)^3 \\ &\quad (b_1-b_2)\,(c_1-c_2)+32\,(b_1-b_2)^3\,(a_1-a_2)\,(c_1-c_2) \\ &\quad +32\,(c_1-c_2)^3\,(a_1-a_2)\,(b_1-b_2)-40\,(a_1-a_2)\,(b_1-b_2) \\ &\quad (c_1-c_2))t^5+(16\,(a_1-a_2)^2\,(b_1-b_2)^2+16\,(a_1-a_2)^2 \\ &\quad (c_1-c_2)^2+16\,(b_1-b_2)^2\,(c_1-c_2)^2-8\,(a_1-a_2)^2 \\ &\quad -8\,(b_1-b_2)^2-8\,(c_1-c_2)^2+4)t^6-8\,(a_1-a_2)\,(b_1-b_2) \\ &\quad (c_1-c_2)\,t^7+t^8, \end{aligned}$$

which represents a generating function of the Chebyshev polynomials of the second kind.

By replacing a_2 by $(-a_2)$, b_1 by $2b_1$, b_2 by $(-2b_2)$, c_1 by $2c_1$ and c_2 by $(-2c_2)$, taking $a_1-a_2=k$, $a_1a_2=1$ and $4b_1b_2=4c_1c_2=-1$ in theorem 2.1, we have the following theorem.

Theorem 3.4. *We have the following a new generating function for the product of k-Fibonacci numbers and Chebyshev polynomials of the second kind as :*

$$\sum_{n=0}^{\infty} F_{k,n}U_n\,(b_1-b_2)\,U_n\,(c_1-c_2)\,t^n=\frac{N_{F_{k,n}U_nU_n}}{D_{F_{k,n}U_nU_n}}, \tag{7}$$

with

$$\begin{aligned} N_{F_{k,n}U_nU_n} &= 1-(-4\,(b_1-b_2)^2-4\,(c_1-c_2)^2+k^2+3)t^2 \\ &\quad -8k\,(b_1-b_2)\,(c_1-c_2)\,t^3+(-4\,(b_1-b_2)^2 \\ &\quad -4\,(c_1-c_2)^2+k^2+3)t^4-t^6. \end{aligned}$$

$$
\begin{aligned}
D_{F_{k,n}U_nU_n} &= 1-4k\left(b_1-b_2\right)\left(c_1-c_2\right)t-(-4k^2\left(c_1-c_2\right)^2 \\
&+(-2+4\left(b_1-b_2\right)^2)(-k^2+4\left(c_1-c_2\right)^2-2))t^2 \\
&-4k\left(b_1-b_2\right)\left(c_1-c_2\right)(k^2-4\left(b_1-b_2\right)^2-4\left(c_1-c_2\right)^2+5)t^3 \\
&+(16\left(b_1-b_2\right)^4+16\left(c_1-c_2\right)^4-16\left(b_1-b_2\right)^2-16\left(c_1-c_2\right)^2 \\
&-16k^2\left(b_1-b_2\right)^2\left(c_1-c_2\right)^2+k^4+4k^2+6)t^4 \\
&+4k\left(b_1-b_2\right)\left(c_1-c_2\right)(k^2-4\left(b_1-b_2\right)^2-4\left(c_1-c_2\right)^2 \\
&+5)t^5-(-4k^2\left(c_1-c_2\right)^2+(-2+4\left(b_1-b_2\right)^2)(-k^2 \\
&+4\left(c_1-c_2\right)^2-2))t^6+4k\left(b_1-b_2\right)\left(c_1-c_2\right)t^7+t^8.
\end{aligned}
$$

Foata in [13] gives the formula (8) which is obtained by taking $k=1$ in the identity (7)

$$
\sum_{n=0}^{\infty} F_n U_n\left(b_1-b_2\right) U_n\left(c_1-c_2\right) t^n=\frac{N_{F_nU_nU_n}}{D_{F_nU_nU_n}}, \tag{8}
$$

with

$$
\begin{aligned}
N_{F_nU_nU_n} &= 1+(4\left(b_1-b_2\right)^2+4\left(c_1-c_2\right)^2-4)t^2-8\left(b_1-b_2\right)\left(c_1-c_2\right)t^3 \\
&-(4\left(b_1-b_2\right)^2+4\left(c_1-c_2\right)^2-4)t^4-t^6. \\
D_{F_nU_nU_n} &= 1-4\left(b_1-b_2\right)\left(c_1-c_2\right)t+(12\left(b_1-b_2\right)^2+12\left(c_1-c_2\right)^2 \\
&-16\left(b_1-b_2\right)^2\left(c_1-c_2\right)^2-6)t^2-(24\left(b_1-b_2\right)\left(c_1-c_2\right) \\
&-16\left(b_1-b_2\right)^3\left(c_1-c_2\right)-16\left(c_1-c_2\right)^3\left(b_1-b_2\right))t^3 \\
&+(16\left(b_1-b_2\right)^4+16\left(c_1-c_2\right)^4-16\left(b_1-b_2\right)^2-16\left(c_1-c_2\right)^2 \\
&-16\left(b_1-b_2\right)^2\left(c_1-c_2\right)^2+11)t^4+(-16\left(b_1-b_2\right)^3\left(c_1-c_2\right) \\
&-16\left(c_1-c_2\right)^3\left(b_1-b_2\right)+24\left(b_1-b_2\right)\left(c_1-c_2\right))t^5 \\
&+(12\left(b_1-b_2\right)^2+12\left(c_1-c_2\right)^2-16\left(b_1-b_2\right)^2\left(c_1-c_2\right)^2-6)t^6 \\
&+4\left(b_1-b_2\right)\left(c_1-c_2\right)t^7+t^8,
\end{aligned}
$$

which represents a generating function of the product of Fibonacci numbers and Chebyshev polynomials of the second kind.

By replacing c_1 by $2c_1$, c_2 by $(-2c_2)$, b_2 by $(-b_2)$ and a_2 by $(-a_2)$, taking $4c_1c_2=-1$, $b_1b_2=a_1a_2=1$ and $b_1-b_2=a_1-a_2=k$ in theorem 2.1, we have the following theorem.

Theorem 3.5. *For $n\in\mathbb{N}$, a new generating function of the product for squares of k-Fibonacci numbers and Chebyshev polynomials of the second kind is given by*

$$
\sum_{n=0}^{\infty} F_{k,n}^2 U_n\left(c_1-c_2\right) t^n=\frac{N_{F_{k,n}^2U_n}}{D_{F_{k,n}^2U_n}}, \tag{9}
$$

with

$$
\begin{aligned}
N_{F_{k,n}^2 U_n} &= 1+(2k^2-4\,(c_1-c_2)^2+3)t^2+4k^2\,(c_1-c_2)\,t^3 \\
&\quad +(2k^2-4\,(c_1-c_2)^2+3)t^4+t^6. \\
D_{F_{k,n}^2 U_n} &= 1-2k^2\,(c_1-c_2)\,t-(4k^2\,(c_1-c_2)^2+\left(2+k^2\right)(4\,(c_1-c_2)^2 \\
&\quad -k^2-2))t^2-2k^2\,(c_1-c_2)\,(4\,(c_1-c_2)^2-2k^2-5)t^3+(2k^4 \\
&\quad +16\,(c_1-c_2)^4-16\,(c_1-c_2)^2+8k^2+4k^4\,(c_1-c_2)^2+6)t^4 \\
&\quad -2k^2\,(c_1-c_2)\,(4\,(c_1-c_2)^2-2k^2-5)t^5-(4k^2\,(c_1-c_2)^2 \\
&\quad +\left(k^2+2\right)(4\,(c_1-c_2)^2-k^2-2))t^6-2k^2\,(c_1-c_2)\,t^7+t^8.
\end{aligned}
$$

Foata in [13] gives the formula (10) which is obtained by taking $k=1$ in the identity (9)

$$\sum_{n=0}^{\infty} F_n^2 U_n\,(c_1-c_2)\,t^n=\frac{N_{F_n^2 U_n}}{D_{F_n^2 U_n}}, \tag{10}$$

with

$$
\begin{aligned}
N_{F_n^2 U_n} &= 1-2\,(c_1-c_2)\,t+4t^2-2\,(c_1-c_2)\,t^3+t^4. \\
D_{F_n^2 U_n} &= (1+2\,(c_1-c_2)\,t+t^2)(1-6\,(c_1-c_2)\,t \\
&\quad +(7+4\,(c_1-c_2)^2)t^2-6\,(c_1-c_2)\,t^3+t^4),
\end{aligned}
$$

which represents a generating function of the product for squares of Fibonacci numbers and Chebyshev polynomials of the second kind .

Proposition 3.6. *For $n\in\mathbb{N}$, a new generating function of the product for squares of k-Fibonacci numbers and Chebyshev polynomials of the first kind is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2 T_n\,(c_1-c_2)\,t^n=\frac{N_{F_{k,n}^2 T_n}}{D_{F_{k,n}^2 T_n}}, \tag{11}$$

with

$$
\begin{aligned}
N_{F_{k,n}^2 T_n} &= 1+(1-2(c_1-c_2)^2k)t^6+k^2(c_1-c_2)t^5+(8(c_1-c_2)^4 \\
&\quad -8(c_1-c_2)^2+2k^2+3)t^4+(-4k^2\,(c_1-c_2)^3+4k^2\,(c_1-c_2))t^3 \\
&\quad +(-4k^2(c_1-c_2)^2-6(c_1-c_2)^2+2k^2+3)t^2-k^2(c_1-c_2)t. \\
D_{F_{k,n}^2 T_n} &= D_{F_{k,n}^2 U_n}.
\end{aligned}
$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} F_{k,n}^{2} T_n\left(c_1-c_2\right) t^n \\
= &\sum_{n=0}^{\infty} F_{k,n}^{2}\left(h_n\left(2c_1,[-2c_2]\right)-\left(c_1-c_2\right) h_{n-1}\left(2c_1,[-2c_2]\right)\right) t^n \\
= &\sum_{n=0}^{\infty} F_{k,n}^{2} h_n(2c_1,[-2c_2]) t^n-\left(c_1-c_2\right) \sum_{n=0}^{\infty} F_{k,n}^{2} h_{n-1}\left(2c_1,[-2_2]\right) t^n \\
= &\sum_{n=0}^{\infty} F_{k,n}^{2} U_n(c_1-c_2) t^n-\left(c_1-c_2\right) \sum_{n=0}^{\infty} F_{k,n}^{2} h_{n-1}\left(2c_1,[-2c_2]\right) t^n .
\end{aligned}$$

By replacing c_1 by $2c_1$, c_2 by $(-2c_2)$, b_2 by $(-b_2)$ and a_2 by $(-a_2)$, taking $4c_1c_2=-1$, $b_1b_2=a_1a_2=1$ and $b_1-b_2=a_1-a_2=k$ in Theorem (2.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} F_{k,n}^{2} h_{n-1}\left(2c_1,[-2c_2]\right) t^n \\
= &\frac{\left(\begin{array}{c} k^2t-2(c_1-c_2)(-2k^2-1)t^2+4k^2(c_1-c_2)^2t^3 \\ -2(c_1-c_2)(4\left(c_1-c_2\right)^2-2)t^4-k^2t^5+2\left(c_1-c_2\right)t^6 \end{array}\right)}{D_{F_{k,n}^{2} T_n}},
\end{aligned}$$

with

$$D_{F_{k,n}^{2} T_n}=D_{F_{k,n}^{2} U_n},$$

then

$$\begin{aligned}
&\sum_{n=0}^{\infty} F_{k,n}^{2} T_n\left(c_1-c_2\right) t^n \\
= &\frac{N_{F_{k,n}^{2} U_n}}{D_{F_{k,n}^{2} U_n}}-\left(c_1-c_2\right) \\
&\times \frac{\left(\begin{array}{c} k^2t-2(c_1-c_2)(-2k^2-1)t^2+4k^2(c_1-c_2)^2t^3 \\ -2(c_1-c_2)(4\left(c_1-c_2\right)^2-2)t^4-k^2t^5+2\left(c_1-c_2\right)t^6 \end{array}\right)}{D_{F_{k,n}^{2} U_n}} \\
= &\frac{N_{F_{k,n}^{2} T_n}}{D_{F_{k,n}^{2} U_n}},
\end{aligned}$$

with

$$\begin{aligned}
N_{F_{k,n}^{2} T_n} = \; & 1+(1-2(c_1-c_2)^2)t^6+k^2(c_1-c_2)t^5+(8(c_1-c_2)^4 \\
& -8(c_1-c_2)^2+2k^2+3)t^4+(-4k^2(c_1-c_2)^3 \\
& +4k^2(c_1-c_2))t^3+(2(c_1-c_2)^2(-2k^2-1)+2k^2 \\
& -4(c_1-c_2)^2+3)t^2-k^2(c_1-c_2)t.
\end{aligned}$$

□

Foata in [13] gives the formula (12) which is obtained by taking $k = 1$ in the identity (11)

$$\sum_{n=0}^{\infty} F_n^2 T_n (c_1 - c_2) t^n = \frac{N_{F_n^2 T_n}}{D_{F_n^2 T_n}}, \tag{12}$$

with

$$\begin{aligned} F_n^2 T_n &= 1 - 3(c_1 - c_2) t + (4 - 4(c_1 - c_2)^2) t^2 + (4(c_1 - c_2)^3 \\ &\quad - (c_1 - c_2)) t^3 + (1 - 2(c_1 - c_2)^2) t^4. \\ D_{F_n^2 T_n} &= D_{F_n^2 U_n}, \end{aligned}$$

which represents a generating function of the product for squares of Fibonacci numbers and Chebyshev polynomials of the first kind.

Proposition 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of k-Fibonacci numbers and Chebyshev polynomials of the first and the second kind is given by*

$$\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) T_n (c_1 - c_2) t^n = \frac{N_{F_{k,n} U_n T_n}}{D_{F_{k,n} U_n T_n}}, \tag{13}$$

with

$$\begin{aligned} N_{F_{k,n} U_n T_n} &= 1 + \left(-1 + 2(c_1 - c_2)^2\right) t^6 + 2k(c_1 - c_2)(b_1 - b_2) t^5 \\ &\quad + (-4(b_1 - b_2)^2 + 8(c_1 - c_2)^4 - 8(c_1 - c_2)^2 + k^2 + 3) t^4 \\ &\quad + (-8k(b_1 - b_2)(c_1 - c_2) + 8k(c_1 - c_2)^3 (b_1 - b_2)) t^3 \\ &\quad + (4(b_1 - b_2)^2 + 6(c_1 - c_2)^2 - 3 - k^2 - 8(c_1 - c_2)^2 (b_1 - b_2)^2 \\ &\quad + 2k^2 (c_1 - c_2)) t^2 - 2k(c_1 - c_2)(b_1 - b_2) t. \\ D_{F_{k,n} U_n T_n} &= D_{F_{k,n} U_n U_n}. \end{aligned}$$

Proof. We have

$$\begin{aligned} &\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) T_n (c_1 - c_2) t^n \\ &= \sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) \left(\begin{array}{c} h_n (2c_1, [-2c_2]) \\ -(c_1 - c_2) h_{n-1} (2c_1, [-2c_2]) \end{array} \right) t^n \\ = &\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) h_n (2c_1, [-2c_2]) t^n \\ &- (c_1 - c_2) \sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) h_{n-1} (2c_1, [-2c_2]) t^n \\ = &\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) U_n (c_1 - c_2) t^n \\ &- (c_1 - c_2) \sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) h_{n-1} (2c_1, [-2c_2]) t^n. \end{aligned}$$

By replacing c_1 by $2c_1$, c_2 by $(-2c_2)$, b_1 by $2b_1$, b_2 by $(-2b_2)$ and a_2 by $(-a_2)$, taking $4c_1c_2 = -1, 4b_1b_2 = -1$, $a_1a_2 = 1$ and $a_1 - a_2 = k$ in Theorem (2.3), we have

$$\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) h_{n-1} (2c_1, [-2c_2]) t^n = \frac{\left(\begin{array}{c} 2k (b_1 - b_2) t - 2 (c_1 - c_2) \left(-4 (b_1 - b_2)^2 + k^2 + 1\right) t^2 \\ -8k (c_1 - c_2)^2 (b_1 - b_2) t^3 + \left(-8 (c_1 - c_2)^3 + 4 (c_1 - c_2)\right) t^4 \\ -2k (b_1 - b_2) t^5 - 2 (c_1 - c_2) t^6 \end{array}\right)}{D_{F_{k,n} U_n T_n}},$$

with

$$D_{F_{k,n} U_n T_n} = D_{F_{k,n} U_n U_n},$$

then

$$\sum_{n=0}^{\infty} F_{k,n} U_n (b_1 - b_2) T (c_1 - c_2) t^n = \frac{N_{F_{k,n} U_n U_n}}{D_{F_{k,n} U_n U_n}} - (c_1 - c_2)$$

$$\times \quad \frac{\left(\begin{array}{c} 2k (b_1 - b_2) t - 2 (c_1 - c_2) (-4 (b_1 - b_2)^2 + k^2 + 1) t^2 \\ -8k (c_1 - c_2)^2 (b_1 - b_2) t^3 + (-8 (c_1 - c_2)^3 + 4 (c_1 - c_2)) t^4 \\ -2k (b_1 - b_2) t^5 - 2 (c_1 - c_2) t^6 \end{array}\right)}{D_{F_{k,n} U_n U_n}}$$

$$= \frac{N_{F_{k,n} U_n T_n}}{D_{F_{k,n} U_n U_n}},$$

with

$$\begin{aligned} N_{F_{k,n} U_n T_n} &= 1 + \left(-1 + 2 (c_1 - c_2)^2\right) t^6 + 2k (c_1 - c_2) (b_1 - b_2) t^5 \\ &\quad + (-4 (b_1 - b_2)^2 + 8 (c_1 - c_2)^4 - 8 (c_1 - c_2)^2 + k^2 + 3) t^4 \\ &\quad + (-8k (b_1 - b_2) (c_1 - c_2) + 8k (c_1 - c_2)^3 (b_1 - b_2)) t^3 + \\ &\quad (4 (b_1 - b_2)^2 + 6 (c_1 - c_2)^2 - 3 - k^2 - 8 (c_1 - c_2)^2 (b_1 - b_2)^2 \\ &\quad + 2k^2 (c_1 - c_2)^2) t^2 - 2k (c_1 - c_2) (b_1 - b_2) t. \end{aligned}$$

□

Foata in [13] gives the formula (14) which is obtained by taking $k = 1$ in the identity (13)

$$\sum_{n=0}^{\infty} F_n U_n (b_1 - b_2) T_n (c_1 - c_2) t^n = \frac{N_{F_n U_n T_n}}{D_{F_n U_n T_n}}, \tag{14}$$

with

$$\begin{aligned} N_{F_nU_nT_n} &= 1-2(b_1-b_2)(c_1-c_2)t+(4(b_1-b_2)^2+8(c_1-c_2)^2 \\ &\quad -8(c_1-c_2)^2(b_1-b_2)^2-4)t^2+(-8(c_1-c_2)(b_1-b_2) \\ &\quad +8(c_1-c_2)^3(b_1-b_2))t^3+(8(c_1-c_2)^4-8(c_1-c_2)^2 \\ &\quad -4(b_1-b_2)^2+4)t^4+2(c_1-c_2)(b_1-b_2)t^5 \\ &\quad +(2(c_1-c_2)^2-1)t^6. \\ D_{F_nU_nT_n} &= D_{F_nU_nU_n}. \end{aligned}$$

Case 3.8. *By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2=1$, $b_1b_2=c_1c_2=k$ and $a_1-a_2=k$, $b_1-b_2=c_1-c_2=2$ in theorem 2.2, we obtain the following theorem.*

Theorem 3.9. *For all $n\in\mathbb{N}$, a new generating function of the product of k-Fibonacci numbers and the squares of k-Pell numbers is given by*

$$\sum_{n=0}^{\infty}F_{k,n}P_{k,n}^2t^n=\frac{N_{F_{k,n}P_{k,n}^2}}{D_{F_{k,n}P_{k,n}^2}}, \tag{15}$$

with

$$\begin{aligned} N_{F_{k,n}P_{k,n}^2} &= kt+4t^2-k^3\left(2+k^2\right)t^3-4k^4t^4+k^3\left(8k+k^2\right)t^5-4k^4t^6. \\ D_{F_{k,n}^2P_{k,n}} &= 1-4kt-\left(4k^3+(2k+4)\left(k^3+2k+4\right)\right)t^2-4k(k^4+8k \\ &\quad +5k^2)t^3+(32k^2+k^8+k^2\left(4k^4+32k-16k^2\right)+6k^4)t^4 \\ &\quad +4k^3\left(k^4+8k+5k^2\right)t^5-(4k^7+k^4(2k+4)\left(k^{3\cdot}+2k+4\right))t^6 \\ &\quad +4k^7t^7+k^8t^8. \end{aligned}$$

By replacing a_1 by $2a_1$, a_2 by $(-2a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $4a_1a_2=-1$, $b_1b_2=c_1c_2=k$ and $b_1-b_2=c_1-c_2=2$ in theorem 2.2, we obtain the following theorem.

Theorem 3.10. *For $n\in\mathbb{N}$, a new generating function of the product of squares of k-Pell numbers and Chebyshev polynomials of the second kind is given by*

$$\sum_{n=0}^{\infty}P_{k,n}^2U_n(a_1-a_2)t^n=\frac{N_{P_{k,n}^2U_n}}{D_{P_{k,n}^2U_n}}, \tag{16}$$

with

$$\begin{aligned} N_{P_{k,n}^2U_n} &= 2(a_1-a_2)t-4t^2-2k^2(a_1-a_2)(4(a_1-a_2)^2-2)t^3 \\ &\quad +16k^2(a_1-a_2)^2t^4+2k^2(a_1-a_2)(8k+k^2)t^5+4k^4t^6. \\ D_{P_{k,n}^2U_n} &= 1-8(a_1-a_2)t-(16k(a_1-a_2)^2+(2k+4)(4k(a_1-a_2)^2 \\ &\quad -2k-4))t^2-8(a_1-a_2)(4k^2(a_1-a_2)^2-8k-5k^2)t^3 \\ &\quad +(32k^2+16k^4(a_1-a_2)^4-k^2(16k^2(a_1-a_2)^2-32k \\ &\quad -64(a_1-a_2)^2)+6k^4)t^4-8k^2(a_1-a_2)(4k^2(a_1-a_2)^2 \\ &\quad -8k-5k^2)t^5-(16k^5(a_1-a_2)^2+k^4(2k+4)(4k(a_1-a_2)^2 \\ &\quad -2k-4))t^6-8k^6(a_1-a_2)t^7+k^8t^8. \end{aligned}$$

Putting $k=1$ in the relationship (16) we obtain a new generating function of the product of squares of Pell numbers and Chebyshev polynomials of the second kind:

$$\sum_{n=0}^{\infty} P_n^2 U_n\left(a_1-a_2\right) t^n=\frac{N_{P_n^2 U_n}}{D_{P_n^2 U_n}},$$

with

$$\begin{aligned}
N_{P_n^2 U_n} &= 2\left(a_1-a_2\right) t-4 t^2-2\left(a_1-a_2\right)\left(-2+4\left(a_1-a_2\right)^2\right) t^3 \\
&\quad +16\left(a_1-a_2\right)^2 t^4+18\left(a_1-a_2\right) t^5+4 t^6 . \\
D_{P_n^2 U_n} &= 1-8\left(a_1-a_2\right) t-\left(40\left(a_1-a_2\right)^2-36\right) t^2-8\left(a_1-a_2\right) \\
&\quad \left(4\left(a_1-a_2\right)^2-13\right) t^3+\left(70+16\left(a_1-a_2\right)^4+48\left(a_1-a_2\right)^2\right) t^4 \\
&\quad -8\left(a_1-a_2\right)\left(4\left(a_1-a_2\right)^2-13\right) t^5-\left(40\left(a_1-a_2\right)^2-36\right) t^6 \\
&\quad -8\left(a_1-a_2\right) t^7+t^8 .
\end{aligned}$$

Case 3.11. *By replacing a_2 by $\left(-a_2\right)$, b_2 by $\left(-b_2\right)$ and c_2 by $\left(-c_2\right)$, taking $a_1 a_2=b_1 b_2=c_1 c_2=k$ and $a_1-a_2=b_1-b_2=c_1-c_2=2$ in theorem 3.1, we obtain the following theorem.*

Theorem 3.12. *We have the following a new generating function for the cubes of k-Pell numbers as*

$$\sum_{n=0}^{\infty} P_{k, n}^3 t^n=\frac{N_{P_{k, n} P_{k, n} P_{k, n}}}{D_{P_{k, n} P_{k, n} P_{k, n}}}, \tag{17}$$

with

$$\begin{aligned}
N_{P_{k, n} P_{k, n} P_{k, n}} &= t-\left(12 k^2+3 k^3\right) t^3-16 k^3 t^4+\left(12 k^5+3 k^6\right) t^5-k^9 t^7 . \\
D_{P_{k, n} P_{k, n} P_{k, n}} &= 1-8 t-\left(16 k+(2 k+4)\left(8 k+2 k^2\right)\right) t^2-8\left(12 k^2+5 k^3\right) t^3 \\
&\quad +\left(48 k^4+k^3\left(48 k^2-64\right)+6 k^6\right) t^4+8 k^3\left(12 k^2+5 k^3\right) t^5 \\
&\quad -\left(16 k^7+k^6(2 k+4)\left(8 k+2 k^2\right)\right) t^6+8 k^9 t^7+k^{12} t^8 .
\end{aligned}$$

Mansour in [26] gives the formula (18) which is obtained by taking $k=1$ in the identity (17)

$$\sum_{n=0}^{\infty} P_n^3 t^n=\frac{t\left(1-4 t-t^2\right)}{\left(1+2 t-t^2\right)\left(1-14 t-t^2\right)}, \tag{18}$$

representing a generating function of the cubes of Pell numbers.

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LMAM Laboratory and Department of Mathematics, University of MSB Jijel, BP 98 Ouled Aissa, Jijel 18000, Algeria
E-mail address: boubelouta.khadidja@gmail.com

LMAM Laboratory and Department of Mathematics, University of MSB Jijel, BP 98 Ouled Aissa, Jijel 18000, Algeria
E-mail address: aboussayoud@yahoo.fr

Department of Economics, Faculty of Economics,Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey
E-mail address: mtsrkn@hotmail.com

LMAM Laboratory and Department of Mathematics, University of MSB Jijel, BP 98 Ouled Aissa, Jijel 18000, Algeria
E-mail address: mkerada@yahoo.fr