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IRREDUCIBLE LOCALLY BOUNDED FINITE-DIMENSIONAL PSEUDOREPRESENTATIONS OF CONNECTED LIE GROUPS

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Abstract. Bounded quasirepresentations always admit small (in norm) perturbations that define irreducible bounded quasirepresentations. The pseudorepresentations corresponding to these quasirepresentations (if they exist) are defined more rigidly, and therefore it is reasonable to seek for irreducible pseudorepresentations. In this note, we describe the irreducible locally bounded finite-dimensional pseudorepresentations of connected locally compact groups.

§ 1. Introduction

For the motivation concerning the interest to irreducible pseudorepresentations of groups, see the introduction to [1].

The paper [1] promised to give a description of irreducible locally bounded finite-dimensional pseudorepresentations of connected locally compact groups. Instead, as follows from the text of the proofs of the theorem in [1], the statement of the theorem is related to semisimple Lie groups only rather than connected locally compact groups (everywhere in the text of [1], the words "connected locally compact group" should read "semisimple Lie group"). In the present note we clarify the matter for connected Lie groups

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and describe the irreducible locally bounded finite-dimensional pseudorepresentations of all connected Lie groups. We recall that these pseudorepresentations are pure, i.e., their restrictions to amenable subgroups are ordinary representations of these subgroups [4, 5].

§ 2. Preliminaries

Recall the general result concerning the structure of quasirepresentations of arbitrary groups.

Theorem 1 [2–5]. *Let G be a group and let π be a quasirepresentation of G on a finite-dimensional vector space E_π. Let E_π^* be the space dual to E_π. Let L be the set of vectors $\xi \in E_\pi$ whose orbit*

$$\{\pi(g)\xi \mid g \in G\}$$

is bounded in E; let M be the set of functionals $f \in E_\pi^*$ whose orbit $\{\pi(g)^* f \mid g \in G\}$ is bounded in E_π^*; then L and the annihilator $M^\perp$ are π-invariant vector subspaces in E_π. Let us consider an increasing family of subspaces $\{0\}$, $L \cap M^\perp$, $M^\perp$, $L + M^\perp$, $E = E_\pi$ and write out the matrix $t(g)$ of the operator $\pi(g)$, $g \in G$, in a block form corresponding to the decomposition of the space E into a direct sum of subspaces $L \cap M^\perp$, $M^\perp \setminus (L \cap M^\perp)$, $L \setminus (L \cap M^\perp)$, and $E \setminus (L + M^\perp)$, where the symbol "$\setminus$" stands for the passage to a complementary subspace:

$$t(g) = \begin{pmatrix} \alpha(g) & \varphi(g) & \sigma(g) & \tau(g) \\ 0 & \beta(g) & 0 & \rho(g) \\ 0 & 0 & \gamma(g) & \chi(g) \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G.$$

(Here $t_{23}(g) = 0$, since L is invariant with respect to π.) Then the following assertions hold:

1) *the mappings α, δ, γ, σ, and χ are bounded;*

2) *the matrix-valued mappings t_1 and t_2 defined by the formulas $t_1(g) = \begin{pmatrix} \alpha(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}$ and $t_2(g) = \begin{pmatrix} \beta(g) & \rho(g) \\ 0 & \delta(g) \end{pmatrix}$, are representations of G;*

3) *the mapping τ is a quasicocycle with respect to the representations t_1 and t_2, that is, the mapping $(g, h) \mapsto \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h)$, $g, h \in G$, is bounded.*

§ 3. Main theorem

Theorem 1 plays a crucial role in the proof of the following main result of this note, which gives a kind of a list the irreducible locally bounded finite-dimensional finally precontinuous pseudorepresentations of connected locally compact groups.

Theorem 2. *Let G be a connected Lie group, let R be the radical of G, and let S be a Levi subgroup of G, and let π be an irreducible locally bounded finite-dimensional pseudorepresentation of G.*

Let the center of S be finite. Then π is an ordinary continuous finite-dimensional representation of G, namely, a product of a central character of R (i.e., a character χ of R such that $\chi(grg^{-1}) = \chi(r)$ for every $r \in R$ and $g \in G$) and an irreducible unitary representation of the compact part of S.

Let the center of G be infinite. Then π is either an ordinary continuous finite-dimensional representation of G or a product of a real exponential of some Guichardet–Wigner pseudocharacter related to simple components with infinite center of S by a central character of R (i.e., a character χ of R such that

$$\chi(grg^{-1}) = \chi(r)$$

for every $r \in R$ and $g \in G$) and by an irreducible unitary representation of the compact part of S.

Proof. Let the center of G be finite. It follows from Theorem 1 that every irreducible locally bounded finite-dimensional pseudorepresentation of G is an irreducible mapping of one of the forms α, β, γ, or δ. According to Theorem 3.3.17 of [6], the mapping γ is the direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G and some G-central unitary characters of the group R. This proves the first assertion of the theorem.

If the center of S is infinite (if the group G (or S) has a nontrivial Hermitian symmetric quotient group), then the information concerning the representations α, β, and δ is quite similar, and, if π is a representation of one of these types, then, as in the previous case, π is an ordinary finite-dimensional representation of G. However, since the center is infinite, it follows that, by Theorem 3.3.17 of [6], the mapping γ is a direct sum, then the mapping γ is a direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of S, some one-dimensional Guichardet–Wigner pseudorepresentations (i.e., one-dimensional mappings of the form

$$g \to \exp(ir\theta(g)), \qquad g \in G,$$

for some $r \in \mathbb{R}$, where θ stands for a Guichardet–Wigner pseudocharacter on G, see [4–6]), and some G-central unitary characters of the group R. This completes the proof of Theorem 2.

§ 4. Concluding remarks

The general case of irreducible finite-dimensional finally precontinuous locally bounded pseudorepresentations of connected locally compact groups will be considered elsewhere.

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