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# GENERALIZED DIFFERENCE SEQUENCE SPACES

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ABSTRACT. In the present paper, we will bring out the new techniques of combining the generalized $B$- difference sequence spaces with binomial mappings and the $\triangle$ operator. Moreover, the basic properties of these spaces will be taken care of.

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## 1. PRELIMINARIES, BACKGROUND AND NOTATION

A sequence space is said to be a linear space of real or complex sequences. The symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will represent the set of non-negative integers, real numbers and complex numbers, respectively. With $\Upsilon$ we mark the space of all sequences; $l_\infty$, $c$ and $c_0$ respectively, abbreviates the set of all bounded sequences , the set of convergent sequences and the null sequences. By $cs$ and $bs$, we mean the set of all convergent series and set of bounded series, respectively. With$l_1$ and $l_p$, $(1 < p < \infty)$, we design the spaces of all absolutely convergent and $p$-absolutely convergent series, respectively. Further, by $l(p)$ we mean the $p_k$th summable sequence, where $p = (p_k)$ is a bounded sequence of positive reals. .

A sequence space $X$ is called an $FK$-space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbf{C}$ defined by $p_n(x) = x_n$ for all $x \in X$ and every $n \in \mathbf{N}$. A $BK$-space is a normed $FK$-space, that is, a $BK$- space is a Banach space with continuous coordinates.

We call a linear space $E$ over $\mathbb{R}$ to be a paranormed space if there exists a sub-additive function $\Gamma : E \to \mathbb{R}$ so that $\Gamma(\theta) = 0$, $\Gamma(-\zeta) = \Gamma(\zeta)$ and scalar multiplication is continuous, i.e,, $|\xi_n - \xi| \to 0$ and $\Gamma(\zeta_n - \zeta) \to 0$ imply $\Gamma(\xi_n\zeta_n - \xi\zeta) \to 0 \ \forall \ \xi's$ in $\mathbb{R}$ and $\zeta's$ in $E$ , where $\theta$ denotes the zero element in $E$. For spaces $E$ and $F$, set

$$\Psi(E : F) = \{\kappa = (\kappa_i) : \zeta\kappa = (\zeta_i\kappa_i) \in F \ \forall \ \zeta = (\zeta_i) \in E\}. \tag{1}$$

By (1), we re-write the $\alpha$-, $\beta$- and $\gamma$- duals of $E$ as follows;

$$E^\alpha = \Psi(E : l_1), \ E^\beta = \Psi(E : cs) \ and \ G^\gamma = \Psi(E : bs).$$

As in [14], if space $G$ paranormed by $\Gamma$ admits a sequence $(\wp_n)$ with the character that for all $g \in G$ there exists one and only one sequence of scalars

$(\xi_n)$ in such a way that

$$\lim_n \Gamma(g - \sum_{k=0}^{n} \xi_k \wp_k) = 0,$$

then $(\wp_n)$ defines a Schauder basis for $G$. Here $\sum \xi_k \wp_k$ having the sum as $g$ is then known as the expansion of $g$ w.r.t. $(\wp_n)$ and it is expressed as $g = \sum \xi_k \wp_k$.

For the matrix $\mathcal{C} = (c_{i,j})$ and $\nu = (\nu_k) \in \Upsilon$, the $\mathcal{C}$-transform of $\nu$ is given by $\mathcal{C}\nu = \{(\mathcal{C}\nu)_i\}$ for if it survives (i.e., it does not diverges) $\forall\ i \in \mathbb{N}$, where $(\mathcal{C}\nu)_i = \sum_{j=0}^{\infty} c_{i,j}\nu_j$.

For such a matrix $\mathcal{C} = (c_{i,j})$, the set $G_{\mathcal{C}}$, where

$$G_{\mathcal{C}} = \{\nu = (\nu_j) \in \Upsilon \ : \ \mathcal{C}\nu \in G\}, \tag{2}$$

is referred as the domain/region of $\mathcal{C}$ in $G$. As in [1], [4], [17], we shall designate all such classes by $(G : H)$ with $G \subseteq H_{\mathcal{C}}$

In[9] the author has constructed new techniques and introduced the following spaces:

$$T(\Delta) = \{\rho = (\rho_i) : \Delta\rho \in T\},$$

where $T = \{l_\infty, c, \ c_0\}$ and which was further analysed as in [6], [10], [12], [15], [16].

This space was further analysed and generalizations of it were studied in [1], [5], [7]-[18], [23], [26]-[32], [25] and put forward the following:

$$V(\Delta^m) = \{(x_k) \in w : (\Delta^m x_k) \in Z\}$$

for $V \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$, $\Delta^0 x_k = x_k$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ for each $k \in \mathbb{N}$.

Now by $E^b$ we mean Euler means of order $b$ and it is a matrix of the type $E^b = (e^b_{n,j})$, where $0 < b < 1$ and

$$e^b_{n,j} = \begin{cases} \binom{n}{j} (1-b)^{n-j} b^j & \text{for } 0 \le j \le n, \\ 0 & \text{for } j > n. \end{cases}$$

It was further taken by Altay et al [2] and defined the spaces $e^b_p$ and $e^b_\infty$ as follows:

$$e^b_p = \left\{ \upsilon = (\upsilon_j) \in \Omega : \sum_n \left| \sum_{j=0}^{n} \binom{n}{j} (1-b)^{n-j} b^k \upsilon_j \right|^p < \infty \right\}$$

and

$$e^b_\infty = \left\{ \upsilon = (\upsilon_j) \in \Omega : \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \binom{n}{j} (1-b)^{n-j} b^j \upsilon_j \right| < \infty \right\}.$$

In [3], the author has further studied it and analysed the following spaces :

$$e_0^b(\Delta) = \{\upsilon = (\upsilon_j) \in \Omega : (\Delta\upsilon_j) \in e_0^b\},$$
$$e_c^b(\Delta) = \{\upsilon = (\upsilon_j) \in \Omega : (\Delta\upsilon_j) \in e_c^b\}$$

and

$$e_\infty^b(\Delta) = \{\upsilon = (\upsilon_j) \in \Omega : (\Delta\upsilon_j) \in e_\infty^b\}.$$

It was studied by many authors viz., Kara et. al. [9], Polat et. al. [16]-[18] and many others. In [23]-[24], the authors have investigated and synthesised the spaces $b_0^{r_1,r_2}$, $b_c^{r_1,r_2}$, $b_\infty^{r_1,r_2}$ and $b_p^{r_1,r_2}$. For $r_1, r_2 \in \mathbb{R}$ and $r_1 + s_2 \neq 0$, the binomial matrix $B^{r_1,r_2} = (b_{n,t}^{r_1,r_2})$ is given by:

$$b_{n,t}^{r_1,r_2} = \begin{cases} \frac{1}{(r_2+r_1)^n} \binom{n}{t} r_2^{n-t} r_1^t & \text{when } 0 \leq t \leq n, \\ 0 & \text{when } t > n, \end{cases} \quad \forall\ t, n \in \mathbb{N}$$

In addition, for $r_2 r_1 > 0$ it is evident that (i) $\|B^{r_1,r_2}\| < \infty$,

(ii) $\lim_{n\to\infty} b_{n,j}^{r_1,r_2} = 0$ for each $j \in \mathbb{N}$,

(iii)$\lim_{n\to\infty} \sum_j b_{n,j}^{r_1,r_2} = 1$. Consequently, the matrix $B^{r_1,r_2}$ is regular for $r_2 r_1 > 0$. Unless expressed elsewhere, we let $r_2 r_1 > 0$. Note choosing $r_2 + r_1 = 1 = g$, we shall get the Euler matrix $E^r$. The author in [24] has constructed new techniques and define various spaces viz.,

$$b_\infty^{r_1,r_2} = \left\{ \upsilon = (\upsilon_j) \in \Omega : \sup_{n\in\mathbb{N}} \left| \frac{1}{(r_2+r_1)^n} \sum_{j=0}^{n} \binom{n}{j} r_2^{n-j} r_1^j \upsilon_j \right| < \infty \right\}.$$

The purpose of this manuscript is to introduce and investigate the space $b_0^{r_1,r_2}(\nabla_m^g)$, $b_c^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$ of the binomial sequence for which the $B^{r_1,r_2}(\nabla_m^g)$-transforms are respectively in the spaces $c_0$, $c$ and $\ell_\infty$. Also, for these spaces, we shall calculate the bases and $\beta$-duals, where $g = (g_k)$ is defined in such a way that $g_k \neq 0$ for all $k \in \mathbb{N}$. .

## 2. The spaces $b_p^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$

The current portion deals with the interpret of spaces $b_p^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$. We shall show their linear isomorphism property with the spaces $\ell_p$ and $\ell_\infty$, respectively. Following [4], [8], [22], [26], [32], we interpret the following new sequence spaces $b_p^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$ as:

$$b_p^{r_1,r_2}(\nabla_m^g) = \{\upsilon = (\upsilon_k) \in \Omega : (\nabla_m^g \upsilon_k) \in b_p^{r_1,r_2}\}$$

and

$$b_\infty^{r_1,r_2}(\nabla_m^g) = \{\upsilon = (\upsilon_k) \in \Omega : (\nabla_m^g \upsilon_k) \in b_\infty^{r_1,r_2}\}.$$

We consider the $B^{r_1,r_2}(\nabla_m^g)$-transform of a sequence $\upsilon = (\upsilon_k)$ as the sequence $y = (y_i)$, that is,

$$(3) \qquad y_i = \left[B^{r_1,r_2}(\nabla_m^g \upsilon_k)\right]_i = \frac{1}{(r_2+r_1)^i} \sum_{k=0}^{i} \binom{i}{k} r_2^{i-k} r_1^k (\nabla_m^g \upsilon_k)\ \forall\ i \in \mathbb{N}.$$

Clearly, $b_p^{r_1,r_2}(\nabla_m^g)$ or $b_\infty^{r_1,r_2}(\nabla_m^g)$ spaces can be seen to be sequences with $B^{r_1,r_2}(\nabla_m^g)$- transforms in $\ell_p$ or $\ell_\infty$.

It is important to note that if we choose $m = 1$, we get results derived in [15].

**Theorem 2.1.** *For $1 \leq p < \infty$ the spaces the spaces $b_p^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$ with the norm*

$$f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon) = \|y\|_p = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

*and*

$$f_{b_\infty^{r_1,r_2}(\nabla_m^g)}(\upsilon) = \|y\|_\infty = \sup_{i\in\mathbb{N}} |y_i|,$$

*where $y = (y_i)$ is $B^{r_1,r_2}(\nabla_m^g)$-transform of $\upsilon$, are complete.*

**Proof** : Trivial part is linearity. Clearly $f_{b_p^{r_1,r_2}}(\kappa\upsilon) = |\kappa| f_{b_p^{r_1,r_2}}(\upsilon)$ and $f_{b_p^{r_1,r_2}}(\upsilon) = 0$ if and only if $\upsilon = \theta$ for all $\upsilon \in b_p^{r_1,r_2}(\nabla_m^g)$, where $\theta$ speaks zero entry in $b_p^{r_1,r_2}$ and $\kappa \in \mathbb{R}$. Choosing $u, \upsilon \in b_p^{r_1,r_2}(\nabla_m^g)$ so that

$$\begin{aligned} f_{b_p^{r_1,r_2}(\nabla_m^g)}(u+\upsilon) &= \left(\sum_n \left|\left(B^{r_1,r_2}\left[\nabla_m^g(u_k+\upsilon_k)\right]\right)_n\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_n \left|\left[B^{r_1,r_2}(\nabla_m^g u_k)\right]_n\right|^p\right)^{\frac{1}{p}} + \left(\sum_n \left|\left[B^{r_1,r_2}(\nabla_m^g \upsilon_k)\right]_n\right|^p\right)^{\frac{1}{p}} \\ &= f_{b_p^{r_1,r_2}(\nabla_m^g)}(u) + f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon). \end{aligned}$$

Consequently $f_{b_p^{r_1,r_2}(\nabla_m^g)}$ is a norm on the space $b_p^{r_1,r_2}(\nabla_m^g)$.

We choose a Cauchy sequence $(\upsilon_i)$ in $b_p^{r_1,r_2}(\nabla_m^g)$, where $\upsilon_i = (\upsilon_{i_k})_{k=1}^\infty \in b_p^{r_1,r_2}(\nabla_m^g)$ for each $i \in \mathbb{N}$. Now given $\varepsilon > 0$, we can find a natural number $i_0$ in such a way that $f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon_i - \upsilon_j) < \varepsilon$ for $i, j \geq i_0$. Therefore, we see

$$\left|\left(B^{r_1,r_2}\left[\nabla_m^g(\upsilon_{i_k} - \upsilon_{j_k})\right]\right)_n\right| \leq \left(\sum_n \left|\left(B^{r_1,r_2}\left[\nabla_m^g(\upsilon_{i_k} - \upsilon_{j_k})\right]\right)_n\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for $i, j \geq i_0$ and $\forall\ k \in \mathbb{N}$. It follows that $(B^{r_1,r_2}(\nabla_m^g \upsilon_{i_k}))_{i=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Thus the completeness of $\mathbb{R}$ yields $\lim_{i\to\infty} B^{r_1,r_2}(\nabla_m^g \upsilon_{i_k}) = B^{r_1,r_2}(\nabla_m^g \upsilon_k)\ \forall\ k \in \mathbb{N}$. We now clearly see that

$$\sum_{n=0}^{m} \left|\left(B^{r_1,r_2}\left[\nabla_m^g(\upsilon_{i_k} - \upsilon_{j_k})\right]\right)_n\right| \leq f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon_i - \upsilon_j) < \varepsilon \tag{4}$$

for $i > i_0$. We take $j \to \infty$ , so that 4 yields that

$$f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon_i - \upsilon) \to 0.$$

Consequently, we see that

$$f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon) \leq f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon_i - \upsilon) + f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon_i) < \infty,$$

i.e.,, $\upsilon \in b_p^{r_1,r_2}(\nabla_m^g)$. Thus, the space $b_p^{r_1,r_2}(\nabla_m^g)$ is complete. In a similar fashion , we are through with the space $b_\infty^{r_1,r_2}(\nabla_m^g)$.$\diamond$

**Theorem 2.2.** *The sequence spaces $b_p^{r_1,r_2}(\nabla_m^g)$ and $b_\infty^{r_1,r_2}(\nabla_m^g)$ are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively, where $1 \leq p < \infty$.*

**Proof** : We shall produce the result for $b_p^{r_1,r_2}(\nabla_m^g)$ and by employing same technique will follow the rest. Thus, to prove $b_p^{r_1,r_2}(\nabla_m^g) \cong \ell_p$, there must exist a linear bijection between $b_p^{r_1,r_2}(\nabla_m^g)$ and $\ell_p$. Choose $T : b_p^{r_1,r_2}(\nabla_m^g) \to \ell_p$ by $T(\upsilon) = B^{r_1,r_2}(\nabla_m^g \upsilon_k)$. $T$ is linear is trivial. Now $\upsilon = \theta$ if $T(\upsilon) = \theta$. Consequently, $T$ is injective.

Let $y = (y_i) \in \ell_p$ and define the sequence $\upsilon = (\upsilon_k)$ $\forall k \in \mathbb{N}$ by

$$\upsilon_k = \sum_{i=0}^{k} (r_2 + r_1)^i \sum_{j=i}^{k} \binom{j}{i} r_1^{-j} (-r_2)^{j-i} g_k^{-1} y_i. \tag{5}$$

Hence, we can see that

$$\begin{aligned} f_{b_p^{r_1,r_2}(\nabla_m^g)}(\upsilon) &= \left\| \left[B^{r_1,r_2}(\nabla_m^g \upsilon_k)\right]_n \right\|_p \\ &= \left( \sum_{n=1}^{\infty} \left| \frac{1}{(r_2 + r_1)^n} \sum_{k=0}^{n} \binom{n}{k} r_2^{n-k} r_1^k (\nabla_m^g x_k) \right|^p \right)^{\frac{1}{p}} \\ &= \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}} = \|y\|_p < \infty, \end{aligned}$$

which implies that $\upsilon \in b_p^{r_1,r_2}(\nabla_m^g)$ and $T(\upsilon) = y$. Hence, we get $T$ as surjective and norm preserving. Therefore, it follows that $b_p^{r_1,r_2}(\nabla_m^g) \cong \ell_p$.$\diamond$

## 3. The Schauder basis for $b_p^{r_1,r_2}(\nabla_m^g) \cong \ell_p$

In 1927, Juliusz Schauder determined Schauder basis [27].

We call a sequence $\{b_i : b_i \in X\}_{i \in \mathbb{N}}$ to be a Schauder basis [6] if $\forall$ $v \in X$, we can find one and only one scalar sequence $(\mu_i)$ such that $\|v - \sum_{i=0}^{n} \mu_i b_i\| \to 0$ as $i \to \infty$.

We now move to find out the Schauder basis for $b_p^{r_1,r_2}(\nabla_m^g)$.

**Theorem 3.1.** *The basis for the space $b_p^{r_1,r_2}(\nabla_m^g)$ is $(v^{(j)}(r_1, r_2))_{j \in \mathbb{N}}$, and for any $x = (x_i) \in b_p^{r_1,r_2}(\nabla_m^g)$ can be written in one and only one way as*

$$x = \sum_{j} \lambda_j(r_1, r_2) v^{(j)}(r_1, r_2), \tag{6}$$

*with $1 \le p < \infty$ and $\lambda_j(r_1, r_2) = [B^{r_1,r_2}(\nabla_m^g x_i)]_j$ and $v^{(j)}(r_1, r_2) = \{v_i^{(j)}(r_1, r_2)\}_{i \in \mathbb{N}}$ given by*

$$v_i^{(j)}(r_1, r_2) = \begin{cases} 0 & \text{if } 0 \le i < j, \\ (r_2 + r_1)^j \sum_{z=k}^{i} \binom{z}{k} r_1^{-z} (-r_2)^{z-j} & \text{if } i \ge j, \end{cases}$$

*for all $j \in \mathbb{N}$.*

**Proof** : It is evident that, $B^{r_1,r_2}(\nabla_m^g v_i^{(j)}(r_1, r_2)) = e_j \in \ell_p$, where $e_j$ is taken to be 1 at $j$th place and zeros elsewhere $\forall$ $j \in \mathbb{N}$. So, it follows that $v^{(j)}(r_1, r_2) \in b_p^{r_1,r_2}(\nabla_m^g)$ for each $j \in \mathbb{N}$. Now for $x \in b_p^{r_1,r_2}(\nabla_m^g)$ and $m \in \mathbb{N}$, we choose

$$x^{(m)} = \sum_{j=0}^{m} \lambda_j(r_1, r_2) v^{(j)}(r_1, r_2).$$

So linearity of $B^{r_1,r_2}(\nabla_m^g)$ implies that

$$B^{r_1,r_2}\big(\nabla_m^g x_i^{(m)}\big) = \sum_{j=0}^{m} \lambda_j(r_1,r_2) B^{r_1,r_2}\big(\nabla_m^g v_i^{(j)}(r_1,r_2)\big) = \sum_{j=0}^{m} \lambda_j(r_1,r_2) e_j$$

and

$$\big[B^{r_1,r_2}\big(\nabla_m^g\big(x_i - x_i^{(m)}\big)\big)\big]_j = \begin{cases} 0 & \text{if } 0 \le j \le m, \\ [B^{r_1,r_2}(\nabla_m^g x_i)]_j & \text{if } j > m, \end{cases}$$

for every $j \in \mathbb{N}$.

Now choose $\varepsilon > 0$ in such a way that

$$\sum_{j=m_0+1}^{\infty} \big|\big[B^{r_1,r_2}(\nabla_m^g x_i)\big]_j\big|^p < \left(\frac{\varepsilon}{2}\right)^p$$

for each natural number $m_0$ with $j \ge m_0$. We thus have

$$\begin{aligned} f_{b_p^{r_1,r_2}(\nabla_m^g)}\big(x - x^{(m)}\big) &= \left(\sum_{j=m+1}^{\infty} \big|\big[B^{r_1,r_2}(\nabla_m^g x_i)\big]_j\big|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{j=m_0+1}^{\infty} \big|\big[B^{r_1,r_2}(\nabla_m^g x_i)\big]_j\big|^p\right)^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

which yiels us that $x \in b_p^{r_1,r_2}(\nabla_m^g)$ is represented as 6.

Now, it remains to show this representation is unique. Let us assume that

$$x = \sum_j \mu_j(r,s) v^{(j)}(r_1,r_2).$$

Then, we see

$$\begin{aligned} \big[B^{r_1,r_2}(\nabla_m^g x_i)\big]_j &= \sum_j \mu_j(r_1,r_2)\big[B^{r_1,r_2}\big(\nabla_m^g v_i^{(j)}(r_1,r_2)\big)\big]_j \\ &= \sum_j \mu_j(r_1,r_2)(e_j)_j = \mu_j(r_1,r_2), \end{aligned}$$

which is a clear contradiction for the assumption that $\lambda_j(r_1,r_2) = [B^{r_1,r_2}(\nabla_m^g x_i)]_j \;\forall\, j \in \mathbb{N}$. Consequently, our representation is unique.$\diamond$

## 4. Köthe-duals for the spaces $b_p^{r_1,r_2}(\nabla_m^g)$

In this part, we will be concerning in computing the Köthe-Toeplitz duals for the space $b_p^{r_1,r_2}(\nabla_m^g) \cong \ell_p$.

For the spaces $G$ and $H$, we choose the space $M(G,H)$ as follows:

$$M(G,H) = \big\{v = (v_j) \in \Omega : vx = (v_j x_j) \in H \;\forall\; x = (x_j) \in G\big\}.$$

In [4], [20], [21], Köthe duals are

$$G^\alpha = M(G,\ell_1), \qquad G^\beta = M(G,c) \quad \text{and} \quad G^\gamma = M(G,\ell_\infty),$$

respectively. We write the following equations that are essential in our text:

(7) $$\sup_{i\in\mathbb{N}}\sum_{j}|a_{i,j}|^{q}<\infty$$

(8) $$\sup_{j\in\mathbb{N}}\sum_{i}|a_{i,j}|<\infty,$$

(9) $$\sup_{i,j\in\mathbb{N}}|a_{i,j}|<\infty$$

(10) $$\lim_{i\to\infty}a_{i,j}=a_{j}\ \forall\ j\in\mathbb{N}$$

(11) $$\sup_{K\in\Gamma}\sum_{j}\left|\sum_{i\in K}a_{i,j}\right|^{q}<\infty$$

(12) $$\lim_{i\to\infty}\sum_{j}|a_{i,j}|=\sum_{j}\left|\lim_{i\to\infty}a_{i,j}\right|$$

where $\Gamma$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\leq\infty$.

**Lemma 4.1.** [17]: *We have following well known results for an infinite matrix $A=(a_{n,k})$ :*
*(i) $A\in(\ell_1:\ell_1)$ iff 8 remain true.*
*(ii) $A\in(\ell_1:c)$ iff 9 and 10 remain true.*
*(iii) $A\in(\ell_1:\ell_\infty)$ iff 9 remain true.*
*(iv) For $1<p\leq\infty$, $A\in(\ell_p:\ell_1)$ iff 11 remain true with $\frac{1}{p}+\frac{1}{q}=1$.*
*(v) For $1<p<\infty$, $A\in(\ell_p:c)$ iff 7 and 10 remain true with $\frac{1}{p}+\frac{1}{q}=1$.*
*(vi) For $1<p<\infty$ $A\in(\ell_p:\ell_\infty)$ iff 7 remain true with $\frac{1}{p}+\frac{1}{q}=1$.*
*(vii) For $1<p<\infty$, $A\in(\ell_\infty:c)$ iff 10 and 12 remain true with $\frac{1}{p}+\frac{1}{q}=1$.*
*(viii) For $q=1$, $A\in(\ell_\infty:\ell_\infty)$ iff 7 remain true.*

**Theorem 4.2.** *We consider the sets $\mho_1^{r_1,r_2}$ and $\mho_2^{r_1,r_2}$ given by*

$$\mho_1^{r_1,r_2}=\left\{\xi=(\xi_k)\in\Omega:\sup_{i\in\mathbb{N}}\sum_{k}\left|(r_2+r_1)^{i}\sum_{j=i}^{k}\binom{j}{i}r_1^{-j}(-r_2)^{j-i}g_k^{-1}\xi_k\right|<\infty\right\}$$

*and*

$$\mho_2^{r_1,r_2}=\left\{\xi=(\xi_k)\in\Omega:\sup_{K\in\Gamma}\sum_{i}\left|\sum_{k\in K}(r_2+r_1)^{i}\sum_{j=i}^{k}\binom{j}{i}r_1^{-j}(-r_2)^{j-i}g_k^{-1}\xi_k\right|^{q}<\infty\right\}.$$

*Then $[b_1^{r_1,r_2}(\nabla_m^g)]^\alpha=\mho_1^{r_1,r_2}$ and $[b_p^{r_1,r_2}(\nabla_m^g)]^\alpha=\mho_2^{r_1,r_2}$, where $1<p\leq\infty$.*

**Proof** : Let $\xi=(\xi_k)\in\Omega$ and choose $\upsilon=(\upsilon_k)$ given by 5, so that

$$\xi_k\upsilon_k=\sum_{i=0}^{k}(r_2+r_1)^{i}\sum_{j=i}^{k}\binom{j}{i}r_1^{-j}(-r_2)^{j-i}\xi_k g_k^{-1}y_i=\left(G^{r_1,r_2}y\right)_k\ \forall k\in\mathbb{N},$$

where $G^{r_1,r_2} = (g^{r_1,r_2}_{k,i})$ is given by

$$g^{r_1,r_2}_{k,i} = \begin{cases} (r_2+r_1)^i \sum_{j=i}^{k} \binom{j}{i} r_1^{-j}(-r_2)^{j-i}\xi_k g_k^{-1} & \text{if } 0 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

Consequently, we deduce that $\xi\upsilon = (\xi_k \upsilon_k) \in \ell_1$ whenever $\upsilon \in b_1^{r_1,r_2}(\nabla_m^g)$ or $b_p^{r_1,r_2}(\nabla_m^g)$ iff $G^{r_1,r_2}y \in \ell_1$ whenever $y \in \ell_1$ or $\ell_p$, this shows that $\xi = (\xi_k) \in [b_1^{r_1,r_2}(\nabla_m^g)]^\alpha$ or $[b_p^{r_1,r_2}(\nabla_m^g)]^\alpha$ iff $G^{r_1,r_2} \in (\ell_1 : \ell_1)$ and $G^{r_1,r_2} \in (\ell_p : \ell_1)$ by employing (i) and (iv) of Lemma 4.1, we see $u = (u_k) \in [b_1^{r_1,r_2}(\nabla_m^g)]^\alpha$ iff

$$\sup_{i\in\mathbb{N}} \sum_k \left| (r_2+r_1)^i \sum_{j=i}^{k} \binom{j}{i} r_1^{-j}(-r_2)^{j-i} u_k \right| < \infty$$

and $\xi = (\xi_k) \in [b_p^{r_1,r_2}(\nabla_m^g)]^\alpha$ iff

$$\sup_{K\in\Gamma} \sum_i \left| \sum_{k\in K} (r_2+r_1)^i \sum_{j=i}^{k} \binom{j}{i} r_1^{-j}(-r_2)^{j-i} \xi_k \right|^q < \infty.$$

Thus, we have $[b_1^{r_1,r_2}(\nabla_m^g)]^\alpha = \mho_1^{r_1,r_2}$ and $[b_p^{r_1,r_2}(\nabla_m^g)]^\alpha = \mho_2^{r_1,r_2}$, where $1 < p \leq \infty$.$\diamond$

Now, we consider the sets viz., $\mho_3^{r_1,r_2}$, $\mho_4^{r_1,r_2}$, $\mho_5^{r_1,r_2}$, $\mho_6^{r_1,r_2}$ and $\mho_7^{r_1,r_2}$ by

$$\mho_3^{r_1,r_2} = \left\{ \xi = (\xi_i) \in \Omega : \lim_{n\to\infty} (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \text{ exists } \ \forall\ k \in \mathbb{N} \right\},$$

$$\mho_4^{r_1,r_2} = \left\{ \xi = (\xi_i) \in \Omega : \sup_{n,k\in\mathbb{N}} \left| (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \right| < \infty \right\},$$

$$\mho_5^{r_1,r_2} = \left\{ \xi = (\xi_i) \in \Omega : \lim_{n\to\infty} \sum_k \left| (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \right| \right.$$
$$\left. = \sum_k \left| \lim_{n\to\infty} (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \right| \right\},$$

$$\mho_6^{r_1,r_2} = \left\{ \xi = (\xi_k) \in \Omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \right|^q < \infty \right\},$$

$1 < q < \infty$ and

$$\mho_7^{r_1,r_2} = \left\{ \xi = (\xi_k) \in \Omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| (r_2+r_1)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r_1^{-j}(-r_2)^{j-k} g_i^{-1}\xi_i \right| < \infty \right\}.$$

**Theorem 4.3.** *We have the following relations:*

*(i).* $[b_1^{r_1,r_2}(\nabla_m^g)]^\beta = \mho_3^{r_1,r_2} \cap \mho_4^{r_1,r_2}$.

*(ii).* $[b_p^{r_1,r_2}(\nabla_m^g)]^\beta = \mho_3^{r_1,r_2} \cap \mho_6^{r_1,r_2}$, *where* $1 < p < \infty$.

*(iii).* $[b_\infty^{r_1,r_2}(\nabla_m^g)]^\beta = \mho_3^{r_1,r_2} \cap \mho_5^{r_1,r_2}$.

*(iv).* $[b_1^{r_1,r_2}(\nabla_m^g)]^\gamma = \mho_4^{r_1,r_2}$.

*(v).* $[b_p^{r_1,r_2}(\nabla_m^g)]^\gamma = \mho_6^{r_1,r_2}$, *where* $1 < p < \infty$.
*(vi).* $[b_\infty^{r_1,r_2}(\nabla_m^g)]^\gamma = \mho_7^{r_1,r_2}$.

**Proof** : Suppose $\xi = (\xi_l) \in \Omega$ and $\upsilon = (\upsilon_l)$ be given by 5, then, we see

$$\sum_{l=0}^{s} \xi_l \upsilon_l = \sum_{l=0}^{s} \xi_l \left[ \sum_{i=0}^{l} (r_2 + r_1)^i \sum_{j=i}^{l} \binom{j}{i} r_1^{-j} (-r_2)^{j-i} g_i^{-1} y_i \right]$$
$$= \sum_{l=0}^{s} \left[ (r_2 + r_1)^l \sum_{i=l}^{s} \sum_{j=l}^{i} \binom{j}{l} r_1^{-j} (-r_2)^{j-l} \xi_i \right] y_l = \left(\mho^{r_1,r_2} y\right)_s,$$

where $\mho^{r_1,r_2} = (\xi_{s,l}^{r_1,r_2})$ is defined by

$$u_{s,l} = \begin{cases} (r_2 + r_1)^l \sum_{i=l}^{s} \sum_{j=l}^{i} \binom{j}{l} r_1^{-j} (-r_2)^{j-l} \xi_i g_i^{-1} & \text{if } 0 \le l \le s, \\ 0 & \text{if } l > s. \end{cases}$$

Therefore, we deduce that $\xi\upsilon = (\xi_l \upsilon_l) \in cs$ whenever $\upsilon \in b_1^{r_1,r_2}(\nabla_m^g)$ iff $\mho^{r_1,r_2} y \in c$ whenever $y \in \ell_1$, which yields us $\xi = (\xi_l) \in [b_1^{r_1,r_2}(\nabla_m^g)]^\beta$ iff $\mho^{r_1,r_2} \in (\ell_1 : c)$. So, using (ii) of Lemma 4.1 yields $[b_1^{r_1,r_2}(\nabla_m^g)]^\beta = \mho_3^{r_1,r_2} \cap \mho_4^{r_1,r_2}$. In a similar fashion, the complete proof will follow by employing Lemma 4.1 (i) and (iii)-(viii) rather that of (ii).$\diamond$

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