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A NEW CHARACTERIZATION OF LOCALLY COMPACT FIR GROUPS

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Abstract. A new characterization of locally compact FIR groups is given. This characterization pays special attention to the connected component of the identity.

§ 1. Introduction

The $[FIR]$ groups (locally compact groups all of whose irreducible continuous unitary representations are finite-dimensional), which are also often called $[MOORE]$ groups, are described in [1, 2, 3].

The objective of this paper is to give another structure theorem for this class of locally compact groups which stresses the role of the identity component of the group.

§ 2. Preliminaries

Let us cite the main known results concerning the $[FIR]$ groups.

Theorem 1.

1 *(Moore, [1]). A group G satisfies $G \in [FIR]$ if and only if there is a family $\{K_\alpha\}$ of compact normal subgroups of G such that $\cap K_\alpha = \{e\}$, and each $G_\alpha = G/K_\alpha$ is a finite extension of a group whose quotient group by the center is compact. In particular, the class $[FIR]$ is stable under finite extensions.*

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2 *(Robertson [2]). A locally compact group G satisfies $G \in [FIR]$ if and only if G contains a characteristic subgroup H such that H has finite index in G and the derived group of H has compact closure and H can be embedded in a compact group.*

3 *(Robertson [2]). A locally compact group G satisfies $G \in [FIR]$ if and only if G is a semidirect product $G = \mathbb{R}^n \ltimes B$, where $B \in [FIR]$ has a compact identity component B_0 and B contains a normal subgroup H of finite index such that $\mathbb{R}^n \ltimes H$ is a direct product $\mathbb{R}^n \times H$.*

4 *(Shtern [3]) A locally compact group G satisfies $G \in [FIR]$ if and only if G is a projective limit of Lie groups G_α each of which contains a normal subgroup of finite index L_α isomorphic to a direct product of the additive group of a finite-dimensional vector space V_α and a product of a compact connected Lie group K_α and a central discrete group D_α, i.e.,*

$$L_\alpha \quad \text{is isomorphic to} \quad (D_\alpha \cdot K_\alpha) \times V_\alpha.$$

§ 3. Main theorem

Theorem 2. *A locally compact group G satisfies $G \in [FIR]$ if and only if the following assertions hold:*

(i) *the connected identity component G_0 of G is isomorphic to the direct product of a vector group V of a finite-dimensional vector space and a compact connected group K,*

(ii) *the quotient group G/G_0 (which is automatically $[FIR]$ if G is) is a projective limit of Abelian-by-finite groups,*

(iii) *G contains a subgroup of finite index that acts on V trivially, and*

(iv) *there is a base filter $\mathcal{N}_1$, $\bigcap_{\mathcal{N}_1} N_1 = \{e\}$, of compact subgroups N_1 of G_0 (we have automatically $N_1 \subset K$) such that N_1 is a normal subgroup of G and the quotient group K/N_1 is a Lie group and a base $\mathcal{N}_2$, $\bigcap_{\mathcal{N}_2} N_2 = \{e\}$, of compact normal subgroups N_2 of G/G_0 such that the quotient group $(G/G_0)/N_2$ is a (discrete, i.e., zero-dimensional) Lie group, and, if N is the preimage of $N_2 \in \mathcal{N}_2$ in G/N_1, then the group G/N has a subgroup H of finite index in G/N which is a direct product of G_0/N_1 and a commutative discrete group.*

Proof. Let G be a locally compact $[FIR]$ group.

Since every subgroup of an $[FIR]$ group is an MAP group (i.e., can be embedded in a compact group), it follows that G_0 is a product of a connected compact group (which is characteristic) and a vector group (see [6] and [7]).

Since every quotient group of an $[FIR]$ group is also an $[FIR]$ group, it follows that G/G_0 is a totally disconnected $[FIR]$ group. It follows from part 4 of Theorem 1 that this quotient group is a projective limit of discrete groups each of which is $[FIR]$, and hence, by [6], is Abelian-by-finite.

As in [3], it is clear that the action of G on V (which is trivial on G_0 due to the direct product structure) defines a continuous linear representation of G/G_0 on V, which is automatically trivial on some normal subgroup of finite index such that the quotient by this normal subgroup is discrete.

Assertion (iv) describes the obvious structure of the (not necessarily connected) quotient Lie groups of G related to the projective limit structure mentioned in parts 1 and 4 of Theorem 1.

Conversely, it follows from condition (iv) that there is a base of compact subgroups of identity of G_0 that are normal subgroups in G. If N_1 and N_2 satisfy condition (iv), then the preimage N of N_2 in G/N_1 is obviously a compact normal subgroup of G indeed.

Therefore, for the normal subgroups of the form N, the action of the corresponding normal subgroups of G/G_0 on the corresponding quotient Lie groups of G_0 is trivial. This shows that G admits a family of normal subgroups N for which $\cap N = \{e\}$ and $N_1 = K \cap N$ is as small as possible and the image N_2 of N in G/G_0 is as small as possible and such that G_0/N_1 and $(G/G_0)/N_2$ are Lie groups. Hence, G/N is a Lie group.

To sum up, by the projective limit structure (with Lie groups as quotients) and the direct product condition in (iv), G/N is a Lie group for every normal subgroup constructed as in (iv), and it immediately follows from conditions (i)–(iv) that this Lie group is $[FIR]$; therefore, G itself is an $[FIR]$ group.

§ 4. Concluding remarks

Theorem 2 generalizes Theorem 1' of [4] and, in particular, gives a first publication of a proof of this result. The connection between the conditions is established by the famous Thoma theorem [6]: a discrete group is of type I if and only if it is Abelian-by-finite.

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