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A NOTE ON THE TYPE 2 POLY-APOSTOL-BERNOULLI POLYNOMIALS

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Abstract. Recently, Kim-Kim ([6]) have studied the type2 poly-Bernoulli polynomials by using the polylogarithm functions. In this paper, we define the type 2 poly-Apostol-Bernoulli polynomials and investigate some properties for them. Furthermore, by making use of unipoly function, we define the type 2 unipoly-Apostol-Bernoulli numbers and show some basic properties for them.

1. Introduction

It is well known that the Bernoulli polynomials of order α are defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad \text{(see [1, 2, 3, 4, 14]).} \tag{1.1}$$

We note that for $\alpha = 1$, $B_n(x) = B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials. When $x = 0$, $B_n^{\alpha} = B_n^{\alpha}(0)$ are called the Bernoulli numbers of order α.

He-Araci-Srivastava-Acikgoz ([4]) introduced the Apostol Bernoulli polynomials $B_{n,\lambda}(x)$ as follows:

$$\frac{t}{\lambda e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \qquad \text{(see [1, 2, 3, 4, 15, 16]),} \tag{1.2}$$

where $|t| < 2\pi$ if $\lambda = 1$ and $|t+\log\lambda| < 2\pi$ if $\lambda \neq 0$. When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are the Apostol Bernoulli numbers. Kim-Kim ([6]) introduced the type 2 poly-Bernoulli polynomials which are given by

$$\frac{e_k(\log(1+t))}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x)\frac{t^n}{n!}, \qquad \text{(see [6, 9]),} \tag{1.3}$$

where $e_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}$ is the polyexponential function, as an inverse to the polylogarithm function $Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, $(|x| < 1)$. When $x = 0$, $\beta_n^{(k)} = \beta_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers. Note that $\beta_n^{(1)}(x) = B_n(x)$, $(n \geq 0)$

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are the ordinary Bernoulli polynomials. For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \qquad \text{(see [11, 12, 13])}, \tag{1.4}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\dots(x-n+1),\ (n \geq 1)$. From (1.3), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \qquad \text{(see [11, 12, 13])}. \tag{1.5}$$

For $n \geq 0$, the the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \qquad \text{(see [11, 12, 13])}. \tag{1.6}$$

From (1.6), it is easily to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \qquad \text{(see [11, 12, 13])}. \tag{1.7}$$

In this paper, we define the type 2 poly-Apostol-Bernoulli polynomials which are modified type 2 poly-Bernoulli polynomials and investigate some properties for them. Furthermore, by making use of type 2 unipoly-Bernoulli numbers, we define the type 2 unipoly-Apostol-Bernoulli numbers and show some basic properties for them.

2. The type 2 poly-Apostol-Bernoulli polynomials

For $k \in \mathbb{Z}$, we consider the polyexponential function defined by Kim-Kim ([6]), as an inverse to the polylogarithm function, which is given by

$$e_k(x) == \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \qquad \text{(see [5, 6, 7, 8, 10])}. \tag{2.1}$$

In the viewpoint of (1.3), we define the type 2 poly-Apostol-Bernoulli polynomials by

$$\frac{e_k(\log(1+t))}{\lambda e^t - 1}e^{xt} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $x=0$, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the type 2 poly-Apostol-Bernoulli numbers. From (2.2) and (1.2), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(1)}(x)\frac{t^n}{n!} &= \frac{e_1(\log(1+t))}{\lambda e^t - 1}e^{xt} \\ &= \frac{t}{\lambda e^t - 1}e^{xt}\end{aligned}$$

$$= \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}, \tag{2.3}$$

where $|t| < 2\pi$ when $\lambda = 1$ and $|t + \log \lambda| < 2\pi$. From (2.3), we get

$$\beta_{n,\lambda}^{(1)}(x) = B_{n,\lambda}(x), \quad (n \geq 0) \tag{2.4}$$

are the Apostol Bernoulli polynomials. From (2.2), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} \\
&= \frac{e_k(\log(1+t))}{\lambda e^t - 1} \\
&= \frac{1}{\lambda e^t - 1} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k} \\
&= \frac{1}{\lambda e^t - 1} \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} \\
&= \frac{1}{\lambda e^t - 1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!} \\
&= \frac{t}{\lambda e^t - 1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!} \\
&= \left(\sum_{j=0}^{\infty} B_{j,\lambda} \frac{t^j}{j!} \right) \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} \right) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} B_{n-l,\lambda} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. *For $k \in \mathbb{Z}$, $n \geq 0$, and $\lambda \in \mathbb{R}(\lambda \neq 0)$, we have*

$$\beta_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} B_{n-l,\lambda}. \tag{2.6}$$

Corollary 2.2. *For $k \in \mathbb{Z}$, $n \geq 0$, and $\lambda \in \mathbb{R}(\lambda \neq 0)$, we have*

$$\beta_{n,\lambda}^{(1)} = B_{n,\lambda} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{S_1(l+1, m+1)}{l+1} B_{n-l,\lambda}. \tag{2.7}$$

Moreover,

$$\sum_{l=1}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{S_1(l+1, m+1)}{l+1} B_{n-l,\lambda} = 0. \tag{2.8}$$

Let $k \geq 1$ be an integer. For $s \in \mathbb{C}$ and $\lambda \in \mathbb{R}(\lambda \neq 0)$, we define the function

$$\eta_{k,\lambda}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t)dt. \tag{2.9}$$

From (2.9), we note that if we take $k = 1$ and $\lambda = 1$, then we get

$$\begin{aligned} \eta_{1,1}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} e_1(\log(1+t))dt \\ &= \frac{s}{s\Gamma(s)} \int_0^\infty \frac{t^s}{e^t - 1} dt \\ &= \frac{s}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^t - 1} dt \\ &= s\zeta(s+1), \end{aligned} \tag{2.10}$$

where $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ is the Riemann zeta function.

We see from (2.9) that $\eta_{k,\lambda}(s)$ is holomorphic for $Re(s) > 0$, since $e_k(\log(1+t)) \leq e_1(\log(1+t))$ for $t \geq 0$. From (2.9), we note that for $\lambda \in \mathbb{R}(\lambda \neq 0)$,

$$\begin{aligned} \eta_{k,\lambda}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t))dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t))dt \\ &\quad + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t))dt. \end{aligned} \tag{2.11}$$

The second integral in (2.11) converges absolutely for any $s \in \mathbb{C}$ and hence the right hand side vanishes at non-positive integers. That is,

$$\lim_{s \to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t))dt \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{2.12}$$

On the other hand, for $Re(s) > 0$, the first integral in (2.11) can be written as

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{\lambda e^t - 1} e_k(\log(1+t))dt \\ &= \frac{1}{\Gamma(s)} \sum_{l=0}^\infty \beta_{l,\lambda}^{(k)} \int_0^1 \frac{t^{l+s-1}}{l!} dt \\ &= \frac{1}{\Gamma(s)} \sum_{l=0}^\infty \beta_{l,\lambda}^{(k)} \frac{1}{s+l} \frac{1}{l!} \end{aligned} \tag{2.13}$$

which defines an entire function of s. Thus, we can conclude that $\eta_{k,\lambda}(s)$ can be continued to an entire function of s. From (2.11) and (2.13), we get

$$\begin{aligned}
\eta_{k,\lambda}(-m) &= \lim_{s\to -m}\frac{1}{\Gamma(s)}\int_0^1 \frac{t^{s-1}}{\lambda e^t-1}e_k(\log(1+t))dt \\
&= \frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k)}\int_0^1\frac{t^{t+s-1}}{l!}dt \\
&= \frac{1}{\Gamma(s)}\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k)}\frac{1}{s+l}\frac{1}{l!} \\
&= \cdots+0+\cdots+0+\lim_{s\to -m}\frac{1}{\Gamma(s)}\frac{1}{s+m}\frac{\beta_{m,\lambda}^{(k)}}{m!}+0+\cdots \\
&= \lim_{s\to -m}\frac{\left(\frac{\Gamma(1-s)sin\pi s}{\pi}\right)}{s+m}\frac{\beta_{m,\lambda}^{(k)}}{m!} \\
&= \Gamma(1+m)\cos(\pi m)\frac{\beta_{m,\lambda}^{(k)}}{m!} \\
&= (-1)^m\beta_{m,\lambda}^{(k)}. \qquad (2.14)
\end{aligned}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.3. *Let $k \ge 1$ and $\lambda \in \mathbb{R}(\lambda \neq 0)$, $m \in \mathbb{N}$, we have*

$$\eta_{k,\lambda}(-m) = (-1)^m\beta_{m,\lambda}^{(k)}. \qquad (2.15)$$

In [6], it is well known that for $k \ge 2$,

$$\frac{d}{dx}e_k(x) = \frac{1}{x}e_{k-1}(x), \qquad (2.16)$$

and

$$e_k(x) = \int_0^x \frac{1}{t}\underbrace{\int_0^t\frac{1}{t}\cdots\int_0^t}_{(k-2)times}\frac{1}{t}(e^t-1)dtdt\cdots dt. \qquad (2.17)$$

From (2.17), we obtain the following equation.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}\frac{x^n}{n!}\\
&=\frac{1}{\lambda e^x-1}e_k(\log(1+x))\\
&=\frac{1}{\lambda e^x-1}\int_0^x\frac{1}{\log(1+t)}e_{k-1}(\log(1+t))dt\\
&=\frac{1}{\lambda e^x-1}\int_0^x\frac{1}{(1+t)\log(1+t)}\\
&\quad\underbrace{\int_0^t\frac{1}{(1+t)\log(1+t)}\cdots\int_0^t}_{(k-2)times}\frac{t}{(1+t)\log(1+t)}dtdt\cdots dt,(k\geq 2).
\end{aligned}\tag{2.18}$$

Thus, (2.18), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(2)}\frac{x^n}{n!}&=\frac{1}{\lambda e^x-1}\int_0^x\frac{1}{(1+t)\log(1+t)}dt\\
&=\frac{1}{\lambda e^x-1}\sum_{l=0}^{\infty}\frac{B_l^{(l)}}{l!}\int_0^x t^n dt\\
&=\frac{x}{\lambda e^x-1}\sum_{l=0}^{\infty}\frac{B_l^{(l)}}{l+1}\frac{x^l}{l!}\\
&=\left(\sum_{m=0}^{\infty}B_{m,\lambda}\frac{x^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{B_l^{(l)}}{l+1}\frac{x^l}{l!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{B_l^{(l)}}{l+1}B_{n-l,\lambda}\right)\frac{x^n}{n!}.
\end{aligned}\tag{2.19}$$

Therefore, (2.19), we obtain the following theorem.

Theorem 2.4. *Let $n\geq 0$ and $\lambda\in\mathbb{R}(\lambda\neq 0)$, we have*

$$\beta_{n,\lambda}^{(2)}=\sum_{l=0}^{n}\binom{n}{l}\frac{B_l^{(l)}}{l+1}B_{n-l,\lambda}.\tag{2.20}$$

3. The type 2 unipoly-Apostol-Bernoulli polynomials

Let p be any arithmetic function. That is, it is a real or complex valued function defined on the set of positive integer $\mathbb{N}$. Then we define the unipoly function attached to p by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k},\quad (k\in\mathbb{Z}).\tag{3.1}$$

It is well known that

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = Li_k(x) \tag{3.2}$$

is ordinary polylogarithm function, and for $k \geq 2$,

$$\frac{d}{dx} u_k(x|p) = \frac{1}{x} u_{k-1}(x|p), \tag{3.3}$$

and

$$u_k(x|p) = \int_0^x \frac{1}{t} \underbrace{\int_0^t \frac{1}{t} \cdots \int_0^t}_{(k-2) times} \frac{1}{t} u_1(t|p) dt dt \cdots dt \tag{3.4}$$

Kim-Kim ([6]) introduced the type 2 unipoly-Bernoulli numbers attached to p defined by

$$\frac{1}{e^t - 1} u_k(\log(1+t)|p) = \sum_{n=0}^{\infty} \beta_{n,p}^{(k)} \frac{t^n}{n!}. \tag{3.5}$$

In the viewpoint of (3.5), we define the type 2 unipoly-Apostol-Bernoulli numbers as follows:

$$\frac{1}{\lambda e^t - 1} u_k(\log(1+t)|p) = \sum_{n=0}^{\infty} \beta_{n,\lambda,p}^{(k)} \frac{t^n}{n!}. \tag{3.6}$$

Let us take $p(n) = \frac{1}{\Gamma(n)}$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,\lambda,p}^{(k)} \frac{t^n}{n!} &= \frac{1}{\lambda e^t - 1} u_k \left(\log(1+t) | \frac{1}{\Gamma} \right) \\
&= \frac{1}{\lambda e^t - 1} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k (m-1)!} \\
&= \frac{1}{\lambda e^t - 1} e_k(\log(1+t)) \\
&= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!}.
\end{aligned} \tag{3.7}$$

Thus, by (3.7), we get

$$\beta_{n,\lambda,p}^{(k)} = \beta_{n,\lambda}^{(k)}. \tag{3.8}$$

From (**??**), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\beta_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\\
&=\frac{1}{\lambda e^t-1}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+t))^m\\
&=\frac{1}{\lambda e^t-1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+!)^k}\sum_{l=m+1}^{\infty}S_1(l,m+1)\frac{t^l}{l!}\\
&=\frac{1}{\lambda e^t-1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+!)^k}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{t^{l+1}}{(l+1)!}\\
&=\frac{t}{\lambda e^t-1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{t^l}{(l+1)!}\\
&=\left(\sum_{j=0}^{\infty}B_{n-l,\lambda}\frac{t^j}{j!}\right)\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}\right)\frac{t^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}B_{n-l,\lambda}\right)\frac{t^n}{n!}. \qquad (3.9)
\end{aligned}
$$

Therefore, by comparing the coefficients on both sides of (3.9), we obtain the following theorem.

Theorem 3.1. *Let $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $\lambda\in\mathbb{R}(\lambda\neq 0)$, we have*

$$
\beta_{n,\lambda,p}^{(k)}=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}B_{n-l,\lambda}. \qquad (3.10)
$$

In particular,

$$
\beta_{n,\lambda,\frac{1}{\Gamma}}^{(k)}=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{B_{n-l,\lambda}}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1}. \qquad (3.11)
$$

Finally, we also define the type 2 unipoly-Apostol-Bernoulli polynomials attached to p as follows:

$$
\frac{1}{\lambda e^t-1}u_k(\log(1+t)|p)e^{xt}=\sum_{n=0}^{\infty}\beta_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \qquad (3.12)
$$

From (3.8) and (3.12), we observe that

$$\begin{aligned}
&\frac{1}{\lambda e^t-1}u_k(\log(1+t)|p)e^{xt}\\
&=\left(\sum_{l=0}^{\infty}\beta_{l,\lambda,p}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda,p}^{(k)}x^{n-l}\right)\frac{t^n}{n!}. \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13), we get the following theorem.

Theorem 3.2. *Let $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $\lambda\in\mathbb{R}(\lambda\neq 0)$, we have*

$$\begin{aligned}
\beta_{n,\lambda,p}^{(k)}&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda,p}^{(k)}x^{n-l}\\
&=\sum_{l=0}^{n}\binom{n}{l}\beta_{n-l,\lambda,p}^{(k)}x^{l}.
\end{aligned} \qquad (3.14)$$

In particular, for $\lambda\neq 1$, we get

$$\beta_{0,\lambda,p}^{(k)}(x)=0. \qquad (3.15)$$

From (3.14), we note that

$$\begin{aligned}
\frac{d}{dx}\beta_{n,\lambda,p}^{(k)}(x)&=\sum_{l=1}^{n}\binom{n}{l}\beta_{n-l,\lambda,p}^{(k)}lx^{l-1}\\
&=n\sum_{l=0}^{n-1}\binom{n-1}{l-1}B_{n-1-l,\lambda,p}^{(k)}(x)x^l\\
&=n\beta_{n-1,\lambda,p}^{(k)}(x). \qquad (3.16)
\end{aligned}$$

From (3.16), we obtain the following theorem.

Theorem 3.3. *Let $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $\lambda\in\mathbb{R}(\lambda\neq 0)$, we have*

$$\frac{d}{dx}\beta_{n,\lambda,p}^{(k)}=n\beta_{n-l,\lambda,p}^{(k)}(x). \qquad (3.17)$$

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