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THREE DIMENSIONAL REAL HYPERSURFACE IN A COMPLEX SPACE FORM SATISFYING THE CONDITION OF THE RICCI OPERATOR

HOONJOO KIM*, DONG HO LIM, AND YEONG JAE SUNG

ABSTRACT. In this paper, we characterize some real hypersurfaces in a complex space form $M_2(c)$, $c \neq 0$ in terms of the Ricci operator S, the tangential projection ϕ and the shape operator A.

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1. INTRODUCTION

A complex n-dimensional Kaehlerian manifold of a constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([4]) and that M is called a *Hopf hypersurface*.

R. Takagi [13] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero [9] and so on. Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface M of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, is said to be a *type* A for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [11] for $c > 0$ and Montiel and Romero [9] for $c < 0$.

Theorem A ([9, 11]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the following hypersurface:
(1) In case of $P_n\mathbf{C}$,
(A_1) a tube of radius r over a hyperplane $P_{n-1}\mathbf{C}$, where $0 < r < \frac{\pi}{2}$, $r \neq \frac{\pi}{4}$,

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(A_2) a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$, $r \neq \frac{\pi}{4}$.
(2) In case of $H_n\mathbf{C}$,
(A_0) a horosphere in $H_n\mathbf{C}$, that is, a Montiel tube,
(A_1) a tube of totally geodesic hyperplane $H_k\mathbf{C}$ $(k = 1$ or $n-1)$,
(A_2) a tube of a totally geodesic $H_k\mathbf{C}$ $(1 \le k \le n-2)$.

From the Gauss equation, we define the Ricci operator S by $g(SX, Y) = trace\{Z \longrightarrow R(Z, X)Y\}$ with respect to X and Y which are vector fields on M. It is well known that the concept of Ricci operator involves many important geometric properties ([3], [7], [9] etc). Regarding the Ricci operator on real hypersurfaces in a complex space form, Lim, Sohn and Ahn ([8]) have proved the following;

Theorem B ([8]) *M is a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$, satisfies $S\phi = \phi S$ if and only if M is pseudo-Einstein.*

In this paper, we study another characterization of real hypersurface in a nonflat complex space form $M_2(c)$ which satisfies $S\phi A = A\phi S$ and $SA\phi = \phi AS$, where S denotes the Ricci operator, A is the shape operator and ϕ is the tangential projection.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify the followings from the Gauss and Weingarten formulas:

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following the Gauss, Codazzi equations and the operator of

Lie derivative respectively:

$$(3)\qquad \begin{aligned} R(X,Y)Z &= \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\quad -2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

$$(4)\quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\},$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M. We shall denote the Ricci operator of M by S. Then it follows from (3) that

$$(5)\qquad SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2X,$$

where $m = traceA$ is the mean curvature of M.

Let Ω be the open subset of M defined by

$$(6)\qquad \Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\},$$

where $\alpha = \eta(A\xi)$. We put

$$(7)\qquad A\xi = \alpha\xi + \mu W,$$

where W is the unit vector field orthogonal to ξ and μ does not vanish on Ω.

3. Real hypersurface satisfying $S\phi A = A\phi S$

Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$, satisfying $S\phi A = A\phi S$. Then the above condition together with (1) and (5) implies that

$$(8)\qquad A\phi A^2X - A^2\phi AX + \frac{5c}{4}(\phi A - A\phi)X = 0$$

for any vector field X on Ω, and there are scalar functions γ, ϵ and δ, unit vector fields W and ϕW orthogonal to ξ such that

$$(9)\qquad AW = \mu\xi + \gamma W + \delta\phi W, \qquad A\phi W = \delta W + \epsilon\phi W$$

and

$$(10)\qquad m = traceA = \alpha + \gamma + \epsilon$$

in $M_2(c)$. At first, we shall prove the following lemmas;

Lemma 3.1 *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$ satisfying $S\phi A = A\phi S$. If the open set Ω is not empty, then we have*

$$(11)\qquad AW = \mu\xi + \gamma W, \qquad A\phi W = \epsilon\phi W,$$

$$(12)\qquad \epsilon^2 - (\alpha+\gamma)\epsilon - \frac{5c}{4} = 0$$

on Ω.

Proof. If we put $X = \xi$ into (8) and make use of (7), then we have

$$(13)\qquad A\phi AW + \alpha A\phi W - A^2\phi W + \frac{5c}{4}\phi W = 0.$$

If we substitute the equation (9) into (13) and using the first equation (1), then we obtain

$$-2\mu\delta^2\xi + \mu\delta(\alpha - \gamma - \epsilon)W + \mu(\alpha\epsilon + \gamma\epsilon - 2\delta^2 - \epsilon^2 + \frac{5c}{4})\phi W = 0.$$

If we take inner product of the equation of ξ and ϕW and using (6), then we have

$$\delta = 0, \qquad 2\delta^2 + \epsilon^2 - \epsilon(\alpha + \gamma) - \frac{5c}{4} = 0. \tag{14}$$

Substituting the first equation of (14) into (9) and by the second equation of (14), we obtain (11) and (12). □

Lemma 3.2 *Under the assumptions of Lemmas 3.1, we have*

$$\alpha\gamma + \frac{5c}{4} = \mu^2. \tag{15}$$

Proof. If we put $X = W$ into (8), then we have

$$A\phi A^2W - A^2\phi AW + \frac{5c}{4}(\phi A - A\phi)W = 0. \tag{16}$$

We put (11) into (16) and by using (7) and (11), we obtain

$$\gamma\epsilon^2 + (\frac{5c}{4} - (\mu^2 + \gamma^2))\epsilon - \frac{5c}{4}\gamma = 0. \tag{17}$$

We substitute (12) into (17), then we get

$$\epsilon(\alpha\gamma + \frac{5c}{4} - \mu^2) = 0. \tag{18}$$

If $\epsilon = 0$ on Ω, then we have $c = 0$ from (12), and hence equation (15) is obtained immediately from equation (18). □

4. Real hypersurface satisfying $S\phi A = A\phi S$ and $SA\phi = \phi AS$

In this section, we will continue the discussion on a real hypersurface M in $M_2(c)$ satisfying $S\phi A = A\phi S$ and $SA\phi = \phi AS$. The condition $SA\phi = \phi AS$, implies that

$$\begin{aligned} A^3\phi X - \phi A^3X + m(\phi A^2X - A^2\phi X) + \tfrac{5c}{4}(\phi A - A\phi)X \\ = -\tfrac{3c}{4}w(\phi X)\xi + \tfrac{3c}{4}\mu\eta(X)\phi W \end{aligned} \tag{19}$$

for any vector field X on Ω, where w is the dual 1-form of the unit vector field W.

Lemma 4.1 *Let M be a real hypersurface satisfying $S\phi A = A\phi S$ in a complex space form $M_2(c)$, $c \neq 0$. If $SA\phi = \phi AS$ holds on Ω, then we obtain $\epsilon^2 = 2c$.*

Proof. We put $X = \xi$ into (19) and by using (1) and (7), we get

$$\phi A^2W - (\gamma + \epsilon)\phi AW - \{\alpha(\gamma + \epsilon) + \frac{c}{2}\}\phi W = 0. \tag{20}$$

If we apply (11) into (20) and make use of (7) and (11), then we obtain

$$\alpha\gamma - \mu^2 + \epsilon(\alpha + \gamma) + \frac{c}{2} = 0. \tag{21}$$

We substitute (12) and (15) into (21), then we have the equation $\epsilon^2 = 2c$. □

Lemma 4.2 *Under the assumptions of Lemma 4.1, if the open set Ω is not empty, then we have*

$$(\gamma - \epsilon)(\epsilon^2 - \frac{5c}{4}) = 0. \tag{22}$$

Proof. If we put $X = W$ in (19) and make use of (1), then we obtain

$$A^3\phi W - \phi A^3 W + m(\phi A^2 - A^2\phi)W + \frac{5c}{4}(\phi A - A\phi)W = 0. \tag{23}$$

We take an inner product of ϕW into (19). Using (7), (9) and (10), we have

$$(\gamma - \epsilon)\{-\mu^2 + \alpha(\gamma + \epsilon) + \epsilon\gamma + \frac{5c}{4}\} = 0.$$

Thus, the equation (22) is derived from the above equation by using (12) and (15). □

5. Characterizations of Real Hypersurfaces

In this section, we shall prove the following theorems;

Theorem 5.1 *Let M be a real hypersurface satisfying $S\phi A = A\phi S$ and $SA\phi = \phi AS$ in a complex space form $M_2(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_2(c)$.*

Proof. Assume that the open set Ω given in (6) is not empty. By (22) in Lemma 4.2, $\gamma = \epsilon$ or $\epsilon^2 = \frac{5c}{4}$ holds on Ω.

In case $\epsilon^2 = \frac{5c}{4}$ holds on Ω, by Lemma 4.1, we obtain c is zero and it is a contradiction. Therefore $\gamma = \epsilon$ holds on Ω. If we substitute $\epsilon = \gamma$ into (12), then we obtain

$$\alpha\gamma + \frac{5c}{4} = 0. \tag{24}$$

By comparing (15) and (24), we have $\mu = 0$ and it is also a contradiction.

Thus, the set Ω is empty, and hence M is a Hopf hypersurface. □

Theorem 5.2 *Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. If it satisfies $S\phi A = A\phi S$ and $SA\phi = \phi AS$, then M is a locally congruent to one of the model spaces of type A.*

Proof. By Theorem 5.1, M is a Hopf hypersurface in $M_2(c)$, that is, $A\xi = \alpha\xi$. Therefore, the assumption $S\phi A = A\phi S$ or $SA\phi = \phi AS$ is equivalent to

$$A\phi A^2 - A^2\phi A + \frac{5c}{4}(\phi A - A\phi) = 0, \tag{25}$$

or

$$A^3\phi - \phi A^3 + m(\phi A^2 - A^2\phi) + \frac{5c}{4}(\phi A - A\phi) = 0, \text{respectively.} \tag{26}$$

Next, if we differentiate $A\xi = \alpha\xi$ covariantly and make use of the Codazzi equation (4), then we have

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0. \tag{27}$$

For any vector field X orthogonal to ξ such that $AX = \lambda X$, it follows from (27) that

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{28}$$

If $2\lambda - \alpha \neq 0$, then ϕX is also principal with principal curvature $\mu = \frac{(2\alpha\lambda + c)}{2(2\lambda - \alpha)}$ and we can write $A\phi X = \mu\phi X$. From (25) and (26), it is easily obtained that

$$(\lambda - \mu)(\lambda\mu + \frac{5c}{4}) = 0, \text{ and} \tag{29}$$

$$(\lambda - \mu)(\lambda\mu + \frac{c}{4}) = 0. \tag{30}$$

From the above relationship, we can consider the following two cases:

Case I. If $\lambda - \mu \neq 0$, then we see from (29) that $\lambda\mu + \frac{5c}{4} = 0$. Since we have $\lambda\mu + \frac{c}{4} \neq 0$ in (30), it is a contradiction. Therefore we obtain $\lambda = \mu$ and then M has at most 3 distinct principal curvatures, α and the two roots of the quadratic equation. Hence M is locally congruent to one of the model spaces of type A.

Case II. If $\lambda = \frac{\alpha}{2}$, then we have $c = -\alpha^2$ by using $\alpha\lambda + \frac{c}{2} = 0$. Thus, we obtain $\phi AX = \frac{\alpha}{2}\phi X = A\phi X$.

For case I and II, we obtain the conclusion of Theorem 5.2 from Theorem A. □

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Department of Mathematics Education, Sehan University, Jeollanam-do 58447, Republic of Korea.
Email address: hoonjoo@sehan.ac.kr

Department of Mathematics Education, Sehan University, Jeollanam-do 58447, Republic of Korea.
Email address: dhlim@sehan.ac.kr

Department of Mathematics, Chonbuk National University, Jeollabuk-do 54896, Republic of Korea.
Email address: syjpooh00@naver.com