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ON FINITE TIMES DEGENERATE CHANGHEE NUMBERS AND POLYNOMIALS WITH SOME APPLICATIONS

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ABSTRACT. In this paper, the authors consider finite times degenerate Changhee numbers and polynomials, and degenerate higher-order Changhee numbers and polynomials. We investigate some identities and properties of these numbers and polynomials relate with various special numbers and polynomials. Regarding applications, we show that those Changhee numbers can be expressed by the probability distributions of appropriate random variables.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of an algebraic closure of $\mathbb{Q}_p$,respectively. The p-adic norm is defined as $|p|_p = 1/p$.

It is common knowledge that the Euler polynomials $E_n(x)$ are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. In [2], L. Carlitz considered the degenerate Euler polynomials given by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!},$$

We note that

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.\end{aligned}$$

That is to say

$$\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda}(x) = E_n(x), \quad (n \geq 0).$$

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Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ was defined by Kim [8–10]

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p)$$
$$= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.$$

From the above definition, we can derive

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1}(-1)^{n-l-1} f(l). \tag{1}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$. Consequently, it follows from (1) that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \tag{2}$$

where $Ch_n(x)$ are called the Changhee polynomials and when $x = 0$, we call $Ch_n = Ch_n(0)$, $(n \geq 0)$ the Changhee numbers. It follows that

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x)(n \geq 0),$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$ for $n \geq 1$, (see [6, 16]).

We note that the Euler polynomials $E_n(x)$ may also be represented by

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

and it is easy to see that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x).$$

Thus we can obtain the following observations, which are known (see [6, 16])

$$Ch_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l),$$

and

$$E_n(x) = \sum_{l=0}^{n} Ch_n(x) S_2(n,l),$$

where $S_1(n,l)$ is the Stirling number of the first kind

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0)$$

and $S_2(n,l)$ is the Stirling number of the second kind

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0)$$

respectively [3].

Furthermore, recall from [7] that the higher-order Changhee polynomials $Ch_n^{(r)}(x)$ are given by the multiple p-adic fermionic integral on $\mathbb{Z}_p^r$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\left(\frac{2}{2+t}\right)^r(1+t)^x$$
$$=\sum_{n=0}^{\infty}Ch_n^{(r)}(x)\frac{t^n}{n!}.$$

Specially when $x=0$, $Ch_n^{(r)}=Ch_n^{(r)}(0)$ are called the Changhee numbers of order r.

Let X be an exponential random variable with parameter λ and the probability density function

$$p(x)=\begin{cases}\lambda e^{-\lambda x} & \text{if } x\geq 0\\ 0 & \text{if } x<0.\end{cases}$$

Then the probability generating function of X is given by

$$G(t)=E[e^{Xt}]=\int_{-\infty}^{\infty}e^{xt}p(x)dx=\frac{\lambda}{\lambda-t},\quad \text{(see [4,5,14,18,19])}. \tag{3}$$

Suppose that $X=(X_1,X_2,\cdots,X_k)$ is taking an exponential random variable values in the non-negative real numbers $\mathbb{R}^k$. Then the probability generating function of X is defined by

$$G(t_1,t_2,\cdots,t_k)=E[e^{t_1X_1}e^{t_2X_2}\cdots e^{t_kX_k}]$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{t_1x_1}e^{t_2x_2}\cdots e^{t_kx_k}p(x_1,x_2,\cdots,x_k)dx_1dx_2\cdots dx_k, \tag{4}$$

where $p(x_1,x_2,\cdots,x_k)$ is the probability density function of $X=(X_1,X_2,\cdots,X_k)$, (see [4,5,14,18,19]).

Recently, many researchers have studied Changhee numbers and polynomials, and their degenerated form of Changhee numbers and polynomials (see [1,6,7,11, 12,16,17]). In section 2, we consider the finite times degenerate Changhee numbers and polynomials. In section 3, we consider finite times degenerate higher order Changhee numbers and polynomials. We give some identities and properties of those numbers and polynomials relate with various special numbers and polynomials. Regarding applications, in section 4 we show that those numbers can be expressed by the probability distributions of appropriate random variables.

2. k-TIMES DEGENERATE CHANGHEE POLYNOMIALS

Assume that $\lambda,t\in\mathbb{C}_p$ such that $|\lambda t|_p<p^{-\frac{1}{p-1}}$. From (1), we note that

$$\int_{\mathbb{Z}_p}f(x+1)d\mu_{-1}(x)+\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)=2f(0). \tag{5}$$

Applying $f(x)=(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x$ in (5) reveals

$$\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x+y}d\mu_{-1}(y)=\frac{2}{2+\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x. \tag{6}$$

In the viewpoint of (2) and (6), the degenerate Changhee polynomials are defined by the generating function to be [16,17]

$$\frac{2\lambda}{2\lambda+\log(1+\lambda t)}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x=\sum_{n=0}^{\infty}Ch_{n,\lambda}^{[1]}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, we call $Ch_{n,\lambda}^{[1]} = Ch_{n,\lambda}^{[1]}(0)$ the degenerate Changhee numbers. The degenerate Changhee polynomials are degenerated 1-time for Changhee polynomials $Ch_n(x)$, thus we denote these polynomials by $Ch_{n,\lambda}^{[1]}(x)$.

It is well known that

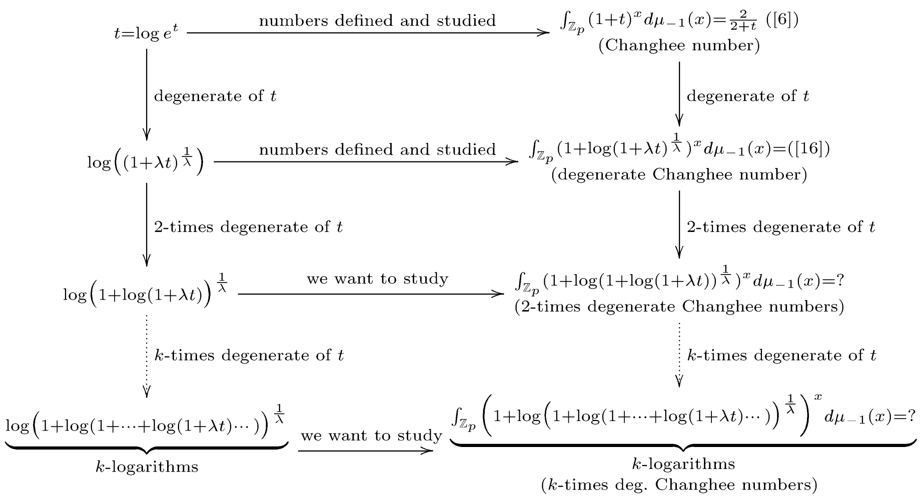
$$e^t = \lim_{\lambda \to 0}(1+\lambda t)^{\frac{1}{\lambda}}.$$

The function $(1+\lambda t)^{\frac{1}{\lambda}}$ is called the degenerate function of e^t. So, for $t = \log e^t$, we have $\log(1+\lambda t)^{\frac{1}{\lambda}}$ as the degenerating function for t. The first quadrant of the following diagram adopted from [11], which we use p-adic fermionic integral on $\mathbb{Z}_p$ instead of Riemann integral on interval $[0,1]$.

We extend the idea to k-times degenerate of $t = \log e^t$, then we obtain

$$\underbrace{\log\Big(1+\log(1+\cdots+\log(1+\lambda t)\cdots)\Big)^{\frac{1}{\lambda}}}_{k\text{-logarithms}}$$

as the k-times degenerating function.



In [16], Kwon et al. defined and investigated some properties of degenerate Changhee numbers and polynomials. We record some important results as theorem in order to relate our studies.

Theorem 2.1. *For $m, n \geq 0$, we have*

$$(1) \quad Ch_{n,\lambda}^{[1]}(x) = \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) Ch_m(x).$$

$$(2) \quad Ch_{n,\lambda}^{[1]}(x) = \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} (x+y)_m d_{\mu_{-1}}(y)$$

$$(3) \quad Ch_{n,\lambda}^{[1]}(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} \lambda^{n-m} S_1(n,m) S_1(m,l) E_l(x)$$

$$(4) \quad Ch_m(x) = \sum_{n=0}^{m} Ch_{n,\lambda}^{[1]}(x) \lambda^{m-n} S_2(m,n).$$

We degenerate $Ch_{n,\lambda}^{[1]}(x)$ one more time, then we get 2-times degenerate Changhee polynomials, which denoted by $Ch_{n,\lambda}^{[2]}(x)$. Then the left hand side of Equation (7) becomes as follows

$$\begin{aligned}\sum_{n_2=0}^{\infty} Ch_{n_2,\lambda}^{[1]}(x)\frac{1}{n_2!}\lambda^{-n_2}\big(\log(1+\lambda t)\big)^{n_2} \\ &= \sum_{n_2=0}^{\infty} Ch_{n_2,\lambda}^{[1]}(x)\lambda^{-n_2}\sum_{n_3=n_2}^{\infty} S_1(n_3,n_2)\frac{\lambda^{n_3}t^{n_3}}{n_3!} \\ &= \sum_{n_3=0}^{\infty}\Big(\sum_{n_2=0}^{n_3} Ch_{n_2,\lambda}^{[1]}(x)\lambda^{n_3-n_2}S_1(n_3,n_2)\Big)\frac{t^{n_3}}{n_3!}.\end{aligned}$$

This implies that

$$Ch_{n_3,\lambda}^{[2]}(x) = \sum_{n_2=0}^{n_3} Ch_{n_2,\lambda}^{[1]}(x)\lambda^{n_3-n_2}S_1(n_3,n_2). \tag{8}$$

Applying this (8) to Theorem 2.1 (1) results in

$$Ch_{n_3,\lambda}^{[2]}(x) = \sum_{n_2=0}^{n_3}\Big[\sum_{n_1=0}^{n_2}\lambda^{n_3-n_1}S_1(n_2,n_1)S_1(n_3,n_2)Ch_{n_1}(x)\Big]. \tag{9}$$

Inductively, we get the k-times degeneration of Changhee polynomials $Ch_{n_{k+1},\lambda}^{[k]}(x)$ as follows:

Theorem 2.2. *For $n_i \geq 0$, for each $i = 0, 1, \dots, k+1$, we have*

$$\begin{aligned}Ch_{n_{k+1},\lambda}^{[k]}(x) &= \sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_1=0}^{n_2}\lambda^{n_{k+1}-n_1}\Big(\prod_{j=1}^{k}S_1(n_{j+1},n_j)\Big)Ch_{n_1}(x) \\ &= \sum_{n_k=0}^{n_{k+1}}\cdots\sum_{n_0=0}^{n_1}\lambda^{n_{k+1}-n_1}\Big(\prod_{j=0}^{k}S_1(n_{j+1},n_j)\Big)E_{n_0}(x).\end{aligned}$$

On the other hand, we apply $f(x) = \Big(\big(1+\log(1+\log(1+\lambda t)\big)^{\frac{1}{\lambda}}\Big)^{x+y}$ in (5), we have the generating function of 2-times degeneration of Changhee polynomials

$$\begin{aligned}\int_{\mathbb{Z}_p}\Big(\big(1+\log(1+\log(1+\lambda t)\big)^{\frac{1}{\lambda}}\Big)^{x+y}d\mu_{-1}(y) \\ &= \frac{2\lambda}{2\lambda+\log(1+\log(1+\lambda t))}\Big(1+\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}\Big)^{x} \\ &= \sum_{n=0}^{\infty} Ch_{n_3,\lambda}^{[2]}(x)\frac{t^{n_3}}{n_3!}.\end{aligned} \tag{10}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in the middle term of (10), we have

$$\begin{aligned}\frac{2\lambda}{2\lambda+\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^x &= \frac{2\lambda}{2\lambda+\log(1+\lambda t)}\big(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\big)^x \\ &= \sum_{n_2=0}^{\infty} Ch_{n_2,\lambda}^{[1]}(x)\frac{t^{n_2}}{n_2!}.\end{aligned} \tag{11}$$

On the other hand, the right side of (10), we have

$$\begin{aligned}
&\sum_{n_3=0}^{\infty} Ch^{[2]}_{n_3,\lambda}(x)\lambda^{-n_3}\frac{(e^{\lambda t}-1)^{n_3}}{n_3!}\\
&=\sum_{n_3=0}^{\infty} Ch^{[2]}_{n_3,\lambda}(x)\lambda^{-n_3}\sum_{n_2=n_3}^{\infty} S_2(n_2,n_3)\frac{\lambda^{n_2}t^{n_2}}{n_2!}\\
&=\sum_{n_2=0}^{\infty}\left[\sum_{n_3=0}^{n_2} Ch^{[2]}_{n_3,\lambda}(x)\lambda^{n_2-n_3}S_2(n_2,n_3)\right]\frac{t^{n_2}}{n_2!}.
\end{aligned} \tag{12}$$

Comparing the coefficients of (11) and (12) leads to the inversion formula for (8);

$$Ch^{[1]}_{n_2,\lambda}(x)=\sum_{n_3=0}^{n_2} Ch^{[2]}_{n_3,\lambda}(x)\lambda^{n_2-n_3}S_2(n_2,n_3).$$

Inductively, we obtain the following identity

Theorem 2.3. *For each $n_i \geq 0$, where $i = 0, 1, \dots, k+1$, we have*

(1) $$Ch^{[1]}_{n_2,\lambda}(x)=\sum_{n_3=0}^{n_2}\sum_{n_4=0}^{n_3}\cdots\sum_{n_{k+1}=0}^{n_k} Ch^{[k]}_{n_{k+1},\lambda}(x)\lambda^{n_2-n_{k+1}}\prod_{i=2}^{k} S_2(n_i,n_{i+1})$$

(2) $$Ch_{n_1}(x)=\sum_{n_2=0}^{n_1}\sum_{n_3=0}^{n_2}\cdots\sum_{n_{k+1}=0}^{n_k} Ch^{[k]}_{n_{k+1},\lambda}(x)\lambda^{n_1-n_{k+1}}\prod_{i=1}^{k} S_2(n_i,n_{i+1}).$$

3. k-times degenerate higher-order Changhee polynomials

In this section, we consider finite times degeneration of higher order Changhee polynomials and investigate some interesting identities relate to various special polynomials.

Recall from [16] that for $r \in \mathbb{N}$, the generating function of the degenerate Changhee polynomials of order r, $Ch^{(r)}_{n,\lambda}(x)$ are defined by

$$\left(\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\right)^{r}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x}=\sum_{n=0}^{\infty} Ch^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{13}$$

When $x = 0$, $Ch^{(r)}_{n,\lambda} = Ch^{(r)}_{n,\lambda}(0)$ are called the degenerate Changhee numbers of order r.

From (13), we note that these degenerate Changhee polynomials of order r can be presented by the multiple p-adic fermionic integral on $\mathbb{Z}_p^r$ as follows

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x_1+\cdots+x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\left(\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\right)^{r}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x}\\
&=\sum_{n=0}^{\infty} Ch^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}.
\end{aligned}$$

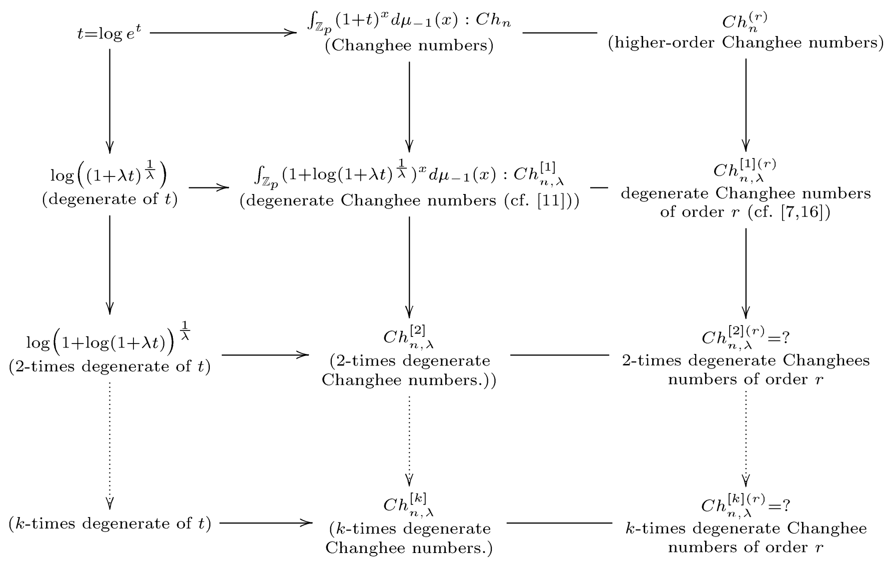
We denote by $Ch^{[1](r)}_{n,\lambda}(x) = Ch^{(r)}_{n,\lambda}(x)$, which means 1-time degeneration of Changhee polynomials of order r.

The following results on degenerate higher order Changhee polynomials are known in [7, 16]. We record here as theorem, which are extended in our result.

Theorem 3.1. *For $m \geq 0$, we have the following identities*

$$(1) \quad Ch_{m,\lambda}^{[1](r)}(x) = \sum_{n=0}^{m} \lambda^{m-n} S_1(m,n) Ch_n^{(r)}(x)$$

$$(2) \quad Ch_{m,\lambda}^{[1](r)}(x) = \sum_{n=0}^{m} \sum_{l=0}^{n} \lambda^{m-n} S_1(m,n) S_1(n,l) E_l^{(r)}(x).$$



Let $Ch_{n,\lambda}^{[k](r)}(x)$ denote k-times degenerate Chaghee polynomials of order r. When $x = 0$, let $Ch_{n,\lambda}^{[k](r)} = Ch_{n,\lambda}^{[k](r)}(0)$ denote k-times degenerate Changhee numbers of order r.

We degenerate one more time in (13), i.e., 2-times degenerate Changhee polynomials of order r, then we can get easily:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,\lambda}^{[2](r)}(x) \frac{t^n}{n!} \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(1 + \log\big(1 + \log(1+\lambda t)\big)^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_r+x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&\quad = \left(\frac{2\lambda}{2\lambda + \log\big(1+\log(1+\lambda t)\big)}\right)^r \left(1 + \log\big(1+\log(1+\lambda t)\big)^{\frac{1}{\lambda}}\right)^x.
\end{aligned} \tag{14}$$

Now we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(1 + \log(1 + \log(1+\lambda t))^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_r+x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&\qquad\qquad\qquad \times \frac{1}{n!} \left(\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}\right)^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \times \sum_{m=n}^{\infty} \lambda^{-n} S_1(m,n) \frac{\log(1+\lambda t)^m}{m!}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left(\sum_{m=0}^{j} \sum_{n=0}^{m} Ch_n^{(r)}(x) \lambda^{i-n} S_1(i,j) S_1(j,m) S_1(m,n) \right) \frac{t^i}{i!}.$$

Combining this identity with (14) arrives at the following theorem.

Theorem 3.2. *For $i \geq 0$, we have*

$$Ch_{i,\lambda}^{[2](r)}(x) = \sum_{j=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{m} Ch_n^{(r)}(x) S_1(i,j) S_1(j,m) S_1(n,m) \lambda^{j-n}.$$

Inductively we try k-times degenerate Changhee polynomials of order r, then we have the following result.

Theorem 3.3. *For $n_i \geq 0$, $i = 1, \ldots, k+1$,*

$$Ch_{n_{k+1},\lambda}^{[k](r)}(x) = \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n_1} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \times \prod_{i=1}^{k} S_1(n_{i+1}, n_i) \lambda^{n_{k+1}-n_1}$$

$$= \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} Ch_{n_1}^{(r)}(x) \prod_{i=1}^{k} S_1(n_{i+1}, n_i) \lambda^{n_{k+1}-n_1}$$

In view of Theorem 3.3, by specializing $k = 1$ or $r = 1$, respectively, we have the following known results in this paper.

Corollary 3.4. *For each $n_i \geq 0$ where $i = 0, 1, \ldots, k+1$, we have*

(*i*) $Ch_{n_2,\lambda}^{(r)}(x) = \sum_{n_1=0}^{n_2} Ch_{n_1,\lambda}^{(r)}(x) S_1(n_2, n_1) \lambda^{n_2-n_1}$ (*Theorem 3.1*)

(*ii*) $Ch_{n_{k+1},\lambda}^{[k]}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \Big(\prod_{i=0}^{k} S_1(n_{i+1}, n_i) \Big) Ch_{n_1}(x)$ (*Theorem 2.2*).

Remark 3.5. The following is well known in [7].

(*i*) $\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x), \quad (n \geq 0)$

(*ii*)
$$Ch_n^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= \sum_{l=0}^{n} S_1(n,l) E_l^{(r)}(x).$$

Therefore, from Theorem 3.3 and Remark 3.5 (*i*) and (*ii*), we have the following corollary.

Corollary 3.6. *For $n_i \geq 0$, $i = 0, 1, \ldots, k+1$, we have the following identity.*

$$Ch^{[k](r)}_{n_{k+1},\lambda} = \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_0=0}^{n_1} \lambda^{n_{k+1}-n_1} \prod_{i=0}^{k} S_1(n_{i+1}, n_i) E^{(r)}_{n_0}(x).$$

Taking $k = 1$ in this corollary yields (2) of Theorem 3.1.

Corollary 3.7. *For $n \geq 0$, we have*

$$Ch^{(r)}_{n,\lambda}(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} S_1(n,k) \lambda^{n-k} S_1(k,j) E^{(r)}_j(x).$$

Taking $r = 1$ in the Corollary 3.6, we have the result in Theorem 2.2. And replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (14) gives

$$\begin{aligned} \sum_{n=0}^{\infty} Ch^{[2](r)}_{n,\lambda}(x) \frac{1}{n!} \left(\frac{e^{\lambda t} - 1}{\lambda} \right)^n &= \left(\frac{2\lambda}{2\lambda + \log(1+\lambda t)} \right)^r \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x \\ &= \sum_{n=0}^{\infty} Ch^{[1](r)}_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{15}$$

The left hand side of (15) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} Ch^{[2](r)}_{n,\lambda}(x) \lambda^{-n} \frac{(e^{\lambda t}-1)^n}{n!} &= \sum_{n=0}^{\infty} Ch^{[2](r)}_{n,\lambda}(x) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n) \frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} Ch^{[2](r)}_{n,\lambda}(x) \lambda^{m-n} S_2(m,n) \right) \frac{t^m}{m!}. \end{aligned}$$

Combination of this identity with (15) results in

Theorem 3.8. *For $m \geq 0$, we have*

$$Ch^{[1](r)}_{m,\lambda}(x) = \sum_{n=0}^{m} Ch^{[2](r)}_{n,\lambda}(x) \lambda^{m-n} S_2(m,n).$$

We make inversion one more time of 2-times degenerate Changhee polynomials of order r, we have

$$Ch^{[2](r)}_{n,\lambda}(x) = \sum_{l=0}^{n} Ch^{[3](r)}_{l,\lambda}(x) \lambda^{n-l} S_2(n,l).$$

Thus combining the above equation with Theorem 3.1, we have

$$Ch^{[1](r)}_{m,\lambda}(x) = \sum_{n=0}^{m} \sum_{l=0}^{n} Ch^{[3](r)}_{l,\lambda}(x) \lambda^{m-l} S_2(m,n) S_2(n,l).$$

Therefore, inductively we get the following inversion formula for Theorem 3.1.

Theorem 3.9. *For $n_{k+1} \geq 0$, we have*

$$Ch^{[1](r)}_{n_{k+1},\lambda}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} Ch^{[k+1](r)}_{n_1,\lambda}(x) \lambda^{n_{k+1}-n_1} \prod_{i=1}^{k} S_2(n_{i+1}, n_i).$$

For the case $r = 1$ or $k = 1$, we have the following results, which are known in this paper.

Corollary 3.10. *For $m \geq 0$, we have*

$$(i) \quad Ch^{[1]}_{n_{k+1},\lambda}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} Ch^{[k+1]}_{n_1,\lambda}(x) \lambda^{n_{k+1}-n_1} \prod_{i=1}^{k} S_2(n_{i+1}, n_i) \quad \textit{(Theorem 2.3)}$$

(ii) $Ch_{n_2,\lambda}^{[1](r)}(x) = \sum_{n_1=0}^{n_2} Ch_{n_1,\lambda}^{[2](r)}(x)\lambda^{n_2-n_1}S_2(n_2,n_1)$ *(Theorem 3.8).*

Recall from [16] that the degenerate Changhee polynomials of the second kind of order r, which are denoted by $\widetilde{Ch}_{n,\lambda}^{(r)}(x)$, are given by

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{-x_1-\cdots-x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&=\left(\frac{2}{2+\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r\left(\log(1+\lambda t)^{\frac{1}{\lambda}}+1\right)^{r+x}\\&=\sum_{n=0}^{\infty}\widetilde{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Degenerating one more time the degenerate Changhee polynomials of the second kind leads to

$$\begin{aligned}&\sum_{n=0}^{\infty}\widetilde{Ch}_{n,\lambda}^{[2](r)}(x)\frac{t^n}{n!}\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\log\big(1+\log(1+\lambda t)\big)^{\frac{1}{\lambda}}\right)^{-x_1-\cdots-x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&=\left(\frac{2\lambda}{2\lambda+\log(1+\log(1+\lambda t))}\right)^r\left(\log\big(1+\log(1+\lambda t)\big)^{\frac{1}{\lambda}}+1\right)^{x+r}.\end{aligned}$$

Therefore, we can obtain

$$\widetilde{Ch}_{n,\lambda}^{[2](r)}(x) = Ch_{n,\lambda}^{[2](r)}(x+r).$$

4. Some applications of Changhee numbers

In this section, we try to give some applications of Changhee numbers and degenerate Chaghee numbers by the probability distribution of exponential random variables of parameter 2. The motivation of these ideas come from the personal communications with professor Taekyun Kim. The purpose of this section is to calculate the moment value combined with the probability random variables through the application of Changhee polynomial. The following two papers motivated the study of this section ([13, 15]).

Specially, from (3), let X be an expronential random variable with parameter 2, then we have

$$\begin{aligned}E[e^{Xt}]&=\frac{2}{2-t}\\&=\frac{2}{2+(-t)}=\sum_{n=0}^{\infty}(-1)^nCh_n\frac{t^n}{n!}\end{aligned}\tag{16}$$

and

$$E[e^{Xt}] = \sum_{n=0}^{\infty} E[X^n]\frac{t^n}{n!}. \tag{17}$$

Thus by comparing the coefficients of (16) and (17), we can express Changhee numbers via the momentum of an exponential random variable with parameter 2.

Theorem 4.1. *Let X be an exponential random variable with parameter 2, then we have*

$$E[X^n] = (-1)^nCh_n.$$

From (4), let $X_1, X_2, \cdots, X_k$ be independent exponential random variables with parameter 2, and let $t = t_1 = t_2 = \cdots = t_k$.

$$E[e^{Xt}] = E[e^{(X_1+\cdots+X_k)t}] = \sum_{n=0}^{\infty} E[(X_1+\cdots+X_k)^n]\frac{t^n}{n!} \tag{18}$$

On the other hand, by using (16)

$$\begin{aligned} E[e^{Xt}] &= E[e^{(X_1+\cdots+X_k)t}] = E[e^{X_1 t}]E[e^{X_2 t}]\cdots E[e^{X_k t}] \\ &= \left(\frac{2}{2-t}\right)^k \\ &= \sum_{n=0}^{\infty}(-1)^k Ch_n^{(k)}\frac{t^n}{n!}. \end{aligned} \tag{19}$$

By (18) and (19), we express higher order Changhee numbers by the momentum of sum of independent exponential random variables with parameter 2.

Theorem 4.2. *Let $X_1, X_2, \cdots, X_k$ be independent exponential random variables with parameter 2, and let $X = X_1 + X_2 + \cdots + X_k$, then the momentum of X is the Changhee numbers of order k, i.e,*

$$E[X^n] = E[(X_1 + X_2 + \cdots X_k)^n] = (-1)^n Ch_n^{(k)}.$$

Now using the property of independent random variable property, Theorem 4.2 and (18), we can express Changhee numbers of order k another well known formulas in [16].

$$\begin{aligned} E[e^{Xt}] &= E[e^{(X_1+\cdots+X_k)t}] \\ &= \sum_{n=0}^{\infty} E[(X_1+\cdots+X_k)^n]\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1\cdots m_k}E[X_1^{m_1}]\cdots E[X_k^{m_k}]\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1\cdots m_k}(-1)^{m_1}Ch_{m_1}(-1)^{m_2}Ch_{m_2}\cdots(-1)^{m_k}Ch_{m_k}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m_1+m_2+\cdots+m_k=n}(-1)^n\binom{n}{m_1\cdots m_k}Ch_{m_1}Ch_{m_2}\cdots Ch_{m_k}\right)\frac{t^n}{n!}, \end{aligned} \tag{20}$$

where $\binom{n}{m_1\cdots m_k}$ is a multinomial coefficient of $(x_1+\cdots+x_k)^m$. Comparing the coefficients of (19) and (20), we have a well known identity for higher order Changhee numbers.

Corollary 4.3. *For $n \geq 0$, we have*

$$Ch_n^{(k)} = \sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1\cdots m_k}Ch_{m_1}Ch_{m_2}\cdots Ch_{m_k}.$$

We want to relate degenerate Changhee numbers to an exponential random variable. Let X be an exponential random variable with parameter 2, we replace t

by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (16). Then we have

$$\begin{aligned} E[e^{X\log(1+\lambda t)^{\frac{1}{\lambda}}}] &= E[\log(1+\lambda t)^{\frac{X}{\lambda}}] \\ &= \frac{2}{2-\log(1+\lambda t)^{\frac{1}{\lambda}}} \\ &= \frac{2}{2+\log(1+(-\lambda)(-t))^{-\frac{1}{\lambda}}} \\ &= \sum_{n=0}^{\infty} Ch_{n,-\lambda}\frac{(-t)^n}{n!}. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} E[\log(1+\lambda t)^{\frac{X}{\lambda}}] &= \sum_{n=0}^{\infty} E[(X)_{n,\lambda}]\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} E[X^m]\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{22}$$

where $(X)_{n,\lambda}$ is the falling factorial sequence $(X)_{0,\lambda}=1$ and $(X)_{n,\lambda}=X(X-\lambda)\cdots(X-(n-1)\lambda)$ for $n\geq 1$. Thus comparing (21) and (22), we have the following theorem.

Theorem 4.4. *Let X be an exponential random variable with parameter* 2. *Then we have*

$$\begin{aligned} E[(X)_{n,\lambda}] &= \sum_{m=0}^{n} E[X^m]\lambda^{n-m}S_1(n,m) \\ &= (-1)^n Ch_{n,-\lambda}. \end{aligned}$$

Applying Theorem 4.1 to the above Theorem 4.4, we have the following identity.

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n Ch_{n,-\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} E[x^m]\lambda^{n-m}S_1(n,m) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m Ch_m\lambda^{n-m}S_1(n,m). \end{aligned}$$

Thus we have

$$(-1)^n Ch_{n,-\lambda} = \sum_{m=0}^{n}(-1)^m Ch_m\lambda^{n-m}S_1(n,m).$$

When we replace $-\lambda$ to λ, we have the identity, which is in the Theorem 2.1.

We want to express degenerate Changhee numbers of order k by the momentum of independent exponential random variables $X_1, X_2, \cdots, X_k$ with parameter 2. For the convenience, we denote

$$T = \log(1+\lambda t)^{\frac{1}{\lambda}}.$$

Let $X = X_1 + X_2 + \cdots X_k$, Then

$$\begin{aligned} E[e^{XT}] &= E[e^{(X_1+X_2+\cdots X_k)T}] \\ &= E[e^{X_1T}]E[e^{X_2T}]\cdots E[e^{X_kT}] \\ &= \left(\frac{2}{2-T}\right)^k \\ &= \left(\frac{2}{2-\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^k \\ &= \left(\frac{2}{2+\log(1+(-\lambda)(-t))^{-\frac{1}{\lambda}}}\right)^k \\ &= \sum_{n=0}^{\infty}(-1)^n Ch_{n,-\lambda}^{(k)}\frac{t^n}{n!}. \end{aligned} \tag{23}$$

We note that

$$\begin{aligned} E[e^{XT}] &= E[\log(1+\lambda t)^{\frac{x}{\lambda}}] \\ &= \sum_{n=0}^{\infty} E[(X)_{n,\lambda}]\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} E[(X_1+X_2+\cdots+X_k)_{n,\lambda}]\frac{t^n}{n!}. \end{aligned} \tag{24}$$

By (23) and (24), we have the following theorem.

Theorem 4.5. *Let $X_1, X_2, \cdots, X_k$ be independent exponential random variables with parameter 2, and let $X = X_1 + X_2 + \cdots + X_k$, then the degenerate Changhee numbers of order k is given by*

$$E[(X_1+X_2+\cdots+X_k)_{n,\lambda}] = (-1)^n Ch_{n,-\lambda}^{(k)}.$$

On the other hand, we have the following identity

$$\begin{aligned} E[e^{XT}] &= E[e^{(X_1+X_2+\cdots X_k)T}] \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1m_2\cdots m_k}E[X_1^{m_1}X_2^{m_2}\cdots X_k^{m_k}]\frac{T^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1m_2\cdots m_k}E[X_1^{m_1}]E[X_2^{m_2}]\cdots E[X_k^{m_k}]\frac{T^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2+\cdots+m_k=n}\frac{n!}{m_1!m_2!\cdots m_k!}E[X_1^{m_1}]E[X_2^{m_2}]\cdots E[X_k^{m_k}]\frac{T^n}{n!} \\ &= \sum_{m_k=0}^{\infty}\sum_{m_{k-1}=0}^{\infty}\cdots\sum_{m_1=0}^{\infty}\frac{E[X_1^{m_1}]}{m_1!}T^{m_1}\frac{E[X_2^{m_2}]}{m_2!}T^{m_2}\cdots\frac{E[X_k^{m_k}]}{m_k!}T^{m_k} \\ &= \left(\sum_{m_1=0}^{\infty}\frac{E[X_1^{m_1}]}{m_1!}T^{m_1}\right)\left(\sum_{m_2=0}^{\infty}\frac{E[X_2^{m_2}]}{m_2!}T^{m_2}\right)\cdots\left(\sum_{m_k=0}^{\infty}\frac{E[X_k^{m_k}]}{m_k!}T^{m_k}\right) \\ &= \left(\sum_{m_1=0}^{\infty}(-1)^{m_1}Ch_{m_1,-\lambda}\right)\left(\sum_{m_2=0}^{\infty}(-1)^{m_2}Ch_{m_2,-\lambda}\right)\cdots\left(\sum_{m_k=0}^{\infty}(-1)^{m_k}Ch_{m_k,-\lambda}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m_1+m_2+\cdots+m_k=n}\binom{n}{m_1m_2\cdots m_k}(-1)^n Ch_{m_1,-\lambda}Ch_{m_2,-\lambda}\cdots Ch_{m_k,-\lambda}. \end{aligned} \tag{25}$$

Comparing the coefficients (24) and (25), we have the well known identity of higher order degenerate Changhee numbers in [16]

$$Ch^{(k)}_{n,-\lambda} = \sum_{m_1+m_2+\cdots+m_k=n} \binom{n}{m_1 m_2 \cdots m_k} (-1)^n Ch_{m_1,-\lambda} Ch_{m_2,-\lambda} \cdots Ch_{m_k,-\lambda}.$$

Finally we want to express 2-times degenerate Changhee numbers by using exponential random variable with parameter 2. The equations (8) and (9) useful in here.

$$Ch^{[2]}_{n,\lambda} = \sum_{m=0}^{n} Ch^{[1]}_{m,\lambda} \lambda^{n-m} S_1(n,m)$$

$$Ch^{[2]}_{n,\lambda} = \sum_{m=0}^{n} \left[\sum_{l=0}^{m} \lambda^{n-l} S_1(n,m) S_1(m,l) Ch_l \right].$$

And in Theorem 4.1 and Theorem 4.4, we know the following identities

$$E[X^n] = (-1)^n Ch_n$$

$$E[(X)_{n,\lambda}] = (-1)^n Ch_{n,-\lambda}.$$

Thus by using the above identities, we express 2-times degenerate Changhee numbers by using random variable.

Theorem 4.6. *For $n \geq 0$, let X be the exponential random variable with parameter 2, then we have*

$$(-1)^n Ch^{[2]}_{n,-\lambda} = \sum_{m=0}^{n} E[(X)_{m,\lambda}] \lambda^{n-m} S_1(n,m)$$

$$= \sum_{m=0}^{n} \sum_{l=0}^{m} E[X^l] \lambda^{n-l} S_1(n,m) S_1(m,l).$$

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References

[1] S. Araci, E. Ayz, M. Acikgoz, *On a q-analog of some numbers and polynomials*, J. Inequal Appl. **2015**:19 (2015).

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Doredecht, 1974

[4] Feller, W. An Introduction to Probability Theory and Its Application; John Wiley: Hoboken, NJ, USA, 1970

[5] D. V. Dolgy, D. S. Kim, J. Kwon, T. Kim, *Some identities of ordinary and degenerate Bernoulli numbers and polynomials*, Symmetry **11** (2019), 847.

[6] D. S. Kim, T. Kim, J.-J. Seo, *Note on Changhee polynomials and numbers*, Adv. Studies-Theor. Phys. **7** (2013), no. 20, 993-1003.

[7] D. S. Kim, T. Kim, J.-J. Seo, S.H. Lee, *Higher-order Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **8** (2014), no. 8, 365-373.

[8] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), no. 3, 288299.

[9] T. Kim, *A note on q-Volkenborn integration*, Proc. Jangjeon Math. Soc. **8** (2005), no. 1, 13–17.

[10] T. Kim, *A note on p-adic q-integral on $\mathbb{Z}_p$ associated with q-Euler numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **15** (2007), no. 2, 133-137.

[11] T. Kim, *On the degenerate Cauchy numbers and polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 307–312.

[12] T. Kim, D. S. Kim, *Some identities on derangement and degenerate derangement polynomials*, Russian J. Math. Phys. **25** (2018), 300–308.

[13] T. Kim, D. S. Kim , L.-C. Jang, H. Y. Kim, *A note on discrete degenerate random variables*, Proc. Jangjeon Math. Soc. **23** (2020), no. 1, 125–135.

[14] T. Kim, D. S. Kim, H. Y. Kim, J. Kwon, *Degenerate Stirling Polynomials of the Second Kind and Some Applications*, Symmetry **11** (2019), no. 8, 1046.

[15] T. Kim, Y. Yao, D. S. Kim, H.I. Kwon, *Some identities involving special numbers and moments of random variables*, Rocky Mountain J. Math. **49** (2019), no. 2, 521–538.

[16] H.-I. Kwon, T. Kim, J.-J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 295–305.

[17] J. Kwon, J.-W. Park, *On modified degenerate Changhee polynomials and numbers*, J. Nonlinear Sci. Appl. **9** (2016), no. 12, 6294–6301.

[18] M. Koutras, *Non-central stirling numbers and some applications*, Discret. Math. **42** (1982), 73-89.

[19] S. M. Ross, it Introduction to Probability Models, Academic Press, Cambridge, MA, USA, 2007

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