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A NOTE ON λ-BERNOULLI NUMBERS OF THE SECOND KIND

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ABSTRACT. In this paper, we study the λ-Bernoulli numbers of the second kind which are defined as an integral of the λ-analogue of the falling factorial sequence and express them in terms of the λ-Stirling numbers of the first kind. Then we investigate the generalized λ-Bernoulli numbers of the second kind given as a multiple integral on the unit cube and show, among other things, the generating function of those numbers can be expressed in term of the recently introduced poly-exponential function by Kim-Kim. Finally, we introduce the higher-order λ-Bernoulli numbers of the second kind, again given by another multiple integral on the unit cube, and show those numbers can be given by the degenerate Stirling numbers of the second.

1. INTRODUCTION

It was Carlitz who initiated the study of degenerate versions of Bernoulli and Euler numbers and obtained some interesting arithmetic and combinatorial properties on them. In recent years, the present authors, their colleagues and some other people have explored various degenerate versions of many special polynomials and numbers with their interests not only in arithmetic and combinatorial properties but also in certain symmetric identities, differential equations, umbral calculus and probability. They have been investigated by employing several different methods, such as generating functions, combinatorial methods, umbral calculus techniques, probability theory, p-adic analysis, differential equations, and so on. This line of studying degenerate versions is not just limited to polynomials but also can be extended to transcendental functions like gamma functions. Indeed, the authors studied the degenerate gamma functions and its related degenerate Laplace transforms in [12]. Overall, we may say that studying various degenerate versions of some polynomials are very interesting, fruitful and promising.

In this section, we will recall some necessary facts that will be used in the next section. Then, in the next section we will introduce the λ-Bernoulli numbers of the second kind, the generalized λ-Bernoulli numbers of the second kind and the higher-order λ-Bernoulli numbers of the second kind, all by means of certain integrals. We will obtain some basic results on those special numbers.

The Bernoulli numbers of the second kind are defined in Roman's book [25] as

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}. \tag{1}$$

From (1), we note that

$$\int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}. \tag{2}$$

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By (2), we get

$$\int_0^1 \binom{x}{n} dx = \frac{b_n}{n!}, \quad (n \geq 0). \tag{3}$$

As is well known, the Bernoulli polynomials of order r are defined by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [1-27]). \tag{4}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.
It is well known that

$$\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\frac{t^k}{k!}, \quad (\text{see } [25]). \tag{5}$$

From (1) and (5), we note that $b_n = B_n^{(n)}(1)$, $(n \geq 1)$.
For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad (\text{see } [10,11,12]). \tag{6}$$

By using Taylor expansion, we easily get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad (\text{see } [13,14,15,16]), \tag{7}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.
Here, $(x)_{n,\lambda}$ are called the λ-analogue of the falling factorial sequence.

For $r \in \mathbb{N}$, the degenerate Bernoulli polynomials of order r are defined by Carlitz as

$$\left(\frac{t}{e_\lambda(t) - 1}\right)^r e_\lambda^x(t) = \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the degenerate Bernoulli numbers of order r (see [3,4]). In particular, $r = 1$, $\beta_{n,\lambda} = \beta_{n,\lambda}^{(1)}$ are called the Carlitz's degenerate Bernoulli numbers.

Recently, Kim-Kim introduced the poly-exponential functions which are given by

$$Ei_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \quad (k \in \mathbb{Z}), \quad (\text{see } [10]). \tag{9}$$

Note that $Ei_1(x) = e^x - 1$, $\lim_{\lambda \to 0} e_\lambda(x) = e^x$. From (8), we note that $\lim_{\lambda \to 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x)$, $(n \geq 0)$.
The Stirling numbers of the first kind are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0), \quad (\text{see } [10-27]). \tag{10}$$

As the inversion formula of (10), the Stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see } [5-8]). \tag{11}$$

Recently, Kim introduced the degenerate Stirling numbers of the second kind defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_n, \quad (n \geq 0), \quad (\text{see } [13]). \tag{12}$$

On the other hand, the λ-Stirling numbers of the first kind are given by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k, \quad (n \geq 0), \text{ (see [20])}. \tag{13}$$

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, $\lim_{\lambda\to 1} S_{1,\lambda}(n,k) = S_1(n,k)$.
The λ-analogue of binomial coefficients are defined by

$$\binom{x}{n}_{\lambda} = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda)}{n!}, \ (n \geq 1), \quad \binom{x}{0}_{\lambda} = 1 \tag{14}$$

and

$$\binom{x}{n}_{\lambda} = 0 \quad \text{if} \quad n = -1, -2, -3, \ldots, \quad \text{(see [20])}.$$

2. λ-Bernoulli numbers of second kind

As a λ-analogue of (3), we may consider the λ-Bernoulli numbers of the second kind given by the integral

$$\frac{b_{n,\lambda}}{n!} = \int_0^1 \binom{x}{n}_{\lambda} dx = \frac{1}{n!}\int_0^1 (x)_{n,\lambda}dx, \quad (n \geq 0). \tag{15}$$

From (15), we note that

$$\sum_{n=0}^{\infty} \frac{b_{n,\lambda}}{n!}t^n = \sum_{n=0}^{\infty} \int_0^1 \binom{x}{n}_{\lambda} dxt^n = \int_0^1 e_{\lambda}^x(t)dx = \frac{1}{\log(e_{\lambda(t)})}\big(e_{\lambda}(t)-1\big). \tag{16}$$

Thus, by (16), we get the generating function for the λ-Bernoulli numbers of the second kind to be

$$\frac{1}{\log(e_{\lambda(t)})}(e_{\lambda}(t)-1) = \sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}. \tag{17}$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!} &= \frac{1}{\log(e_{\lambda}(t))}\big(e_{\lambda}(t)-1\big) = \frac{1}{\log(e_{\lambda}(t))}\big(e^{\log(e_{\lambda}(t))}-1\big) \\ &= \sum_{m=1}^{\infty} \frac{\big(\log e_{\lambda}(t)\big)^{m-1}}{m!} = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\Big(\log(e_{\lambda}(t))\Big)^m \\ &= \sum_{m=0}^{\infty}\frac{1}{m+1}\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{S_{1,\lambda}(n,m)}{m+1}\right)\frac{t^n}{n!}. \end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned} \frac{1}{\log(e_{\lambda}(t))}\big(e_{\lambda}(t)-1\big) &= \frac{1}{t}\left(\frac{\lambda t}{\log(1+\lambda t)}(1+\lambda t)^{\frac{1}{\lambda}}-1\right) \\ &= \frac{1}{t}\sum_{n=0}^{\infty}\left(B_n^{(n)}(\frac{1}{\lambda}+1)-B_n^{(n)}(0)\right)\lambda^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\left(B_{n+1}^{(n+1)}(\frac{1}{\lambda}+1)-B_{n+1}^{(n+1)}(0)\right)}{n+1}\lambda^{n+1}\frac{t^n}{n!}. \end{aligned} \tag{19}$$

Therefore, by comparing the coefficients on the both sides of the (18), (19), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$b_{n,\lambda} = \sum_{m=0}^{n} \frac{S_{1,\lambda}(n,m)}{m+1} = \frac{B_{n+1}^{(n+1)}(\frac{1}{\lambda}+1) - B_{n+1}^{(n+1)}(0)}{n+1}\lambda^{n+1}.$$

From (5), we note that

$$\left(\frac{t}{\log e_\lambda(t)}\right)^n e_\lambda^{\lambda(x-1)}(t) = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\lambda^k \frac{t^k}{k!}, \quad (n \geq 0). \tag{20}$$

Thus, by (20), we get

$$\frac{t}{\log e_\lambda(t)} = \sum_{k=0}^{\infty} B_k^{(k)}(1)\lambda^k \frac{t^k}{k!}. \tag{21}$$

On the other hand, by (17), we get

$$\begin{aligned} \frac{t}{\log e_\lambda(t)} &= \frac{e_\lambda(t)-1}{\log e_\lambda(t)} \frac{t}{e_\lambda(t)-1} = \sum_{l=0}^{\infty} b_{l,\lambda}\frac{t^l}{l!} \sum_{m=0}^{\infty} \beta_{m,\lambda}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}b_{n-m,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\lambda^n B_n^{(n)}(1) = \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}b_{n-m,\lambda}.$$

Letting $\lambda \to 0$ on both sides of Theorem 2, we have

$$\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{n-m+1} = \begin{cases} 1, & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

Let n, k be integers with $n \geq 0$ and $k \geq 1$.

Then we consider the generalized λ-Bernoulli numbers of the second kind which are defined by the following multiple integrals on the unit cube

$$\int_0^1 \cdots \int_0^1 \binom{x_1x_2\cdots x_k}{n}_\lambda dx_1\cdots dx_k = \frac{b_{n,\lambda}^{(k)}}{n!}. \tag{23}$$

From (23), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} b_{n,\lambda}^{(k)}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\int_0^1\cdots\int_0^1 \binom{x_1\cdots x_k}{n}_\lambda dx_1\cdots dx_k t^n \\ &= \int_0^1\cdots\int_0^1 e_\lambda^{x_1x_2\cdots x_k}(t)dx_1dx_2\cdots dx_k. \end{aligned} \tag{24}$$

Thus, by (24), we get

$$\frac{\partial}{\partial x_1}e_\lambda^{x_1x_2\cdots x_k} = \log e_\lambda^{x_2\cdots x_k}\cdot e_\lambda^{x_1x_2\cdots x_k}(t) \tag{25}$$

Thus, by (25), we have

$$e_\lambda^{x_1x_2\cdots x_k}(t) = \frac{\partial}{\partial x_1}\left[\frac{e_\lambda^{x_1x_2\cdots x_k}(t)}{\log e_\lambda^{x_2\cdots x_k}(t)}\right]. \tag{26}$$

From (26), we can derive the following equations

$$(27)\quad \begin{aligned}\int_0^1 e_\lambda^{x_1x_2\cdots x_k}(t)dx_1 &= \frac{1}{\log e_\lambda^{x_2\cdots x_k}(t)}\left[e_\lambda^{x_2\cdots x_k}(t)-1\right]\\ &= \frac{1}{x_2\cdots x_k\log e_\lambda(t)}\left(e^{x_2\cdots x_k\log(e_\lambda(t))}-1\right)\\ &= \frac{1}{x_2\cdots x_k\log e_\lambda(t)}\sum_{m=1}^{\infty}\frac{\left(x_2x_3\cdots x_k\log e_\lambda(t)\right)^m}{(m-1)!m}\\ &= \frac{1}{x_2\cdots x_k\log e_\lambda(t)}Ei_1\left(x_2x_3\cdots x_k\log e_\lambda(t)\right).\end{aligned}$$

Thus, by (24) and (27), we get

$$(28)\quad \begin{aligned}\sum_{n=0}^{\infty}b_{n,\lambda}^{(k)}\frac{x^n}{n!} &= \int_0^1\cdots\int_0^1 e_\lambda^{x_1x_2\cdots x_k}(t)dx_1dx_2\cdots dx_k\\ &= \sum_{m=1}^{\infty}\frac{\left(\log e_\lambda(t)\right)^{m-1}}{(m-1)!m}\int_0^1\cdots\int_0^1 x_2^{m-1}x_3^{m-1}\cdots x_k^{m-1}dx_1\cdots dx_k\\ &=\sum_{m=1}^{\infty}\frac{\left(\log e_\lambda(t)\right)^{m-1}}{(m-1)!m^k}=\frac{1}{\log e_\lambda(t)}Ei_k\left(\log e_\lambda(t)\right).\end{aligned}$$

Therefore, we obtain the following generating function of the generalized λ-Bernoulli numbers of the second kind to be

$$(29)\quad \frac{1}{\log e_\lambda(t)}Ei_k\left(\log e_\lambda(t)\right)=\sum_{n=0}^{\infty}b_{n,\lambda}^{(k)}\frac{t^n}{n!}.$$

From (13), we note that

$$(30)\quad \frac{1}{k!}\left(\log e_\lambda(t)\right)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0).$$

By (29) and (30), we get

$$(31)\quad \begin{aligned}\sum_{n=0}^{\infty}b_{n,\lambda}^{(k)}\frac{t^n}{n!} &= \frac{1}{\log e_\lambda(t)}\sum_{m=1}^{\infty}\frac{\left(\log e_\lambda(t)\right)^m}{(m-1)!m^k} = \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\frac{1}{m!}\left(\log(e_\lambda(t))\right)^m\\ &= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{(m+1)^k}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned}$$

Now, we observe that

$$(32)\quad \begin{aligned}\frac{d}{dx}Ei_k\left(\log e_\lambda(x)\right) &= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{\left(\log(e_\lambda(x))\right)^n}{(n-1)!n^k}\\ &= \frac{1}{(1+\lambda x)\log(e_\lambda(x))}\sum_{m=1}^{\infty}\frac{\left(\log(e_\lambda(x))\right)^n}{(n-1)!n^{k-1}}\\ &= \frac{1}{(1+\lambda x)\log(e_\lambda(x))}Ei_{k-1}\left(\log e_\lambda(x)\right)\end{aligned}$$

From (32), we have

$$(33)\quad Ei_k\left(\log e_\lambda(x)\right)=\int_0^x\frac{1}{(1+\lambda t)\log(e_\lambda(t))}Ei_{k-1}\left(\log e_\lambda(t)\right)dt.$$

Thus, by (29) and (33), we get

$$\sum_{n=0}^{\infty} b_{n,\lambda}^{(2)} \frac{x^n}{n!} = \frac{1}{\log e_\lambda(x)} Ei_2\big(\log(e_\lambda(x))\big) \tag{34}$$

$$= \frac{1}{\log\big(e_\lambda(x)\big)} \int_0^t \frac{1}{(1+\lambda t)\log(e_\lambda(t)} \big(e_\lambda(t)-1\big)dt.$$

From (34), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{n,\lambda}^{(2)} \frac{x^n}{n!} &= \frac{1}{\log e_\lambda(x)} \int_0^x \frac{1}{(1+\lambda t)\log(e_\lambda(t))}\big(e_\lambda(t)-1\big)dt \\
&= \frac{1}{\log(1+\lambda x)} \int_0^x \frac{1}{t} \frac{\lambda t}{\log(1+\lambda t)}\big((1+\lambda t)^{\frac{1}{\lambda}-1} - (1+\lambda t)^{-1}\big)dt \\
&= \frac{1}{\log(1+\lambda x)} \int_0^x \frac{1}{t} \sum_{n=1}^{\infty} \big(B_n^{(n)}(1/\lambda) - B_n^{(n)}(0)\big)\lambda^n \frac{t^n}{n!}dt \\
&= \frac{1}{\log(1+\lambda x)} \sum_{n=0}^{\infty} \frac{B_{n+1}^{(n+1)}(1/\lambda) - B_{n+1}^{(n+1)}(0)}{(n+1)n!} \lambda^{n+1} \int_0^x t^n dt \\
&= \sum_{n=0}^{\infty} \frac{B_{n+1}^{(n+1)}(1/\lambda) - B_{n+1}^{(n+1)}(0)}{(n+1)^2 n!} \lambda^n x^n \frac{\lambda x}{\log(1+\lambda x)} \\
&= \sum_{n=0}^{\infty} \frac{B_{n+1}^{(n+1)}(1/\lambda) - B_{n+1}^{(n+1)}(0)}{(n+1)^2} \lambda^n \frac{x^n}{n!} \sum_{l=0}^{\infty} \lambda^l B_l^{(l)}(1) \frac{x^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \frac{1}{(m+1)^2} \big(B_{m+1}^{(m+1)}(1/\lambda) - B_{m+1}^{(m+1)}(0)\big)\lambda^n B_{n-m}^{(n-m)}(1) \right) \frac{x^n}{n!}.
\end{aligned} \tag{35}$$

Therefore, by comparing the coefficients on the both sides of (35), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$\frac{b_{n,\lambda}^{(2)}}{\lambda^n} = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{(m+1)^2} \big(B_{m+1}^{(m+1)}(1/\lambda) - B_{m+1}^{(m+1)}(0)\big) B_{n-m}^{(n-m)}(1).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda x}-1)$ in (29), we get

$$\sum_{n=0}^{\infty} b_{n,\lambda}^{(k)} \frac{\lambda^{-n}}{n!} (e^{\lambda t}-1)^n = \frac{1}{t} Ei_k(t). \tag{36}$$

From (36), we have

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{t^m}{(m+1)^k m!} &= \frac{1}{t} Ei_k(t) = \sum_{n=0}^{\infty} b_{n,\lambda}^{(k)} \frac{\lambda^{-n}}{n!} \big(e^{\lambda t}-1\big)^n \\
&= \sum_{n=0}^{\infty} b_{n,\lambda}^{(k)} \lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n) \lambda^m \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} b_{n,\lambda}^{(k)} \lambda^{m-n} S_2(m,n) \right) \frac{t^m}{m!}.
\end{aligned} \tag{37}$$

Thus, by comparing the coefficients on the both sides of (37), we get the following equation.

For $m \geq 0$, $k \in \mathbb{N}$, we have

$$\frac{1}{(m+1)^k} = \sum_{n=0}^{m} b_{n,\lambda}^{(k)} \lambda^{n-m} S_2(m,n).$$

3. Further Remark

As an additive version of (24), we consider the λ-Bernoulli numbers of the second of order r given by the following multiple integrals on the unit cube

$$\sum_{n=0}^{\infty} \hat{b}_{n,\lambda}^{(r)} \frac{t^n}{n!} = \underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{r-\text{times}} e_\lambda^{x_1+x_2+\cdots+x_r}(t)\, dx_1 dx_2 \cdots dx_r. \tag{38}$$

Thus, by (38), we get

$$\left(\frac{1}{\log e_\lambda(t)}\right)^r \left(e_\lambda(t)-1\right)^r = \sum_{n=0}^{\infty} \hat{b}_{n,\lambda}^{(r)} \frac{t^n}{n!}. \tag{39}$$

From (39), we note that

$$\left(\frac{1}{\log e_\lambda(t)}\right)^r \left(e_\lambda(t)-1\right)^r = \frac{r!}{t^r}\left(\frac{t}{\log e_\lambda(t)}\right)^r \frac{1}{r!}\left(e_\lambda(t)-1\right)^r. \tag{40}$$

By (12), we easily get

$$\frac{1}{r!}\left(e_\lambda(t)-1\right)^r = \sum_{m=r}^{\infty} S_{2,\lambda}(m,r)\frac{t^m}{m!}, \quad (r \geq 0). \tag{41}$$

From (40) and (41), we can derive the following equation.

$$\begin{aligned} \left(\frac{1}{\log e_\lambda(t)}\right)^r \left(e_\lambda(t)-1\right)^r &= \frac{r!}{t^r}\sum_{l=0}^{\infty} \lambda^l B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=r}^{\infty} S_{2,\lambda}(m,r)\frac{t^m}{m!} \\ &= \frac{r!}{t^r}\sum_{l=0}^{\infty} \lambda^l B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty} S_{2,\lambda}(m+r,r)\frac{t^{m+r}}{(m+r)!} \\ &= \sum_{l=0}^{\infty} \lambda^l B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty} S_{2,\lambda}(m+r,r)\frac{m!r!}{(m+r)!}\frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty} \lambda^l B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty} S_{2,\lambda}(m+r,r)\frac{1}{\binom{m+r}{r}}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \lambda^{n-m}\frac{\binom{n}{m}}{\binom{m+r}{r}} B_{n-m}^{(n-m-r+1)}(1) S_{2,\lambda}(m+r,r)\right)\frac{t^n}{n!} \end{aligned} \tag{42}$$

Thus, by (39) and (42), we get

$$\sum_{n=0}^{\infty} \hat{b}_{n,\lambda}^{(r)} \frac{t^n}{n!} = \frac{1}{\left(\log e_\lambda(t)\right)^r}\left(e_\lambda(t)-1\right)^r = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \lambda^{n-m}\frac{\binom{n}{m}}{\binom{m+r}{m}} B_{n-m}^{(n-m-r+1)}(1) S_{2,\lambda}(m+r,r)\right)\frac{t^n}{n!}. \tag{43}$$

Therefore, by comparing the coefficients on both sides of (43), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$\hat{b}_{n,\lambda}^{(r)} = \sum_{m=0}^{n} \lambda^{n-m}\frac{\binom{n}{m}}{\binom{m+r}{m}} B_{n-m}^{(n-m-r+1)}(1) S_{2,\lambda}(m+r,r).$$

Note that

$$(44)\qquad \sum_{n=0}^{\infty}\lambda^n B_n^{(n-r+1)}(1)\frac{t^n}{n!} = \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r = \left(\frac{1}{\log e_\lambda(t)}\right)^r (e_\lambda(t)-1)^r\left(\frac{t}{e_\lambda(t)-1}\right)^r$$
$$= \sum_{l=0}^{\infty}\hat{b}_{l,\lambda}^{(r)}\frac{t^l}{l!}\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(r)}\frac{t^m}{m!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\hat{b}_{l,\lambda}^{(r)}\beta_{n-l,\lambda}^{(r)}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (44), we obtain the following theorem.

Theorem 5. *For integers n,r, with $n \geq 0$, $r \geq 1$, we have*

$$(45)\qquad \lambda^n B_n^{(n-r+1)}(1) = \sum_{l=0}^{n}\binom{n}{l}\hat{b}_{l,\lambda}^{(r)}\beta_{n-l,\lambda}^{(r)}.$$

4. CONCLUSION

As an λ-analogue of the Bernoulli numbers of the second kind and motivated by an integral representation of the generating function of those numbers, we introduced the λ-Bernoulli numbers of the second kind. Among other things, we expressed the λ-Bernoulli numbers of the second kind in terms of the λ-Stirling numbers of the first kind. Then, as a generalization of the λ-Bernoulli numbers of the second kind, we introduced the generalized λ-Bernoulli numbers of the second kind as a multiple integral on the unit cube. We observed that the generating function of those numbers can be expressed in terms of poly-exponential function and also of an iterated integral. Again, we showed that the generalized λ-Bernoulli numbers of the second kind can be expressed in terms of the λ-Stirling numbers of the first kind. Finally, we introduced the high-order λ-Bernoulli numbers of the second kind, as an additive version of the generalized λ-Bernoulli numbers of the second kind and by anther multiple integral on the unit cube. Then we obtained an expression of those numbers which involves the degenerate Stirling numbers of the second.

We would like to continue to study various degenrate versions of many special numbers and polynomais as one of research projects and to find possible applications in other disciplines like physics, science and engineering.

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