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Some Properties of q-Distance Energy

M. R. Rajesh Kanna[1], R. Pradeep Kumar[2,3], Soner Nandappa D[3], Ismail Naci Cangul[4]

[1]Department of Mathematics,
Sri D Devaraj Urs Government First Grade College,
Hunsur - 571105, India.

[2]Department of Mathematics,
GSSS Institute of Engineering and Technology for Women,
KRS Road, Metagalli, Mysuru-570016, Karnataka, India.

[3]Department of Studies in Mathematics, University of Mysore,
Mysuru - 570 006, India.

[4]Corresponding Author, Department of Mathematics,
Bursa Uludag University,
16059 Bursa, Turkey.

e-mails: mr.rajeshkanna@gmail.com, pradeepr.mysore@gmail.com,
ndsoner@yahoo.co.in and cangul@uludag.edu.tr

Abstract

Graphs are nowadays frequently used in modelling. Especially the sub area of graph theory called the molecular graph theory deals with physicochemical properties of chemical substances by means of mathematical methods. Three main methods to study graphs mathematically are to make use of the vertex degrees, distances and matrices. The classical graph energy was defined 1978 by I. Gutman and has a large number of applications in chemistry and physics. In this paper, a recently defined new type of energy called q-distance energy by means of distances and matrices is studied and q-distance energies of star graph, complete graph, crown graph, cocktail party graph, complete bipartite graph, windmill graph are computed.

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Keywords and Phrases: q-distance matrix, q-distance eigenvalues, q-distance energy, join of two graphs.

1 Introduction

The concept of energy of a graph was introduced by I. Gutman in 1978, [6]. Let G be a graph with n vertices and m edges and let $\mathbb{A} = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbb{A}$ in non-increasing order, are called the eigenvalues of the graph G. As $\mathbb{A}$ is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $\mathbb{E}(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e.

$$\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For details on the mathematical aspects of the theory of graph energy, see the review [8], papers [3, 4, 7] and the references cited therein. The basic properties including various upper and lower bounds for the energy of a graph have been established in [12, 13], and the notion of graph energy has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 9].

The distance matrix of G is the square matrix of order n whose (i, j)-th entry is the distance between the vertices v_i and v_j which is defined as the length of the shortest path between these two vertices. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the distance matrix of G. The distance energy $D\mathbb{E}$ is defined by

$$D\mathbb{E} = D\mathbb{E}(G) := \sum_{i=1}^{n} |\mu_i|.$$

Detailed information on distance energy can be found in [2, 10, 11, 16]. The distance energy of the join of two given graphs can be found in [15].

Recently R. B. Bapat et al., [1], defined a new distance matrix, called as the q-distance matrix denoted by

$$\mathbb{A}_q(G) = (q_{ij}).$$

For an indeterminate q, the entries q_{ij} of this new matrix are defined by

$$q_{ij} = \begin{cases} 1 + q + q^2 + \cdots + q^{k-1}, & \text{if } k = d_{ij}, \\ 0, & \text{if } i = j, \end{cases}$$

where $k = d_{ij}$ is the distance between the vertices v_i and v_j. Each entry of $\mathbb{A}_q(G)$ is a polynomial in q. Observe that $\mathbb{A}_q(G)$ is an entry-wise non-negative matrix for all $q \geq -1$.

The characteristic polynomial of $\mathbb{A}_q(G)$ is defined by

$$f_n(G, \mu) = det(\mu I - \mathbb{A}_q(G)).$$

The q-distance eigenvalues of the graph G are similarly the eigenvalues of $\mathbb{A}_q(G)$. Since $\mathbb{A}_q(G)$ is real and symmetric, its eigenvalues are also real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. In [14], the q-distance energy of G was denoted by $\mathbb{E}_q(G)$ and defined by

$$\mathbb{E}_q(G) = \sum_{i=1}^{n} |\mu_i|.$$

Note: **(i)** Trace of $\mathbb{A}_q(G) = 0$ and **(ii)** If $q = 1$, q-distance energy coincides with distance energy of a graph.

To understand these new notion, let us see the following example:

Example 1.1. *Consider a crown graph S_6^0 as in Fig. 1.1.*

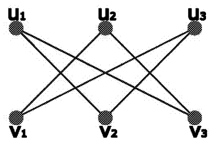

Figure 1.1 *The crown graph S_0^6*

As

$$\mathbb{A}_q(S_6^0) = \begin{pmatrix} 0 & 1+q & 1+q & 1+q+q^2 & 1 & 1 \\ 1+q & 0 & 1+q & 1 & 1+q+q^2 & 1 \\ 1+q & 1+q & 0 & 1 & 1 & 1+q+q^2 \\ 1+q+q^2 & 1 & 1 & 0 & 1+q & 1+q \\ 1 & 1+q+q^2 & 1 & 1+q & 0 & 1+q \\ 1 & 1 & 1+q+q^2 & 1+q & 1+q & 0 \end{pmatrix},$$

the characteristic polynomial of S_0^6 is

$$(\mu + q^2 - q + 1)(\mu - q^2 - 3q - 5)(\mu + q^2 + 2q + 1)^2(\mu - q^2 + 1)^2.$$

Then the q-distance energy of S_6^0 is found as

$$\begin{aligned} \mathbb{E}_q(S_6^0) &= |-(q^2 - q + 1)| + |q^2 + 3q + 5| + 2 \cdot |-(q^2 + 2q + 1)| + 2 \cdot |q^2 - 1| \\ &= 6q^2 + 6q + 6. \end{aligned}$$

2 q-distance energy of some standard graphs and their complements

We now compute the q-distance energies of several important graph classes:

Theorem 2.1. *The q-distance energy of the star graph $K_{1,n-1}$ is*

$$\begin{cases} (n-2)(1+q) + \sqrt{(n-2)^2q^2 + 2(n-2)^2q + n^2}, & \text{if } \ q > -1 \text{ and } n \geq 3, \\ -(n-2)(1+q) + \sqrt{(n-2)^2q^2 + 2(n-2)^2q + n^2}, & \text{if } \ q \leq -1 \text{ and } n \geq 3, \\ \sqrt{5}, & \text{if } \ n = 2. \end{cases}$$

Proof. **Case 1**: For $n \geq 3$, q-distance matrix $\mathbb{A}_q(K_{1,n-1})$ of the star graph $K_{1,n-1}$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is

$$\left(\begin{array}{c|cccccc} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \hline 1 & 0 & 1+q & \cdots & 1+q & 1+q & 1+q \\ 1 & 1+q & 0 & \cdots & 1+q & 1+q & 1+q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1+q & 1+q & \cdots & 0 & 1+q & 1+q \\ 1 & 1+q & 1+q & \cdots & 1+q & 0 & 1+q \\ 1 & 1+q & 1+q & \cdots & 1+q & 1+q & 0 \end{array}\right)_{n\times n}.$$

The characteristic polynomial is

$$(\mu+1+q)^{n-2}(\mu^2-(n-2)(1+q)\mu-(n-1)).$$

Hence the q-distance spectrum of $K_{1,n-1}$ would be
$$\begin{pmatrix} -1-q & \frac{(n-2)(1+q)+\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2} & \frac{(n-2)(1+q)-\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$
As a result, the q-distance energy of $K_{1,n-1}$ can be obtained as follows:

Case 1a: If $q > -1$, then

$$\begin{aligned} \mathbb{E}_q(K_{1,n-1}) &= (n-2)|-q-1| + \left|\frac{(n-2)(1+q)+\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2}\right| \\ &\quad + \left|\frac{(n-2)(1+q)-\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2}\right| \\ &= (n-2)(1+q)+\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2} \end{aligned}$$

Case 1b: If $q \leq -1$, then

$$\begin{aligned} \mathbb{E}_q(K_{1,n-1}) &= (n-2)|-q-1| + \left|\frac{(n-2)(1+q)+\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2}\right| \\ &\quad + \left|\frac{(n-2)(-1-q)-\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}}{2}\right| \\ &= -(n-2)(1+q)+\sqrt{(n-2)^2q^2+2(n-2)^2q+n^2}. \end{aligned}$$

Case 2: For $n = 2$, $\mathbb{A}_q(K_{1,1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then the characteristic polynomial is $\mu^2 - \mu - 1$ and therefore the corresponding q-distance spectrum of $K_{1,1}$ would be found as $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$. As a result of all these, we obtain the q-distance energy of $K_{1,1}$ as $\mathbb{E}_q(K_{1,1}) = \sqrt{5}$. □

Theorem 2.2. *For $n \geq 2$, the q-distance energy of the complement $\overline{K_{1,n-1}}$ of the star graph is $2(n-2)$.*

Proof. For $n \geq 2$, the q-distance matrix $\mathbb{A}_q(\overline{K_{1,n-1}})$ of the star graph $\overline{K_{1,n-1}}$ with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \end{array} \\ \left(\begin{array}{c|cccccc} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 \end{array}\right)_{n\times n} \end{array}.$$

The characteristic polynomial is $\mu(\mu-(n-2))(\mu+1)^{n-2}$ giving the q-distance spectrum of $\overline{K_{1,n-1}}$ as $\begin{pmatrix} 0 & n-2 & -1 \\ 1 & 1 & n-2 \end{pmatrix}$. As a result, the q-distance energy of $\overline{K_{1,n-1}}$ would be found as

$$\begin{aligned} \mathbb{E}_q(\overline{K_{1,n-1}}) &= |0|(1) + |n-2|(1) + |-1|(n-2) \\ &= 2(n-2). \end{aligned}$$

□

Recall that a cocktail party graph denoted by $K_{n\times 2}$ is the graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set

$$E = \{u_i u_j, v_i v_j : i \neq j\} \bigcup \{u_i v_j, v_i u_j : 1 \leq i < j \leq n\}.$$

We have

Theorem 2.3. *For $n \geq 2$, the q-distance energy of cocktail party graph $K_{n\times 2}$ is*

$$\begin{cases} -2n(q+1), & \text{if } q \leq -(2n-1), \\ -nq + 2(q+n-1), & \text{if } -(2n-1) < q < -1, \\ 2(nq+q-1), & \text{if } -1 \leq q < 1, \\ 2n(1+q), & \text{if } q \geq 1. \end{cases}$$

Proof. The q-distance matrix $\mathbb{A}_q(K_{n\times 2})$ of the cocktail party graph $K_{n\times 2}$ having the vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ and the edge set $E=\{u_iu_j, v_iv_j : i\neq j\}\cup\{u_iu_j, v_iv_j : 1\leq i<j\leq n\}$ is

$$\begin{array}{c} \\ u_1\\ u_2\\ u_3\\ \vdots\\ u_n\\ v_1\\ v_2\\ v_3\\ \vdots\\ v_n \end{array}\begin{array}{c} \begin{array}{cccccccccc} u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \end{array}\\ \left(\begin{array}{ccccc|ccccc} 0 & 1 & 1 & \dots & 1 & 1+q & 1 & 1 & \dots & 1\\ 1 & 0 & 1 & \dots & 1 & 1 & 1+q & 1 & \dots & 1\\ 1 & 1 & 0 & \dots & 1 & 1 & 1 & 1+q & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 0 & 1 & 1 & 1 & \dots & 1+q\\ \hline 1+q & 1 & 1 & \dots & 1 & 0 & 1 & 1 & \dots & 1\\ 1 & 1+q & 1 & \dots & 1 & 1 & 0 & 1 & \dots & 1\\ & 1 & 1+q & \dots & 1 & 1 & 1 & 0 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 1+q & 1 & 1 & 1 & \dots & 0 \end{array}\right)_{2n\times 2n} \end{array}.$$

The characteristic polynomial is $(\mu+q+1)^n(\mu-1+q)^{n-1}(\mu-q-(2n-1))$ and therefore the q-distance spectrum of $K_{n\times 2}$ is $\begin{pmatrix} -q-1 & q-1 & q+2n-1\\ n & n-1 & 1\end{pmatrix}$ giving the q-distance energy of $K_{n\times 2}$ as follows:

Case 1: If $q\leq -(2n-1)$, then

$$\begin{aligned}\mathbb{E}_q(K_{n\times 2}) &= n|-q-1|+(n-1)|q-1|+|q+2n-1|\\ &= -2n(q+1).\end{aligned}$$

Case 2: If $-(2n-1)<q<-1$, then

$$\begin{aligned}\mathbb{E}_q(K_{n\times 2}) &= n|-q-1|+(n-1)|q-1|+|q+(2n-1)|\\ &= -nq+2q+2n-2.\end{aligned}$$

Case 3: If $-1\leq q<1$, then

$$\begin{aligned}\mathbb{E}_q(K_{n\times 2}) &= n|-q-1|+(n-1)|q-1|+|q+2n-1|\\ &= 2(2n+q-1).\end{aligned}$$

Case 4 : If $q\geq 1$ then

$$\begin{aligned}\mathbb{E}_q(K_{n\times 2}) &= n|-q-1|+(n-1)|q-1|+|q+2n-1|\\ &= 2n(q+1).\end{aligned}$$

□

Theorem 2.4. *For $n \geq 2$, the q-distance energy of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph is $2n$.*

Proof. The q-distance matrix $\mathbb{A}_q(\overline{K_{n\times 2}})$ of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph having the vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and the edge set $\overline{E} = \{u_i v_i : 1 \leq i \leq n\}$ is given as

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\begin{array}{c} \begin{array}{cccccccccc} u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \end{array} \\ \left(\begin{array}{ccccc|ccccc} 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\ \hline 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \end{array}\right)_{2n\times 2n} \end{array}.$$

Then the characteristic polynomial is $(\mu+1)^n(\mu-1)^n$. Therefore the q-distance spectrum of $\overline{K_{n\times 2}}$ can be obtained as $\begin{pmatrix} 1 & -1 \\ n & n \end{pmatrix}$. Hence the q-distance energy of $\overline{K_{n\times 2}}$ would be found as

$$\begin{aligned} \mathbb{E}_q(\overline{K_{n\times 2}}) &= n|1| + n|-1| \\ &= 2n. \end{aligned}$$

□

Theorem 2.5. *For $n \geq 2$, the q-distance energy of the crown graph S^0_{2n} is $2n(q^2 + q + 1)$ if $q < n-2$ and $2q(nq + 2n - 2)$ if $q \geq n-2$.*

Proof. The q-distance matrix $\mathbb{A}_q(S^0_{2n})$ of the crown graph S^0_{2n} with vertex set $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ is

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\begin{array}{c} \begin{array}{cccccccccc} u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \end{array} \\ \left(\begin{array}{ccccc|ccccc} 0 & 1+q & 1+q & \dots & 1+q & 1+q+q^2 & 1 & 1 & \dots & 1 \\ 1+q & 0 & 1+q & \dots & 1+q & 1 & 1+q+q^2 & 1 & \dots & 1 \\ 1+q & 1+q & 0 & \dots & 1+q & 1 & 1 & 1+q+q^2 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+q & 1+q & 1+q & \dots & 0 & 1 & 1 & 1 & \dots & 1+q+q^2 \\ \hline 1+q+q^2 & 1 & 1 & \dots & 1 & 0 & 1+q & 1+q & \dots & 1+q \\ 1 & 1+q+q^2 & 1 & \dots & 1 & 1+q & 0 & 1+q & \dots & 1+q \\ 1 & 1 & 1+q+q^2 & \dots & 1 & 1+q & 1+q & 0 & \dots & 1+q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1+q+q^2 & 1+q & 1+q & 1+q & \dots & 0 \end{array}\right)_{2n\times 2n} \end{array}.$$

Hence the characteristic polynomial is found as

$$(\mu + q^2 - (n-2)q + 1)(\mu - q^2 - nq - 2n + 1))(\mu + q^2 + 2q + 1)^{n-1}(\mu - q^2 + 1)^{n-1}.$$

Therefore the q-distance spectrum of the graph S^0_{2n} is

$$\begin{pmatrix} -q^2+(n-2)q-1 & q^2+nq+(2n-1) & -q^2-2q-1 & q^2-1 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$$

implying that the q-distance energy of S^0_{2n} is

$$\begin{aligned} \mathbb{E}_q(S^0_{2n}) &= |-((q^2+1)-(n-2)q)|+|q^2+nq+2n-1| \\ &\quad +|-(q^2+2q+1)|(n-1)+|q^2-1|(n-1) \\ &= |-((q^2+1)-(n-2)q)|+2nq^2-q^2+3nq-2q+2n-1. \end{aligned}$$

Hence we have the following situations:

Case 1: If $q<(n-2)$, then

$$\begin{aligned} \mathbb{E}_q(S^0_{2n}) &= q^2+1-nq+2q+2nq^2-q^2+3nq-2q+2n-1 \\ &= 2n(q^2+q+1). \end{aligned}$$

Case 2: If $q\geq(n-2)$, then

$$\begin{aligned} \mathbb{E}_q(S^0_{2n}) &= nq-2q+q^2+1+2nq^2-q^2+3nq-2q+2n-1 \\ &= 2q(nq+2n-2). \end{aligned}$$

□

Theorem 2.6. *For $n\geq 2$, the q-distance energy of the complement $\overline{S^0_{2n}}$ of the crown graph is*

$$\begin{cases} -4qn+4q-2n, & \text{if } q\leq -\frac{2n-1}{n-1}, \\ -2qn+2q+2n-2, & \text{if } -\frac{2n-1}{n-1}<q\leq -1, \\ 4n-4, & \text{if } -1<q\leq -\frac{1}{n-1}, \\ 2nq+4n-2q-2, & \text{if } -\frac{1}{n-1}q<1<1, \\ 4nq+2n-4q, & \text{if } q\geq 1. \end{cases}$$

Proof. The q-distance matrix $\mathbb{A}_q(\overline{S^0_{2n}})$ of the complement $\overline{S^0_{2n}}$ of the crown graph with the vertex set $V=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and the edge set $\overline{E}=\{u_iv_i:1\leq i\leq n\}\cup\{u_iu_j,v_iv_j:1\leq i,j\leq n, i\neq j\}$ is

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccc} u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \end{array} \\ \left(\begin{array}{ccccc|ccccc} 0 & 1 & 1 & \dots & 1 & 1 & 1+q & 1+q & \dots & 1+q \\ 1 & 0 & 1 & \dots & 1 & 1+q & 1 & 1+q & \dots & 1+q \\ 1 & 1 & 0 & \dots & 1 & 1+q & 1+q & 1 & \dots & 1+q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1+q & 1+q & 1+q & \dots & 1 \\ \hline 1 & 1+q & 1+q & \dots & 1+q & 0 & 1 & 1 & \dots & 1 \\ 1+q & 1 & 1+q & \dots & 1+q & 1 & 0 & 1 & \dots & 1 \\ 1+q & 1+q & 1 & \dots & 1+q & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+q & 1+q & 1+q & \dots & 1 & 1 & 1 & 1 & \dots & 0 \end{array}\right)_{2n\times 2n} \end{array}.$$

Hence the characteristic polynomial is

$$(\mu+1+q)^{n-1}(\mu-q+1)^{n-1}(\mu-(n-1)q-(2n-1))(\mu+(n-1)q+1).$$

Therefore the q-distance spectrum of $\overline{S_{2n}^0}$ is

$$\begin{pmatrix} -q-1 & q-1 & (n-1)q+(2n-1) & -(n-1)q-1 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Finally the q-distance energy of $\overline{S_{2n}^0}$ is obtained as follows:

Case 1: If $q \le -\dfrac{2n-1}{n-1}$, then

$$\begin{aligned} \mathbb{E}_q(\overline{S_{2n}^0}) &= |-q-1|(n-1)+|q-1|(n-1)+|(n-1)q+(2n-1)|+|-(n-1)q-1| \\ &= -4qn+4q-2n. \end{aligned}$$

Case 2: If $-\dfrac{2n-1}{n-1} < q \le -1$, then

$$\begin{aligned} \mathbb{E}_q(\overline{S_{2n}^0}) &= |-q-1|(n-1)+|q-1|(n-1)+|(n-1)q+(2n-1)|+|-(n-1)q-1| \\ &= -2qn+2q+2n-2. \end{aligned}$$

Case 3: If $-1 < q \le -\dfrac{1}{n-1}$, then

$$\begin{aligned} \mathbb{E}_q(\overline{S_{2n}^0}) &= |-q-1|(n-1)+|q-1|(n-1)+|(n-1)q+2n-1|+|-(n-1)q-1| \\ &= 4n-4. \end{aligned}$$

Case 4: If $-\dfrac{1}{n-1} < q < 1$, then

$$\begin{aligned} \mathbb{E}_q(\overline{S_{2n}^0}) &= |-q-1|(n-1)+|q-1|(n-1)+|(n-1)q+2n-1|+|-(n-1)q-1| \\ &= 2nq+4n-2q-2. \end{aligned}$$

Case 5: If $q \geq 1$, then

$$\begin{aligned}\mathbb{E}_q(\overline{S_{2n}^0}) &= |-q-1|(n-1)+|q-1|(n-1)+|(n-1)q+2n-1|+|-(n-1)q-1| \\ &= 4nq+2n-4q.\end{aligned}$$

□

Theorem 2.7. *For $n \geq 2$, the q-distance energy of the complete bipartite graph $K_{m,n}$ is $2(q+1)(m+n-2)$.*

Proof. The q-distance matrix $\mathbb{A}_q(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ with vertex set $V = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ is

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\begin{array}{c} \begin{array}{cccccccccc} u_1 & u_2 & u_3 & \ldots & u_m & v_1 & v_2 & v_3 & \ldots & v_n \end{array} \\ \left(\begin{array}{ccccc|ccccc} 0 & 1+q & 1+q & \ldots & 1+q & 1 & 1 & 1 & \ldots & 1 \\ 1+q & 0 & 1+q & \ldots & 1+q & 1 & 1 & 1 & \ldots & 1 \\ 1+q & 1+q & 0 & \ldots & 1+q & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+q & 1+q & 1+q & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 \\ \hline 1 & 1 & 1 & \ldots & 1 & 0 & 1+q & 1+q & \ldots & 1+q \\ 1 & 1 & 1 & \ldots & 1 & 1+q & 0 & 1+q & \ldots & 1+q \\ 1 & 1 & 1 & \ldots & 1 & 1+q & 1+q & 0 & \ldots & 1+q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 1+q & 1+q & 1+q & \ldots & 0 \end{array}\right)_{(m+n)\times(m+n)} \end{array}.$$

Hence the characteristic polynomial is

$$(\mu+q+1)^{m+n-2}(\mu^2-(m+n-2)(q+1)\mu+(m-1)(n-1)(q^2+2q)-(m+n-1))$$

implying that the q-distance energy of $K_{m,n}$ would be obtained as

$$\begin{aligned}\mathbb{E}_q(K_{m,n}) &= |-q-1|(m+n-2)+\left|\frac{(m+n-2)(1+q)+\sqrt{(m-n)^2(1+q)^2+4mn}}{2}\right| \\ &\quad +\left|\frac{(m+n-2)(1+q)-\sqrt{(m-n)^2(1+q)^2+4mn}}{2}\right| \\ &= 2(q+1)(m+n-2).\end{aligned}$$

□

Theorem 2.8. *For $n \geq 2$, the q-distance energy of the complement $\overline{K_{m,n}}$ of the complete bipartite graph is $2(m+n-2)$.*

Proof. The q-distance matrix $\mathbb{A}_q(\overline{K_{m,n}})$ of the complement $\overline{K_{m,n}}$ of the complete bipartite graph is

$$u_1 \;\; u_2 \;\; u_3 \;\ldots\; u_m \;\; v_1 \;\; v_2 \;\; v_3 \;\ldots\; v_n$$

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\left(\begin{array}{ccccc|ccccc} 0 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 0 \end{array}\right)_{(m+n)\times(m+n)}.$$

Hence the characteristic polynomial is

$$(\mu - (m-1))(\mu - (n-1))(\mu + 1).$$

Therefore the q-distance spectrum of $\overline{K_{m,n}}$ is

$$\begin{pmatrix} m-1 & n-1 & -1 \\ 1 & 1 & m+n-2 \end{pmatrix}.$$

Then the q-distance energy of $\overline{K_{m,n}}$ is

$$\begin{aligned} \mathbb{E}_q(\overline{K_{m,n}}) &= |m-1| + |n-1| + |-1|(m+n-2) \\ &= 2(m+n-2). \end{aligned}$$

□

Recall that a windmill graph is the graph obtained by taking t copies of the complete graph K_s with a vertex in common. It is denoted by ${W_s}^{(t)}$ and it consists of $(s-1)t+1$ vertices. The windmill graph is also called as the friendship graph if $s = 3$. Now we have

Theorem 2.9. *For $n \geq 2$, the q-distance energy of the windmill graph ${W_s}^{(t)}$ is*

$$\begin{cases} st - 3t - sqt + qt + sq - q + 1 + \sqrt{Y}, & \text{if} \quad q < -\dfrac{1}{s-1}, \\ st - t + sqt - qt - sq + q - 1 + \sqrt{Y}, & \text{if} \quad q > -\dfrac{1}{s-1}, \\ st - 2t + \sqrt{Y}, & \text{if} \quad q = -\dfrac{1}{s-1}, \end{cases}$$

where $Y = ((t-1)(s-1)q + ts - t - 1)^2 + 4t(s-1)$.

Proof. The q-distance matrix $\mathbb{A}_q({W_s}^{(t)})$ of the windmill graph ${W_s}^{(t)}$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ is

$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_s \\ v_{s+1} \\ v_{s+2} \\ \vdots \\ v_{2s-1} \\ \vdots \\ v_{n-s+2} \\ v_{n-s+3} \\ \vdots \\ v_n
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc}
v_1 & v_2 & v_3 & \dots & v_s & v_{s+1} & v_{s+2} & \dots & v_{2s-1} & \dots & v_{n-s+2} & v_{n-s+3} & \dots & v_n
\end{array} \\
\left(\begin{array}{c|cccc|cccc|c|cccc}
0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 \\ \hline
1 & 0 & 1 & \dots & 1 & 1+q & 1+q & \dots & 1+q & \dots & 1+q & 1+q & \dots & 1+q \\
1 & 1 & 0 & \dots & 1 & 1+q & 1+q & \dots & 1+q & \dots & 1+q & 1+q & \dots & 1+q \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 0 & 1+q & 1+q & 1+q & 1+q & \dots & 1+q & 1+q & \dots & 1+q \\ \hline
1 & 1+q & 1+q & \dots & 1+q & 0 & 1 & \dots & 1 & \dots & 1+q & 1+q & \dots & 1+q \\
1 & 1+q & 1+q & \dots & 1+q & 1 & 0 & \dots & 1 & \dots & 1+q & 1+q & \dots & 1+q \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
1 & 1+q & 1+q & \dots & 1+q & 1 & 1 & \dots & 0 & \dots & 1+q & 1+q & \dots & 1+q \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
1 & 1+q & 1+q & \dots & 1+q & 1+q & 1+q & \dots & 1+q & \dots & 0 & 1 & \dots & 1 \\
1 & 1+q & 1+q & \dots & 1+q & 1+q & 1+q & \dots & 1+q & \dots & 1 & 0 & \dots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1+q & 1+q & \dots & 1+q & 1+q & 1+q & \dots & 1+q & \dots & 1 & 1 & \dots & 0
\end{array}\right)
\end{array}.
$$

Hence the characteristic polynomial is

$(\mu+1)^{(s-2)t}((\mu+(s-1)q+1))^{(t-1)}(\mu^2-(s-1)(t-1)q\mu-((s-1)t-1)\mu-(s-1)t)$

and therefore the q-distance spectrum of ${W_s}^{(t)}$ would be obtained as

$$
\begin{pmatrix}
-1 & -(s-1)q-1 & \dfrac{X+\sqrt{Y}}{2} & \dfrac{X-\sqrt{Y}}{2} \\
(s-2)t & t-1 & 1 & 1
\end{pmatrix}
$$

where $X=(t-1)(s-1)q+ts-t-1$ and $Y=((t-1)(s-1)q+ts-t-1)^2+4t(s-1)$. Then the q-distance energy of ${W_s}^{(t)}$ can be obtained as follows:

Case 1: If $q<-\dfrac{1}{s-1}$, then

$$
\begin{aligned}
\mathbb{E}_q({W_s}^{(t)}) &= (s-2)t|-1|+|-(s-1)q-1|(t-1)+\left|\frac{X+\sqrt{Y}}{2}\right|+\left|\frac{X-\sqrt{Y}}{2}\right| \\
&= st-3t-sqt+qt+sq-q+1+\sqrt{Y}.
\end{aligned}
$$

Case 2: If $q>-\dfrac{1}{s-1}$, then

$$
\begin{aligned}
\mathbb{E}_q({W_s}^{(t)}) &= (s-2)t|-1|+|-(s-1)q-1|(t-1)+\left|\frac{X+\sqrt{Y}}{2}\right|+\left|\frac{X-\sqrt{Y}}{2}\right| \\
&= st-t+sqt-qt-sq+q-1+\sqrt{Y}.
\end{aligned}
$$

Case 3: If $q=-\dfrac{1}{s-1}$, then

$$
\begin{aligned}
\mathbb{E}_q({W_s}^{(t)}) &= (s-2)t|-1|+\left|\frac{X+\sqrt{Y}}{2}\right|+\left|\frac{X-\sqrt{Y}}{2}\right| \\
&= st-2t+\sqrt{Y}.
\end{aligned}
$$

□

Theorem 2.10. *For $n \geq 2$, the q-distance energy of the complement $\overline{W_s^{(t)}}$ of windmill graph is*

$$\begin{cases} -2qst+4qt-2st+4t+2q, & \text{if} \quad q<-1, \\ qt-sq, & \text{if} \quad -1<q<-\dfrac{t+1-st}{s-2}, \\ 2st+2qs-2t-4q-2, & \text{if} \quad -\dfrac{t+1-st}{s-2}<q<\dfrac{1}{s-2}, \\ 2qst-4qt+2st-4t, & \text{if} \quad q>\dfrac{1}{s-2}. \end{cases}$$

Proof. The q-distance matrix $\mathbb{A}_q(\overline{W_s^{(t)}})$ of the complement $\overline{W_s^{(t)}}$ of the windmill graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_s \\ v_{s+1} \\ v_{s+2} \\ \vdots \\ v_{2s-1} \\ \vdots \\ v_{n-s+2} \\ v_{n-s+3} \\ \vdots \\ v_n \end{array}
\begin{array}{c|cccc|cccc|c|cccc} v_1 & v_2 & v_3 & \cdots & v_s & v_{s+1} & v_{s+2} & \cdots & v_{2s-1} & \cdots & v_{n-s+2} & v_{n-s+3} & \cdots & v_n \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
0 & 0 & 1+q & \cdots & 1+q & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & 1+q & 0 & \cdots & 1+q & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1+q & 1+q & 1+q & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
0 & 1 & 1 & \cdots & 1 & 0 & 1+q & \cdots & 1+q & \cdots & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 & 1+q & 0 & \cdots & 1+q & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 1 & 1 & \cdots & 1 & 1+q & 1+q & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & 1+q & \cdots & 1+q \\
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1+q & 0 & \cdots & 1+q \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1+q & 1+q & \cdots & 0
\end{array}.$$

Hence the characteristic polynomial is

$$\mu(\mu+q+1)^{(s-2)t}(\mu-(s-2)q+1)^{t-1}(\mu-(s-2)q-(s-1)t+1)$$

and therefore the q-distance spectrum of $\overline{W_s^{(t)}}$ is

$$\begin{pmatrix} 0 & -q-1 & (s-2)q-1 & (s-2)q+(s-1)t-1 \\ 1 & (s-2)t & t-1 & 1 \end{pmatrix}.$$

As the result, we obtain the q-distance energy of $\overline{W_s^{(t)}}$ as follows:

Case 1: If $q<-1$, then

$$\begin{aligned} \mathbb{E}_q(\overline{W_s^{(t)}}) &= |0|+|-q-1|(s-2)t+|(s-2)q-1|(t-1)+|(s-2)q+(s-1)t-1| \\ &= -2qst+4qt-2st+4t+2q. \end{aligned}$$

Case 2: If $-1<q<-\dfrac{t+1-st}{s-2}$, then

$$\begin{aligned} \mathbb{E}_q(\overline{W_s^{(t)}}) &= |0|+|-q-1|(s-2)t+|(s-2)q-1|(t-1)+|(s-2)q+(s-1)t-1| \\ &= qt-sq. \end{aligned}$$

Case 3: If $-\frac{t+1-st}{s-2} < q < \frac{1}{s-2}$, then

$$\begin{aligned}\mathbb{E}_q(\overline{W_s^{(t)}}) &= |0| + |-q-1|(s-2)t + |(s-2)q-1|(t-1) + |(s-2)q+(s-1)t-1| \\ &= 2st + 2qs - 2t - 4q - 2.\end{aligned}$$

Case 4: If $q > \frac{1}{s-2}$, then

$$\begin{aligned}\mathbb{E}_q(\overline{W_s^{(t)}}) &= |0| + |-q-1|(s-2)t + |(s-2)q-1|(t-1) + |(s-2)q+(s-1)t-1| \\ &= 2qst - 4qt + 2st - 4t.\end{aligned}$$

□

Theorem 2.11. *For $n \geq 2$, the q-distance energy of the complete graph K_n is equal to $2n-2$.*

Proof. Since the q-distance adjacency matrix of K_n is the same as the ordinary adjacency matrix of K_n, the q-distance energy of the complete graph K_n is equal to $2n-2$. □

3 Brief summary and conclusion

Energy is a very important subject of graph theory with many applications in physics and chemistry. Similarly to the classical graph energy, there are a few other types of energy in graphs which are similarly defined by means of some other matrices. In this paper, we have studied a recently defined type of energy called q-distance energy and the q-distance energy has been obtained for some standard graphs. As the distances are calculated between the vertices of the graph representing the atoms in the corresponding molecule, the q-distance energy is expected to have applications in chemistry due to its effect on the intermolcecular forces which effect the graph energy.

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