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GENERALIZED TYPE 2 DEGENERATE EULER NUMBERS

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Abstract. Recently, the type 2 degenerate Euler numbers and polynomials were introduced by Kim-Kim. In this paper, we study the generalized type 2 degenerate Euler numbers, as a further generalization of the type 2 degenerate Euler numbers, and find explicit expressions for these numbers involving the degenerate central factorial numbers of the second kind.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by Kim-Kim as

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t), \text{ (see [12]).} \tag{1}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$.

The generalized degenerate Bernoulli polynomials $\beta_{n,\lambda}^{(\alpha)}(x)$ and the generalized degenerate Euler polynomials $\mathcal{E}_{n,\lambda}^{(\alpha)}(x)$ are defined by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(\alpha)}(x), \quad \left(\frac{2}{e_\lambda(t)+1}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(\alpha)}(x), \tag{2}$$

where $\alpha, x \in \mathbb{R}(\text{or } \mathbb{C})$ and $\lambda \in \mathbb{R}$, (see [1,2,3]).

When $\alpha = 1$, $\beta_{n,\lambda}^{(1)}(x) = \beta_{n,\lambda}(x)$, $\mathcal{E}_{n,\lambda}^{(1)}(x) = \mathcal{E}_{n,\lambda}(x)$ are respectively called Carlitz's Bernoulli and Euler polynomials.

As is well known, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^n, \ (n \geq 0), \ \text{(see [1–12])}, \tag{3}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

From (3), we note that

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \ (k \geq 0), \ \text{(see [8])}. \tag{4}$$

Let n be nonnegative integer. Then the central factorial numbers of the second kind are defined by

$$x^n = \sum_{k=0}^{n} T(n,k)\, x^{(k)}, \tag{5}$$

where $x^{[0]} = 1$, $x^{[n]} = x\left(x+\frac{n}{2}-1\right)\cdots\left(x-\frac{n}{2}+1\right)$, $(n \geq 1)$.

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Recently, Kim-Kim introduced the degenerate central factorial numbers of the second kind given by

$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k=\sum_{n=k}^{\infty}T_\lambda(n,k)\frac{t^n}{n!},\quad \text{(see [4])}. \tag{6}$$

Further, Kim-Kim considered type 2 degenerate Euler polynomials given by

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [6,9,10])}. \tag{7}$$

When $x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the type 2 degenerate Euler numbers.

In this paper, we would like to consider the generalized type 2 degenerate Euler numbers, as a further generalization of the recently studied the type 2 degenerate Euler numbers. Then, among other things, we will find some explicit expressions for these numbers involving the degenerate central factorial numbers of the second kind.

1. Generalized type 2 degenerate Euler numbers

For $x\in\mathbb{R}$ (or $\mathbb{C}$), we consider the generalized type 2 degenerate Euler numbers given by

$$\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^x=\sum_{n=0}^{\infty}E_{n,\lambda}^{(x)}\frac{t^n}{n!}, \tag{8}$$

which are a degenerate version of the generalized type 2 Euler numbers defined by

$$\left(\frac{2}{e^t+e^{-t}}\right)^x=\sum_{n=0}^{\infty}E_n^{(x)}\frac{t^n}{n!}. \tag{9}$$

When $x=1$, $E_{n,\lambda}^{(1)}=E_{n,\lambda}$, $(n\geq 0)$. From (8) and (9), we note that

$$\begin{aligned}\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^x&=\left(\frac{2}{e^{\frac{1}{\lambda}\log(1+\lambda t)}+e^{-\frac{1}{\lambda}\log(1+\lambda t)}}\right)^x\\&=\sum_{k=0}^{\infty}E_{2k}^{(x)}\frac{\lambda^{-2k}}{(2k)!}\big(\log(1+\lambda t)\big)^{2k}\\&=\sum_{k=0}^{\infty}E_{2k}^{(x)}\lambda^{-2k}\sum_{n=2k}^{\infty}S_1(n,2k)\,\lambda^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}E_{2k}^{(x)}\lambda^{n-2k}S_1(n,2k)\right)\frac{t^n}{n!}.\end{aligned} \tag{10}$$

Therefore, we obtain the following theorem.

Theorem 1.1. *Let n be a nonnegative integer. Then we have*

$$E_{n,\lambda}^{(x)}=\sum_{k=0}^{[\frac{n}{2}]}E_{2k}^{(x)}\lambda^{n-2k}S_1(n,2k).$$

From (8), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\lambda}^{(x)} \frac{t^n}{n!} &= \left(\frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \right)^x \\
&= \left(\frac{1}{\frac{e_\lambda(t)+e_\lambda^{-1}(t)-2}{2} + 1} \right)^x \\
&= \left(1 + \frac{e_\lambda(t) + e_\lambda^{-1}(t) - 2}{2} \right)^{-x} \\
&= \sum_{j=0}^{\infty} \binom{-x}{j} \left(\frac{1}{2} \right)^j \left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t) \right)^{2j} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \binom{x+j-1}{j} (2j)! \frac{1}{(2j)!} \left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t) \right)^{2j} \\
&= \sum_{j=0}^{\infty} \left(-\frac{1}{2} \right)^j \binom{x+j-1}{j} (2j)! \sum_{n=2j}^{\infty} T_\lambda(n,2j) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\left[\frac{n}{2}\right]} \left(-\frac{1}{2} \right)^j \binom{x+j-1}{j} (2j)!\, T_\lambda(n,2j) \right) \frac{t^n}{n!}.
\end{aligned} \tag{11}$$

Therefore, by (11), we obtain the following theorem.

Theorem 1.2. *For $n \geq 0$, we have*

$$E_{n,\lambda}^{(x)} = \sum_{j=0}^{\left[\frac{n}{2}\right]} \left(-\frac{1}{2} \right)^j \binom{x+j-1}{j} (2j)!\, T_\lambda(n,2j).$$

It is easy to show that

$$\binom{x+j-1}{j} = \frac{(-1)^j}{j!} (-x)_j. \tag{12}$$

Thus, by (3), Theorem 1.2 and (12), we get

$$\begin{aligned}
E_{n,\lambda}^{(x)} &= \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{(2j)!}{2^j j!} (-x)_j\, T_\lambda(n,2j) \\
&= \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{(2j)!}{2^j j!} T_\lambda(n,2j) \sum_{i=0}^{j} S_1(j,i)(-x)^i \\
&= \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \sum_{j=i}^{\left[\frac{n}{2}\right]} \frac{(2j)!}{2^j j!} T_\lambda(n,2j) S_1(j,i)\, x^i \\
&= \sum_{i=0}^{\left[\frac{n}{2}\right]} \rho_\lambda(n,i)\, x^i,
\end{aligned}$$

where $\rho_\lambda(n,i) = (-1)^i \sum_{j=i}^{\left[\frac{n}{2}\right]} \frac{(2j)!}{2^j j!} T_\lambda(n,2j) S_1(j,i)$. Therefore, we obtain the following theorem.

Theorem 1.3. *For $n \geq 0$, we let*

$$\rho_\lambda(n,i) = (-1)^i \sum_{j=i}^{\left[\frac{n}{2}\right]} \frac{(2j)!}{2^j j!} T_\lambda(n,2j) S_1(j,i).$$

Then we have

$$E_{n,\lambda}^{(x)} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \rho_\lambda(n,i)\, x^i.$$

From (8), for any nonnegative integer m we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\lambda}^{(-m)} \frac{t^n}{n!} &= \left(\frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \right)^{-m} \\ &= \frac{1}{2^m} \left(e_\lambda(t) + e_\lambda^{-1}(t) \right)^m \\ &= \frac{e_\lambda^m(t)}{2^m} \left(1 + e_\lambda^{-2}(t) \right)^m \\ &= \frac{e_\lambda^m(t)}{2^m} \sum_{l=0}^{m} \binom{m}{l} e_\lambda^{-2l}(t) \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} (m-2l)_{n,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{13}$$

Therefore, by (13), we obtain the following theorem.

Theorem 1.4. *For $n, m \geq 0$, we have*

$$\sum_{i=0}^{\left[\frac{n}{2}\right]} \rho_\lambda(n,i)\, (-m)^i = E_{n,\lambda}^{(-m)} = \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} (m-2l)_{n,\lambda}.$$

From Theorem 1.3, we note that

$$\frac{d^k}{dx^k} E_{n,\lambda} \Big|_{x=0} = k! \rho_\lambda(n,k), \quad \left(1 \leq k \leq \left[\frac{n}{2} \right] \right).$$

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