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Bounds for Matching Number of Fundamental Realizations According to New Graph Invariant Omega

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Abstract

Matching number of a graph is one of the intensively studied areas in graph theory due to numerous applications of the matching and related notions. Recently, Delen and Cangul defined a new graph invariant denoted by Ω which helps to determine several graph theoretical and combinatorial properties of the realizations of a given degree sequence. In this paper, using K_2 deletion process, the maximum and minimum matching numbers of all so-called fundamental realizations of a given degree sequence.

1 Introduction

[1 2] Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$. n and m are called order and size of graph G, respectively. If there is an edge e that connects the vertices v and w, then v and w are called adjacent vertices and in such a case e is called incident to v and w. The degree of a vertex v is the number of edges that are connected to v and it is denoted by d_v. If the degree of a vertex v is one, then it is called pendant vertex and a vertex w that is connected to v is called quasi-pendant vertex or support vertex. An edge that is incident to a pendant vertex is called pendant edge. A vertex of degree zero is called an isolated vertex and a

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graph consisting of only isolated vertices will be called a null graph. A null graph of order n will be denoted by N_n. The smallest and biggest degrees of vertices in a graph G are denoted by δ and Δ. If there is a path between every couple of vertices of a graph G, then G is called a connected graph. A graph that is not connected is called a disconnected graph. For example, N_n is disconnected for $n > 1$. A maximal connected subgraph of G is called a component. A connected graph has only one component and if a graph has more than one components, then it is disconnected. A graph without any cycle is called acyclic and a graph with at least one cycle is called cyclic. An edge that connects a vertex v to itself is called a loop. If there are at least two edges between two vertices, then these edges are called multiple edges. An edge that exists between two vertices of a cycle but not along the cycle is called a chord. An area that is bounded by a cycle in the graph is called a region. The area outside of the graph is not counted as a region.

A matching is a set of disjoint edges such that no two of them have a common vertex. A maximum matching is a matching with the maximum possible number of edges. The number of edges that exist in the maximum matching of G is called matching number of G and we denote it with $\nu(G)$. Several relations between matching number, nullity, chromatic number, independence number and also the minimum and maximum nullity conditions are studied in [2, 8, 11, 16–18].

Written with multiplicities, a degree sequence is shown as

$$D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$$

where a_i's are non-negative integers. If there are some isolated vertices in the corresponding graph, then we also have the term $0^{(a_0)}$ in the degree sequence D of the corresponding graph. If the degree sequence of a graph G is equal to D, then G is called a realization of the degree sequence D. To avoid ambiguity, a degree sequence can be stated in a non-decreasing order without multiplicities as $D = \{d_1, d_2, \cdots, d_n\}$.

For a realizable degree sequence, several graphs having the same degree sequence may exist. Shortly, realization of a degree sequence is not unique. For instance, there are two completely different graphs in Fig. 1 but their degree sequences are the same.

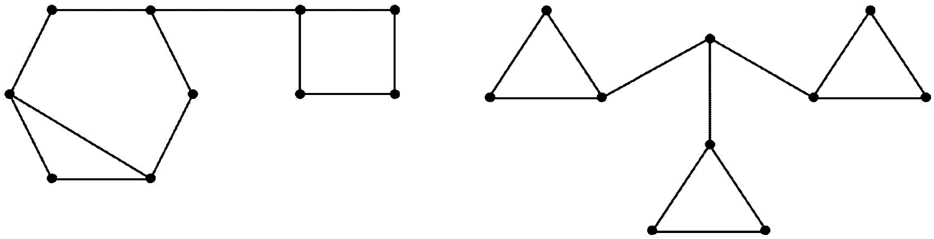

Figure 1: Two realizations of the same degree sequence

Also, in many cases, a degree sequence may not have a realization. To determine the realizability, the most well-known result is known as the Havel-Hakimi process. In addition to this, some algorithms, can be found in [1, 3, 4, 9, 10, 13, 14, 19]. Besides realizability, determining the topological and combinatorial properties of realizations of a degree sequence is really important in graph theory and it is related to research areas such as molecular chemistry, computer sciences, etc.

Up to now, several topological invariants were defined by many mathematicians and chemists. The most famous one is Euler characteristic. Euler characteristic of a graph G is defined as $\chi(S) = n - m + r$ where n, m, and r are the numbers of vertices, edges and regions of graph G and it must be noted that in calculating r, the infinite region outside the graph is also counted. Recently, Delen and Cangul defined a new graph invariant in relation to Euler characteristic. This invariant gives several combinatorial and topological information about the realizations of a degree sequence and it is denoted by $\Omega(D)$ or $\Omega(G)$. They classified all graphs into three main classes according to their omega values. In each of these three groups, they defined the notion of fundamental realizations. In this paper, we study the maximum and minimum matching numbers amongst all corresponding fundamental realizations.

2 Omega invariant

Delen and Cangul defined a new graph invariant $\Omega(G)$ for a graph G that gives lots of topological and combinatorial information about graph realizations of a degree sequence. Let us give the definition of $\Omega(G)$ and some properties first.

Definition 2.1. *[5] Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ be the degree sequence of a graph G. $\Omega(G)$ is defined as*

$$\begin{aligned}\Omega(G) &= -a_1 + a_3 + 2a_4 + 3a_5 + 4a_6 + 5a_7 + \cdots + (\Delta - 2)a_\Delta \\ &= \textstyle\sum_{i=1}^{\Delta}(i-2)a_i.\end{aligned}$$

Theorem 2.1. *[5] Let G be a graph. Then $\Omega(G) = 2(m - n)$.*

It is clear that for any graph G, $\Omega(G)$ is even by the previous theorem. Also, for a degree sequence D, if $\Omega(D)$ is odd, then D is not realizable. In the literature, next lemma is well-known and called the hand-shaking Lemma. Hand-shaking Lemma is the most practical criteria for realizability.

Lemma 2.1. *Let $G=(V,E)$ be a graph with n vertices and m edges. Then*

$$\sum_{i=1}^{n} d_{v_i} = 2m.$$

Theorem 2.2. *[5] Let G be a graph with c components. Then $r=\frac{\Omega(G)}{2}+c$ where r is the number of regions in G.*

Let us calculate the $\Omega(G)$ and r in the next example.

Example 2.1. *A graph G with* 13 *edges and* 8 *vertices is given in Fig. 2. The degree sequence of graph G is $D=\left\{1^{(1)},2^{(1)},3^{(3)},4^{(1)},5^{(2)}\right\}$*

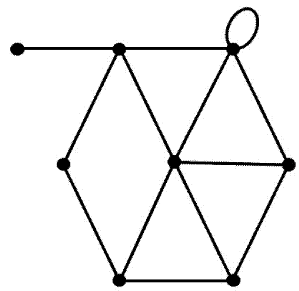

Figure 2: A graph G with $\Omega(G)=10$

$$\begin{aligned}\Omega(G) &= -a_1+a_3+2a_4+3a_5+4a_6+5a_7+\cdots+(\Delta-2)a_\Delta \\ &= -1\cdot 1+1\cdot 3+2\cdot 1+3\cdot 2=10.\end{aligned}$$

Moreover, we can calculate $\Omega(G)=2(m-n)=2(13-8)=10$ similarly. By Theorem 2.2, $r=\frac{\Omega(G)}{2}+c=\frac{10}{2}+1=6$. In Fig. 2, we see that the number of regions is 6 *and* 4 *of them is bounded by C_3's, one of them is bounded by a C_4 and one of them is bounded by a loop.*

In the next theorem, Delen and Cangul gave the relation between Euler characteristic and Omega invariant.

Theorem 2.3. *[5] Let G be a graph. Then $\chi(G)=r-\frac{\Omega(G)}{2}$ where r is the number of regions in G.*

3 Upper bounds and lower bounds for the matching number of fundamental realizations of a given degree sequence

Recently, Delen and Cangul defined three fundamental forms (realizations) of graphs with respect to a given degree sequence D. These forms were defined for $\Omega(D) \geq 0$, $\Omega(D) = -2$ and $\Omega(D) \leq -4$. These forms are respectively called a cyclic fundamental realization, an acyclic fundamental realization and a mixed fundamental realization of D, see [5–7].

Definition 3.1. *[5] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. If $\Omega(D) \geq 0$, then a connected realization of D that consists of a cycle of length $n - a_1$ having a_1 pendant edges around the cycle and a total of $\frac{\Omega}{2}$ loops, multiple edges and chords is called a cyclic fundamental realization of D.*

Definition 3.2. *[5] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. If $\Omega(D) = -2$, then a connected realization of D that consists of a path of order $a_2 + a_3 + \cdots a_\Delta + 2$ where a_1 pendant edges are incident to the vertices of the path is called an acyclic fundamental realization of D. In the literature, this acyclic realization is called a caterpillar tree.*

Definition 3.3. *[5] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. If $\Omega(D) \leq -4$, then a realization of D that consists of a main component as a cycle with length $a_2 + a_3 + \cdots a_\Delta$ and $-\Omega(D)/2$ times K_2 complete graphs is called a mixed fundamental realization of D.*

We now recall a useful process for determining matching number of a graph by using some induced subgraphs of the given graph.

Let G be a graph that has at least one pendant vertex and inherently one pendant edge. The operation of taking out a pendant vertex together with its adjacent vertex from G is called a K_2 deletion operation, see [15] and [12]. Recall that when we are taking out a vertex from G, all the incident edges are also taken out. The process of applying consecutive K_2 deletion operations until having an induced subgraph without pendant vertices is called a K_2 deletion process. A crucial subgraph is defined as follows, see [15]: If there is no pendant vertex in G, then the crucial subgraph of G is itself. If there are pendant vertices in G, then the subgraph obtained at the end of a K_2 deletion process is called a crucial subgraph of G. A crucial subgraph of G is denoted by G'. For the sake of simplicity, in this paper the number of K_2 deletions that we need to get G' from G is denoted by $\gamma(G)$. Note that all crucial subgraphs of G are shown to be isomorphic, see for details, [12, 15]. In Fig. 3 we have a graph G, we apply K_2 deletion process to G and we get a crucial subgraph.

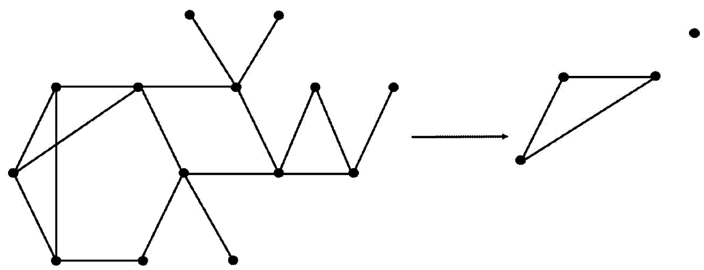

Figure 3: Illustration of K_2 deletion process

In our study, we need next lemma that was given by Tam and Huang in [12].

Lemma 3.1. *Let G be a graph. Then $\nu(G) = \nu(G') + \nu(G - G')$ and $\gamma(G) = \nu(G - G')$.*

Theorem 3.1. *Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ be a realizable degree sequence. If $\Omega(D) \geq 0$ and $a_1 = 0$, then the minimum matching number and the maximum matching number amongst all cyclic fundamental realizations of D are equal and given by*

$$\lfloor \frac{a_2 + a_3 + \cdots + a_\Delta}{2} \rfloor.$$

That is, all fundamental realizations of D will have the same matching number.

In the case of $a_1 \neq 0$, the minimum and maximum matching numbers which can be obtained for all the cyclic fundamental realizations of D are different and given in the next two results. In the proofs of these results, we shall need a number $p(a_1)$ which is defined by the algorithm below:

Algorithm 3.1. *Let us rewrite the degree sequence D in a non-decreasing order without multiplicities as $D = \{d_1,\ d_2,\ \cdots,\ d_n\}$. Then*

- *Calculate the difference $a_1 - (d_n - 2)$. If this difference is positive, continue with the second step.*
- *Calculate the difference $a_1 - (d_n - 2) - (d_{n-1} - 2)$. If this difference is positive, continue with the third step.*
- *Continue calculating the differences $a_1 - (d_n - 2) - (d_{n-1} - 2) - \cdots - (d_{n-k} - 2)$ and stop when this difference is negative or zero. Define $p(a_1) = k + 1$.*

For $\Omega(D) \geq 0$, $p(a_1)$ is the number of vertices that exist on the cycle and incident to at least one pendant edge in the corresponding cyclic fundamental realization G of D.

For $\Omega(D) = -2$, $p(a_1)$ is the number of vertices existing on the path and incident to at least one pendant edge that is adjacent to the main path of length $a_2 + a_3 + \cdots + a_\Delta + 2$ in the corresponding acyclic fundamental realization G of D.

Example 3.1. *Let* $D = \left\{1^{(9)}, 2^{(3)}, 3^{(1)}, 4^{(1)}, 5^{(2)}\right\}$ *be a degree sequence. It is clearly realizable. Let us rewrite* D *as* $D = \{1, 1, \cdots, 1, 2, 2, 2, 3, 4, 5, 5\}$ *. Let us calculate* $p(a_1)$.

- $p(a_1) = a_1 - (d_n - 2) = a_1 - (d_{16} - 2) = 9 - (5 - 2) = 6$. *Since the difference is positive, we continue with the second step.*
- $a_1 - (d_n - 2) - (d_{n-1} - 2) = a_1 - (d_{16} - 2) - (d_{15} - 2) = 9 - (5 - 2) - (5 - 2) = 3$. *Since the difference is positive, we continue with the third step.*
- $a_1 - (d_n - 2) - (d_{n-1} - 2) - (d_{n-2} - 2) = a_1 - (d_{16} - 2) - (d_{15} - 2) - (d_{14} - 2) = 9 - (5 - 2) - (5 - 2) - (4 - 2) = 1$. *Since the difference is positive, we continue with the fourth step.*
- $a_1 - (d_n - 2) - (d_{n-1} - 2) - (d_{n-2} - 2) - (d_{n-3} - 2) = a_1 - (d_{16} - 2) - (d_{15} - 2) - (d_{14} - 2) - (d_{13} - 2) = 9 - (5 - 2) - (5 - 2) - (4 - 2) - (3 - 2) = 0$. *Since, at this step, the difference is zero, we stop and find that* $p(a_1) = 3 + 1 = 4$.

Theorem 3.2. *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence. Let* $\Omega(D) \geq 0$ *and let* $a_1 \neq 0$. *Also let* G_1 *be the realization of* D *having the minimum matching number amongst all cyclic fundamental realizations of* D. *Then*

$$\nu(G_1) = \begin{cases} p(a_1) + \lfloor \frac{n - a_1 - (2p(a_1) - 1)}{2} \rfloor & p(a_1) < a_2, \\ p(a_1) + \lfloor \frac{n - a_1 - a_2 - p(a_1) + 1}{2} \rfloor & p(a_1) = a_2, \\ p(a_1) + \lfloor \frac{n - a_1 - a_2 - p(a_1)}{2} \rfloor & p(a_1) > a_2. \end{cases}$$

Proof. Let D be a realizable degree sequence such that $\Omega(D) \geq 0$. We shall use a constractive proof. By the definition of a cyclic fundamental realization of D, we get a cycle with order $a_2 + a_3 + \cdots + a_\Delta$ and a_1 pendant edges adjacent to the vertices on cycle. We add the pendant edges according to the following rule: Let us demonstrate the degree sequence of D in a non-decreasing order without multiplicities as $D = \{d_1,\ d_2,\ \cdots,\ d_n\}$. Add $d_n - 2$ pendant edges to a vertex, say v_1, $d_{n-1} - 2$ pendant edges to another vertex, say v_2, and continue up to the vertex $v_{p(a_1)}$ on the cycle. To get the minimum matching number amongst all cyclic fundamental realizations, we must search for the minimal value of $\nu(G')$ by Lemma 3.1 since $\gamma(G)$ already is fixed. For minimality of $\nu(G')$, we must choose a cyclic fundamental realization of D having maximum number of null graphs in the G'. Therefore, we must place a vertex of degree 2 between every vertex pair on the cycle as far as possible where we already added pendant edges. Hence, if $p(a_1) \leq a_2$, then we start applying K_2 deletion processs. As during this process, we deleted $p(a_1)$ vertices $v_1,\ v_2,\ \cdots,\ v_{p(a_1)}$ from the main cycle and $p(a_1) - 1$ vertices of degree two between them, the remaining part of the main cycle which consists of a path of length $a_2+a_3+\cdots+a_\Delta-(2p(a_1)-1)$ together with some loops, pendant edges and chords. Since the loops, multiple edges and chords are not counted in the calculation of the matching number, we may ignore them and continue the K_2 deletion process until we get the desired result that is $\nu(G_1)=p(a_1)+\lfloor\frac{a_2+a_3+\cdots+a_\Delta-(2p(a_1)-1)}{2}\rfloor$. For the case of equality $p(a_1)=a_2$, by cancellation of a_2 and $p(a_1)$, we have $\lfloor\frac{a_3+\cdots+a_\Delta-p(a_1)+1}{2}\rfloor$. Finally, if $p(a_1)>a_2$, then we have the result $\nu(G_1)=p(a_1)+\lfloor\frac{a_2+a_3+\cdots+a_\Delta-(p(a_1)+a_2)}{2}\rfloor$ by using the same method and we complete the proof by means of $n=a_1+a_2+a_3+\cdots+a_\Delta$. $\square$

Theorem 3.3. *Let $D=\{1^{(a_1)},2^{(a_2)},3^{(a_3)},\cdots,\Delta^{(a_\Delta)}\}$ be a realizable degree sequence. Let $\Omega(D)\geq 0$ and let the realization of D having the maximum matching number amongst all cyclic fundamental realizations of D be G_2. Then*

$$\nu(G_2)=\begin{cases} a_1+\lfloor\frac{n-2a_1}{2}\rfloor & 2a_1<n-a_2,\\ a_1+\lfloor\frac{a_2}{2}\rfloor & 2a_1=n-a_2,\\ n-a_1-a_2+\lfloor\frac{a_2}{2}\rfloor & 2a_1>n-a_2.\end{cases}$$

Proof. Let D be a realizable degree sequence and $\Omega(D)\geq 0$. It is well known by [5–7] that each realization of D must be a cyclic one. A cyclic fundamental realization of D is formed by a cycle of order $a_2+a_3+\cdots+a_\Delta$ and a_1 pendant edges incident to the vertices on the cycle. For finding the maximum matching number amongst all cyclic fundamental realizations, we must search for the maximal value of $\nu(G')$ of all the crucial subgraphs G' by Lemma 3.1. Hence, we must determine a cyclic fundamental realization of D having minimum number of null graphs in the G' for maximality of $\nu(G')$. For this reason, for both cases $a_1\leq a_3+a_4+\cdots+a_\Delta$ and $a_1>a_3+a_4+\cdots+a_\Delta$, we place pendant edges consequtively without having any vertex between them on the cycle part. In the first case, we add one pendant edge to a_1 vertices having degree greater than 2 without having any vertex of degree two between them and cyclic fundamental realization is completed by using chords, loops and multiple edges. Hence, if $a_1\leq a_3+a_4+\cdots+a_\Delta$, then we get $\nu(G_2)=a_1+\lfloor\frac{a_2+a_3+\cdots+a_\Delta-a_1}{2}\rfloor$ by using K_2 deletion process (we may ignore loops, multiple edges and chords for the reason that is given in the proof of previous theorem). Also for equality $a_1=a_3+a_4+\cdots+a_\Delta$, it is clear that we have $\nu(G_2)=a_1+\lfloor\frac{a_2}{2}\rfloor$. In the second case, if $a_1>a_3+a_4+\cdots+a_\Delta$, then first, we add one pendant edge to $a_3+a_4+\cdots+a_\Delta$ vertices having degree greater than 2 without having any vertex of degree two between them. The required cyclic fundamental realization is completed by adding some loops, multiple edges, chords and remaining pendant edges to the vertices having degree greater than 3 and the process is continued in this way. Thus, we get $\nu(G_2)=a_3+a_4+\cdots+a_\Delta+\lfloor\frac{a_2}{2}\rfloor$ and since $n=a_1+a_2+a_3+\cdots+a_\Delta$, we obtain required result. $\square$

In the next example, we find $\nu(G_1)$ and $\nu(G_2)$ by means of the theorems above.

Example 3.2. *Let $D=\{1^{(7)},2^{(4)},3^{(3)},4^{(1)},5^{(2)}\}$. We want to find the fundamental realizations of D with minimum and maximum matching numbers. First we have $\Omega(D)=-1\cdot7+1\cdot3+2\cdot1+3\cdot2=4$, so we are in the case $\Omega(D)\geq 0$. First of all, let us find the $\nu(G_1)$. We see that $a_2=4$ and*

$p(a_1) = 3$. *Since* $p(a_1) < a_2$, $\nu(G_1) = p(a_1) + \lfloor \frac{n-a_1-(2p(a_1)-1)}{2} \rfloor = 3 + \lfloor \frac{10-(2\cdot3-1)}{2} \rfloor = 3+2 = 5$ *as shown in the first cyclic fundamental realization of* D *of Fig. 4. Secondly, we find the* $\nu(G_2)$*. We have* $a_1 = 7$*, and* $n-a_1-a_2 = 6$*. Since* $2a_1 > n-a_2$, $\nu(G_2) = n-a_1-a_2+\lfloor \frac{a_2}{2} \rfloor = 6+\lfloor \frac{4}{2} \rfloor = 8$ *as shown in the second cyclic fundamental realization of* D *of Fig. 4.*

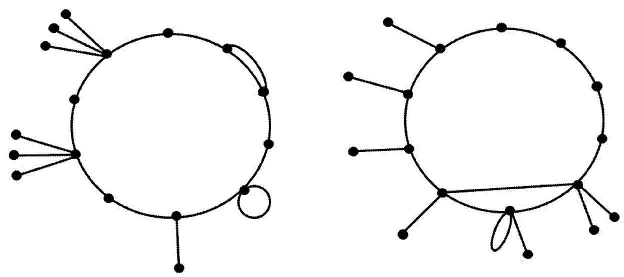

Figure 4: The fundamental realizations of D having minimum and maximum matching numbers

The following result shows that in the special case of $\Omega(D) = 0$, we have a direct formula for maximum matching number amongst all cyclic fundamental realizations of D:

Theorem 3.4. *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ *be a realizable degree sequence. Let* $\Omega(D) = 0$ *and let* G_2 *be the realization of* D *having the maximum matching number amongst all cyclic fundamental realizations of* D*. Then the maximum matching number is obtained as*

$$\nu(G_2) = n - a_1 - a_2 + \lfloor \frac{a_2}{2} \rfloor.$$

Proof. Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. By the definition of omega invariant we know the fact that $\Omega(D) = -a_1 + a_3 + 2a_4 + 3a_5 + \cdots + (\Delta-2)a_\Delta$. Since $\Omega(D) = 0$ it means that $a_1 > a_3 + a_4 + \cdots + a_\Delta$, namely $2a_1 > n - a_2$, so by the Theorem 3.3 we obtain the formula for the maximum matching number between all cyclic fundamental realizations that is

$$\nu(G_2) = n - a_1 - a_2 + \lfloor \frac{a_2}{2} \rfloor.$$

□

Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. If $\Omega(D) = 0$, then it is clear that $\nu(G_1)$ amongst all cyclic fundamental realizations of D corresponds to the Theorem 3.2.

Now, in the case of $\Omega(D) = 0$ we give another theorem related to the matching number by using the next theorem that was given by Delen and Cangul.

Theorem 3.5. *[7] Let G be a graph having the degree sequence $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. If $\Omega(D) = 0$, then G is potentially connected and has at least one cycle. If G is connected, then G has exactly one cycle and the order of the cycle can be every number between 1 and $a_2 + a_3 + \cdots + a_\Delta$, including 1 and $a_2 + a_3 + \cdots + a_\Delta$.*

Theorem 3.6. *Let G be a graph having the degree sequence $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. If $\Omega(D) = 0$, then the matching number of a connected realization of D that has exactly one cycle of length 1, which is a loop, is equal to the maximum matching number amongst all cyclic fundamental realizations of D.*

Proof. Let G be a graph having the degree sequence $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. Firstly, if $\Omega(D) = 0$, then we can obtain a connected realization of D. Let us draw a cycle with order $a_2 + a_3 + \cdots + a_\Delta$ and let us add a pendant edge to each one of a_3 vertices, add two pendant edges to each of a_4 vertices. In this way, finally we add $\Delta - 2$ pendant edges to each of a_Δ vertices. At the end of this adding operation, there are $a_3 + 2a_4 + \cdots + (\Delta - 2)a_\Delta$ pendant edges connected to the cycle with order $a_2 + a_3 + \cdots + a_\Delta$. Therefore, the degree sequence of this realization is

$$\left\{1^{(a_3 + 2a_4 + \cdots + (\Delta-2)a_\Delta)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}.$$

Since $a_3 + 2a_4 + \cdots + (\Delta - 2)a_\Delta - a_1 = 0$, it means that all of the pendant edges are used. Hence degree sequence of this realization is $D(G) = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. Secondly, we delete a vertex from the cycle part that has the smallest degree in the realization and add a new vertex with the same degree with the deleted vertex onto a pendant edge whose incident vertex has the maximum degree in the realization. After that, in the new realization, we delete a vertex in the same way and carry this vertex to the same pendant edge, similarly. We continue this process until reaching up to only one cycle that is loop. Finally, we get a connected realization that consists of a tree part and a loop. As a result, matching number of this connected realization of D is $p(a_1) + \lfloor \frac{a_2}{2} \rfloor$. Since $a_3 + 2a_4 + \cdots + (\Delta - 2)a_\Delta - a_1 = 0$, we get $p(a_1) = a_3 + a_4 + a \cdots + a_\Delta$. Hence, we get the matching number of this connected realization of D as $n - a_1 - a_2 + \lfloor \frac{a_2}{2} \rfloor$ which is the maximum matching number by Theorem 3.4. □

Example 3.3. *Let $D = \{1^{(12)}, 2^{(4)}, 3^{(3)}, 4^{(1)}, 5^{(1)}, 6^{(1)}\}$ with $\Omega(D) = 0$. In Fig. 5, we have two connected realizations of D respectively having a cycle of order 10 and a cycle of order 1. Matching number of first connected realization is equal to $\nu(G_2) = 6 + \lfloor \frac{4}{2} \rfloor = 8$ as shown above. Also, we see that matching number of second connected realization is equal to $\nu(G_2)$. That is, both realizations have the maximum matching numbers amongst all fundamental realizations having a unique cycle of length 10 and length 1, respectively.*

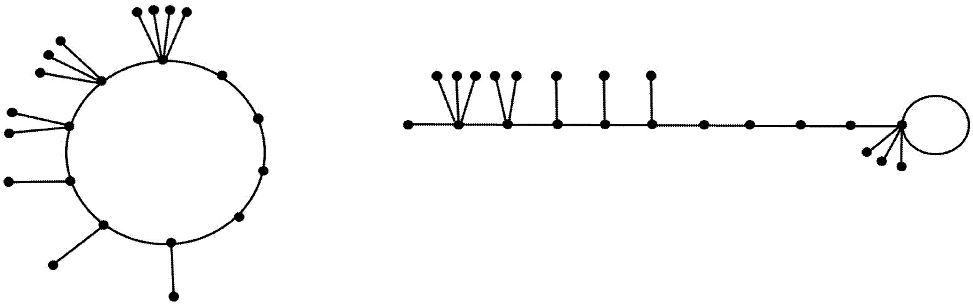

Figure 5: Two connected realizations of D

Theorem 3.7. *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ *be a realizable degree sequence. Let* $\Omega(D) = -2$ *and let the realization of D having the minimum matching number amongst all acyclic fundamental realizations of D be* G_1*. Then*

$$\nu(G_1) = \begin{cases} n - a_1 - a_2 + \lfloor \frac{-n+a_1+2a_2+2}{2} \rfloor & 2a_2 > n - a_1, \\ n - a_1 - a_2 + 1 & 2a_2 = n - a_1, \\ n - a_1 - a_2 & 2a_2 < n - a_1. \end{cases}$$

Proof. Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence with $\Omega(D) = -2$. By the definition of acyclic fundamental realization of D, we have a path with length $a_2 + a_3 + \cdots + a_\Delta + 2$ and a_1 pendant edges that are incident to the vertices of the path. Similar to the proof of Theorem 3.2, we demonstrate the degree sequence of D in a non-decreasing order without multiplicities as $D = \{d_1, d_2, \cdots, d_n\}$ after that we place $d_n - 2$ pendant edges to a vertex, say v_1, $d_{n-1} - 2$ pendant edges to another vertex, say v_2, and we continue up to the vertex $v_{a_3+a_4+\cdots+a_\Delta}$ on the path. For minimal matching number we must minimize the $\nu(G')$ by using Lemma 3.1. In other words, we must choose an acyclic fundamental realization of D that exists maximum number of null graph in G'. Considering this situation, there must add a vertex of degree 2 between every vertex pair on the path as far as possible where already placed pendant edges. Hence, if $a_2 \geq a_3 + a_4 + \cdots + a_\Delta$, then we start applying K_2 deletion process and we get $a_3+a_4+\cdots+a_\Delta+(4-3)a_4+(5-3)a_5+\cdots+(\Delta-3)a_\Delta = a_3+a_4+\cdots+a_\Delta+\sum_{i=4}^{\Delta}(i-3)a_i$ null graphs that $a_3+a_4+\cdots+a_\Delta-1$ of them are vertices of degree 2 in the desired realization, (namely on the path) and a path $P_{a_2-(a_3+a_4+\cdots+a_\Delta)+2}$. Hence, we continue to apply K_2 deletion process on $P_{a_2-(a_3+a_4+\cdots+a_\Delta)+2}$. As a result, we have $\nu(G_1) = a_3+a_4+\cdots+a_\Delta+\lfloor\frac{a_2-(a_3+a_4+\cdots+a_\Delta-1)+1}{2}\rfloor$, namely, we get $\nu(G_1) = a_3 + a_4 + \cdots + a_\Delta + \lfloor\frac{a_2-(a_3+a_4\cdots+a_\Delta)+2}{2}\rfloor$. For equality $a_2 = a_3 + a_4 + \cdots + a_\Delta$, it is clear that we have $\nu(G_1) = a_3 + a_4 + \cdots + a_\Delta + 1$. Finally, in the case of $a_2 < a_3 + a_4 + \cdots + a_\Delta$, after the K_2 deletion process G' only have null graphs without any path because of the deficiency of a_2 so the result is $\nu(G_1) = a_3 + a_4 + \cdots + a_\Delta$. Consequently we complete the proof by using $a_3 + a_4 + \cdots + a_\Delta = n - a_1 - a_2$. □

Theorem 3.8. *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence. Let* $\Omega(D) = -2$ *and let the realization of* D *having the maximum matching number amongst all acyclic fundamental realizations of* D *be* G_2*. Then*

$$\nu(G_2) = n - a_1 - a_2 + \lfloor \frac{a_2 + 2}{2} \rfloor.$$

Note that $n - a_1 - a_2 = a_3 + a_4 + a_5 \cdots + a_\Delta$ which is the total number of non-pendant vertices of degree at least three.

Proof. Let us consider a realizable degree sequence $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$. If $\Omega(D) = -2$, a acyclic fundamental realization of D is constructed a path with length of $a_2 + a_3 + \cdots + a_\Delta + 2$ and a_1 pendant edges that are incident to the vertices of the path. As mentioned before, when $\nu(G')$ is maximum, matching number $\nu(G)$ of G reaches the maximum value by the Lemma 3.1. Therefore, we must set an acyclic fundamental realization of D in the way of existing minimum number of null graphs in G'. To do this, contrary to previous theorem, we add pendant edges consequtively to the path without existing any vertex of degree two between them. Also, we start adding pendant edges from third vertex of the path in order not to get more null graphs in G'. Since $\Omega(G) = -a_1 + a_3 + 2a_4 + 3a_5 + \cdots + a_\Delta = -2$, there are $a_1 - 2$ pendant edges incident to the path. By the definition of an acyclic fundamental realization of D has $a_3 + a_4 + \cdots + a_\Delta$ vertex having at least one pendant edge that is adjacent to path. Thus, we start applying the K_2 deletion process, then we have $\sum_{i=4}^{\Delta}(i-3)a_i$ null graphs, one edge, and a path P_{a_2}. Consequently, we continue to K_2 deletion process and we get the result $\nu(G_2) = a_3 + a_4 + \cdots + a_\Delta + \lfloor \frac{a_2+2}{2} \rfloor$. Hence, we get required result. □

Theorem 3.9. *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence. Let* $\Omega(D) \leq -4$ *and let the realization of* D *having the minimum matching number amongst all mixed fundamental realizations of* D *be* G_1*. Then*

$$\nu(G_1) = \begin{cases} n - a_1 - a_2 + \lfloor \frac{2a_2 - n + a_1 + 1}{2} \rfloor - \frac{\Omega(D)}{2} & n - a_1 < 2a_2, \\ n - a_1 - a_2 - \frac{\Omega(D)}{2} & n - a_1 = 2a_2, \\ n - a_1 - a_2 - \frac{\Omega(D)}{2} & n - a_1 > 2a_2. \end{cases}$$

Proof. Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ be a realizable degree sequence and $\Omega(D) \leq -4$. A mixed fundamental realization of D is formed as $\frac{-\Omega(D)}{2}$ times K_2 complete graphs and a cycle with length $a_2 + a_3 + \cdots a_\Delta$ by the definition of mixed fundamental realization of D. Since $\frac{-\Omega(D)}{2}$ is fixed number and matching number of a mixed fundamental realization of D is added by one for every K_2 complete graphs. Thus, we consider the cyclic part of the realization for minimality of matching number. To do this, in the cyclic part we deal with the degree sequence $D^c = \left\{1^{(a_1 + \Omega(D))}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ because of the added K_2 complete graphs. By Theorem 3.2, the minimum matching number $\nu(G_1)$ amongst all mixed fundamental realizations of D

is

$$\nu(G_1) = \begin{cases} p(a_1+\Omega(D)) + \lfloor \frac{a_2+a_3+\cdots+a_\Delta-(2p(a_1+\Omega(D))-1)}{2} \rfloor - \frac{\Omega(D)}{2} & p(a_1+\Omega(D)) < a_2, \\ p(a_1+\Omega(D)) + \lfloor \frac{a_3+\cdots+a_\Delta-p(a_1+\Omega(D))+1}{2} \rfloor - \frac{\Omega(D)}{2} & p(a_1+\Omega(D)) = a_2, \\ p(a_1+\Omega(D)) + \lfloor \frac{a_3+\cdots+a_\Delta-p(a_1+\Omega(D))}{2} \rfloor - \frac{\Omega(D)}{2} & p(a_1+\Omega(D)) > a_2. \end{cases}$$

Note that, by the definition of omega invariant $\Omega(D^c) = -(a_1+\Omega(D)) + \sum_{i=3}^{\Delta}(i-2)a_i = 0$ so it means that $p(a_1+\Omega(D)) = a_3+a_4+\cdots+a_\Delta$ and since $a_3+a_4+\cdots+a_\Delta = n-a_1-a_2$ the proof is completed by placing $p(a_1+\Omega(D))$ in $\nu(G_1)$. $\square$

Theorem 3.10. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. Let $\Omega(G) \leq -4$ and let the realization of D having the maximum matching number amongst all mixed fundamental realizations of D be G_2. Then*

$$\nu(G_2) = n - a_1 - a_2 + \lfloor \frac{a_2}{2} \rfloor - \frac{\Omega(D)}{2}.$$

Proof. Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence. If $\Omega(D) \leq -4$, then we get a cycle with length $a_2+a_3+\cdots a_\Delta$ and $\frac{-\Omega(D)}{2}$ times K_2 complete graphs. In this situation, omega value of the degree sequence $D^c = \{1^{(a_1+\Omega(D))}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ that we attend the draw cyclic part according to it equals to 0 since $\Omega(D^c) = -(a_1+\Omega(D)) + a_3 + 2a_4 + \cdots + (\Delta-2)a_\Delta$. By Theorem 3.4, we obtain the result

$$\nu(G_2) = n - a_1 - a_2 + \lfloor \frac{a_2}{2} \rfloor - \frac{\Omega(D)}{2}.$$

$\square$

Example 3.4. *The degree sequence of graph G is $D = \{1^{(15)}, 2^{(3)}, 3^{(1)}, 4^{(1)}, 5^{(2)}\}$*

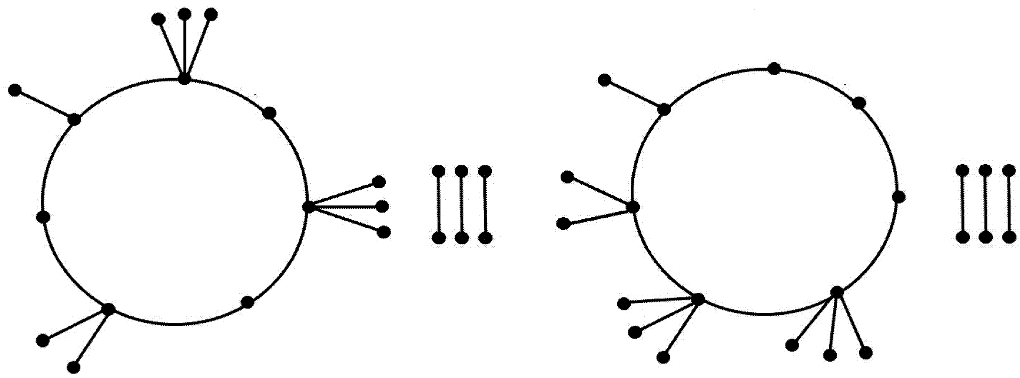

Figure 6: A graph G with $\Omega(G) = -6$

In Fig. 6, first graph is formed for $\nu(G_1)$ *and second graph is formed for* $\nu(G_2)$. $n = a_1 + a_2 + a_3 + a_4 + a_5 = 22$ *and since* $n - a_1 > 2a_2$*, by Theorem 3.9*

$$\begin{aligned}\nu(G_1) &= n - a_1 - a_2 - \tfrac{\Omega(D)}{2} \\ &= 4 - \tfrac{-6}{2} = 7.\end{aligned}$$

Let us calculate $\nu(G_2)$ *by using Theorem 3.10 as follows:*

$$\begin{aligned}\nu(G_2) &= n - a_1 - a_2 + \lfloor \tfrac{a_2}{2} \rfloor - \tfrac{\Omega(D)}{2} \\ &= 4 + \lfloor \tfrac{3}{2} \rfloor - \tfrac{-6}{2} = 8.\end{aligned}$$

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