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CERTAIN SEQUENCE SPACES USING △-OPERATOR

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Abstract. The aim of this paper is to introduce the new type of generalized B- difference sequence spaces by the combination of binomial mappings and the △ operator. We also study their topological properties.

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1. Preliminaries, Background and Notation

Sequence space is referred to be a function space with entries as functions from positive numbers $\mathbb{N}$ to the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ the complex numbers. The set of every sequences (real or complex) will be given symbol as Υ. The bounded sequences, p-absolutely sequence, convergent sequences and null sequences will be symbolized by l_∞, l_p, c and c_0 respectively as in [20].

A linear Topological space G over $\mathbb{R}$ is called a paranormed space if there exists a sub-additive function $\Gamma : G \to \mathbb{R}$ so that $\Gamma(\theta) = 0$, $\Gamma(-\zeta) = \Gamma(\zeta)$ and scalar multiplication is continuous, i.e,, $|\xi_n - \xi| \to 0$ and $\Gamma(\zeta_n - \zeta) \to 0$ imply $\Gamma(\xi_n\zeta_n - \xi\zeta) \to 0 \; \forall \; \xi's$ in $\mathbb{R}$ and $\zeta's$ in G , where θ denotes the zero element in G. For spaces G and H, set

$$\Psi(G : H) = \{\kappa = (\kappa_i) : \zeta\kappa = (\zeta_i\kappa_i) \in H \; \forall \; \zeta = (\zeta_i) \in G\}. \tag{1}$$

By (1), we re-write the α-, β- and γ- duals of G as follows;

$$G^\alpha = \Psi(G : l_1),\; G^\beta = \Psi(G : cs) \; and \; G^\gamma = \Psi(X : bs).$$

As in [14], if space G paranormed by Γ admits a sequence $(\wp_n)$ with the character that for all $g \in G$ there exists one and only one sequence of scalars (ξ_n) in such a way that

$$\lim_n \Gamma(g - \sum_{k=0}^{n} \xi_k \wp_k) = 0,$$

then $(\wp_n)$ defines a Schauder basis for G. Here $\sum \xi_k\wp_k$ having the sum as g is then known as the expansion of g w.r.t. $(\wp_n)$ and it is expressed as $g = \sum \xi_k \wp_k$.

For the matrix $\mathcal{C} = (c_{i,j})$ and $\nu = (\nu_k) \in \Upsilon$, the $\mathcal{C}$-transform of ν is given by $\mathcal{C}\nu = \{(\mathcal{C}\nu)_i\}$ for if it survives (i.e., it does not diverges) $\forall \; i \in \mathbb{N}$, where $(\mathcal{C}\nu)_i = \sum_{j=0}^{\infty} c_{i,j}\nu_j$.

For such a matrix $\mathcal{C} = (c_{i,j})$, the set $G_{\mathcal{C}}$, where

$$G_{\mathcal{C}} = \{\nu = (\nu_j) \in \Upsilon \ : \ \mathcal{C}\nu \in G\}, \tag{2}$$

is referred as the domain/region of $\mathcal{C}$ in G. As in [1], [4], [17], we shall designate all such classes by $(G : H)$ with $G \subseteq H_{\mathcal{C}}$

In[9] the author has constructed new techniques and introduced the following spaces:

$$T(\Delta) = \{\rho = (\rho_i) : \Delta\rho \in T\},$$

where $T = \{l_\infty, c, \ c_0\}$ and which was further analysed as in [6], [10], [12], [15], [16].

Let $a, \ b \in \mathbf{R}$ with $a + b \neq 0$, then recently Bişign [2] introduced and studied the matrix $B^{a,b} = \left(\vartheta^{a,b}_{n,k}\right)$ and is defined as follows:

$$\vartheta^{a,b}_{n,k} = \begin{cases} \frac{1}{(a+b)^n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k & \text{if } \ 0 \leq k \leq n, \\ \\ 0 & \text{if } k > n, \end{cases}$$

for all $n, \ k \in \mathbf{N}$. It is obvious that for $ab > 0$, we have

$$(i) \ \|B^{a,b}\| < \infty$$

$$(ii) \ \lim_{n\to\infty} \vartheta^{a,b}_{n,k} = 0 \ \forall \ k \in \mathbf{N}$$

$$(iii) \ \lim_{n\to\infty} \sum_k \vartheta^{a,b}_{n,k} = 1.$$

Hence, we conclude that the binomial matrix $B^{a,b}$ is regular for $ab > 0$ [11], [13], [14], [18].

Quite recently in [2] we have the following sequence spaces:

$$\vartheta^{a,b}_p = \left\{ \zeta = (\zeta_k) \in \Upsilon : \ \sum_n \left| \frac{1}{(a+b)^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k \zeta_k \right|^p < \infty \right\}$$

and

$$\vartheta^{a,b}_\infty = \left\{ \zeta = (\zeta_k) \in \Upsilon : \ \sup_n \left| \frac{1}{(a+b)^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k \zeta_k \right| < \infty \right\}.$$

An approach of constructing of a new sequence space by means of the matrix domain of a particular limitation method was used by various authors viz., Başar [1], M. Başarir [2], Bişgin [3], Ganie et al [4] - [6], [17]-[19], Jarrah [8], Mursaleen [11]-[12], Ng and Lee [14] and many more. Following the authors cited, we would like to introduce the binomial difference sequence spaces $B^{a,b}_p(\Delta_u)$ and $B^{a,b}_\infty(\Delta_u)$, whose $B^{a,b}(\Delta)$-transforms are respectively in the spaces ℓ_p and ℓ_∞ and $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbf{N}$.

2. The difference sequence spaces $\vartheta^{a,b}_p(\Delta_u)$ and $\vartheta^{a,b}_\infty(\Delta_u)$

In this section, we have introduced the spaces $\vartheta^{a,b}_p(\Delta_u)$ and $\vartheta^{a,b}_\infty(\Delta_u)$ and show that these spaces are BK-spaces.

We define the spaces $\vartheta_p^{a,b}(\Delta_u)$ and $\vartheta_\infty^{a,b}(\Delta_u)$ as follows:

$$\vartheta_p^{a,b}(\Delta_u) = \left\{\zeta = (\zeta_k) \in \Upsilon : \ (\Delta_u \zeta_k) \in \vartheta_p^{a,b}\right\}$$

and

$$\vartheta_\infty^{a,b}(\Delta_u) = \left\{\zeta = (\zeta_k) \in \Upsilon : \ (\Delta_u \zeta_k) \in \vartheta_\infty^{a,b}\right\}.$$

By the definition of matrix domain (2), we re-define these spaces as follows:

$$b_p^{a,b}(\Delta_u) = \left(\vartheta_p^{a,b}\right)_{\Delta_u} \text{ and } \vartheta_\infty^{a,b}(\Delta_u) = \left(\vartheta_\infty^{a,b}\right)_{\Delta_u}.$$

For each $n \in \mathbf{N}$, we define the sequence $y = \{y_n\}$, which will be frequently used as the $B^{a,b}\Delta_u$-transform of a sequence $\zeta = \{\zeta_k\}$, i.e.,

$$(3) \qquad y_n = \left[B^{a,b}\left(\Delta_u \zeta_k\right)\right]_n = \frac{1}{(a+b)^n}\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^k \left(\Delta_u \zeta_k\right).$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The spaces $\vartheta_p^{a,b}(\Delta_u)$ and $\vartheta_\infty^{a,b}(\Delta_u)$ are BK- spaces with their norm defined by*

$$f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta) = \|y\|_p = \left(\sum_{k=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

and

$$f_{\vartheta_\infty^{a,b}(\Delta_u)}(\zeta) = \|y\|_\infty = \sup_{n \in \mathbf{N}} |y_n|,$$

where $1 \leq p < \infty$ and the sequence $y = (y_n)$ is defined by the $B^{a,b}(\Delta_u)$-transform of ζ.

Proof : The trivial part is linearity. Further, it is clear that $f_{\vartheta_p^{a,b}(\Delta_u)}(\beta\zeta) = \beta f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta)$ and $f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta) = 0$ if and only if $\zeta = \theta$ for all $\zeta \in f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta)$, where θ is the zero element of $f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta)$ and $\beta \in \mathbf{R}$.

Now for any $\zeta, \ \eta \in f_{\vartheta_p^{a,b}(\Delta_u)}$, we have

$$\begin{aligned} f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta + \eta) &= \left(\sum_n \left|\left(B^{a,b}\left[\Delta_u(\zeta_k + \eta_k)\right]\right)_n\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_n \left|\left(B^{a,b}\left[\Delta_u \zeta_k\right]\right)_n\right|^p\right)^{\frac{1}{p}} + \left(\sum_n \left|\left(B^{a,b}\left[\Delta_u \eta_k\right]\right)_n\right|^p\right)^{\frac{1}{p}} \\ &= f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta) + f_{\vartheta_p^{a,b}(\Delta_u)}(\eta). \end{aligned}$$

This shows that the $f_{\vartheta_p^{a,b}(\Delta_u)}$ is a norm on the space $\vartheta_p^{a,b}(\Delta_u)$.

Now let (ζ_i) be a Cauchy sequence in $\vartheta_p^{a,b}(\Delta_u)$, where $\zeta_i = \{\zeta_{i_k}\}_{k=1}^{\infty}$ for each $i \in \mathbb{N}$. Then for every $\epsilon > 0$, there exists a positive integer i_0 such that

$$f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta_i - \zeta_j) < \epsilon \text{ for } \ i, \ j \geq i_0.$$

Therefore, we have

$$\left|\left(B^{a,b}\left[\Delta_u\left(\zeta_{i_k}-\zeta_{j_k}\right)\right]\right)_n\right| \leq \left(\sum_n \left|\left(B^{a,b}\left[\Delta_u\left(\zeta_{i_k}-\zeta_{j_k}\right)\right]\right)_n\right|^p\right)^{\frac{1}{p}} < \epsilon,$$

for every $i, j \geq i_0$ and for each $k \in \mathbb{N}$. This shows that $\left(B^{a,b}\left(\Delta_u \zeta_{i_k}\right)\right)_{i=1}^{\infty}$ is a Cauchy sequence in the set of real numbers $\mathbb{R}$. But $\mathbb{R}$ is complete, therefore, we have

$$\lim_{i\to\infty} B^{a,b}\left(\Delta_u \zeta_{i_k}\right) = B^{a,b}\left(\Delta_u \zeta_k\right)$$

for each $k \in \mathbb{N}$. Thus,

$$\sum_{n=0}^{m}\left|\left(B^{a,b}\left[\Delta_u\left(\zeta_{i_k}-\zeta_{j_k}\right)\right]\right)_n\right| \leq f_{\vartheta_p^{a,b}(\Delta_u)}\left(\zeta_i-\zeta_j\right) < \epsilon, \tag{4}$$

for every $i \geq i_0$. Now letting m and $j \to \infty$, then from (4), we have

$$f_{\vartheta_p^{a,b}(\Delta_u)}\left(\zeta_i-\zeta\right) \to 0.$$

$$\therefore\ f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta) \leq f_{\vartheta_p^{a,b}(\Delta_u)}\left(\zeta_i-\zeta\right) + f_{\vartheta_p^{a,b}(\Delta_u)}\left(\zeta_i\right) < \infty,$$

that is $x \in \vartheta_p^{a,b}(\Delta_u)$. This shows the completeness of the space $\vartheta_p^{a,b}(\Delta_u)$ and the proof is complete.

In a similar fashion, the space $\vartheta_\infty^{a,b}(\Delta_u)$ is to be shown as complete.$\diamond$

Theorem 2.2. *The space $\vartheta_p^{a,b}(\Delta_u)$ and $\vartheta_\infty^{a,b}(\Delta_u)$ are linearly isomorphic to the space ℓ_p and ℓ_∞ respectively, i.e., $\vartheta_p^{a,b}(\Delta_u) \cong l_p$ and $\vartheta_\infty^{a,b}(\Delta_u) \cong l_\infty$; where $1 \leq p < \infty$.*

Proof : To establish the result we shall consider $\vartheta_p^{a,b}(\Delta_u) \cong l_p$ and rest will follow by similar fashion.

Thus, to show get the result, we must show that the existence of a linear bijection between the space $\vartheta_p^{a,b}(\Delta_u)$ and l_p for $1 \leq p < \infty$. Consider the transformation $T : \vartheta_p^{a,b}(\Delta_u) \to l_p$ defined by $T\zeta = B^{a,b}\left(\Delta_u \zeta_k\right)$.

The linearity of T is trivial. Further, it is trivial that $\zeta = \theta$ whenever $T\zeta = \theta$ and hence T is injective.

Let $y = (y_n) = l_p$ for $1 \leq p < \infty$ and define the sequence $\zeta = (\zeta_k)$ by

$$\zeta_k = \sum_{j=0}^{k}(a+b)^j \sum_{r=j}^{k} b^{-r}(-a)^{r-j} u_j^{-1} y_j \text{ for each } k \in \mathbb{N}. \tag{5}$$

Therefore, we have

$$\begin{aligned} f_{\vartheta_p^{a,b}(\Delta_u)}(\zeta) &= \left\| \left[B^{a,b} \left(\Delta_u \zeta_k \right) \right]_n \right\| \\ &\leq \left[\sum_{n=1}^{\infty} \left| \frac{1}{(a+b)^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k \left(\Delta_u \zeta_k \right) \right|^p \right]^{\frac{1}{p}} \\ &= \left[\sum_{n=1}^{\infty} |y_n|^p \right]^{\frac{1}{p}} \\ &= \|y\|_p < \infty. \end{aligned}$$

Hence, we see that $\zeta \in \vartheta_p^{a,b}(\Delta_u)$ and $T(\zeta) = y$. Consequently, T is surjective and is norm preserving, where $1 \leq p < \infty$. Hence, T is linear bijection which implies that the spaces $\vartheta_p^{a,b}(\Delta_u)$ and l_p are linearly isomorphic for $1 \leq p < \infty$.$\diamond$

3. Schauder basis

In this section, we compute the Schauder basis of the space $\vartheta_p^{a,b}(\Delta_u)$ for $1 \leq p < \infty$.

Theorem 3.1. *Define the sequence* $g^{(k)}(a,b) = \{g_i^{(k)}(a,b)\}_{n\in\mathbb{N}}$ *of the elements of the space* $\vartheta_p^{a,b}(\Delta_u)$ *for each* $k \in \mathbb{N}$ *as*

$$g_i^{(k)}(a,b) = \begin{cases} 0 & \text{if } 0 \leq i < k, \\ (a+b)^k \sum_{j=k}^{i} \left(\begin{array}{c} j \\ k \end{array} \right) b^{-j} (-a)^{j-k} u^{-j} & \text{if } i \geq k, \end{cases}$$

for each $k \in \mathbb{N}$. *Then, the sequence* $\{g^{(k)}(a,b)\}_{k\in\mathbb{N}}$ *is a Schauder basis for the space* $\vartheta_p^{a,b}(\Delta_u)$ *and any* $\zeta = (\zeta_i) \in \vartheta_p^{a,b}(\Delta_u)$ *has a unique representation of the form*

$$\zeta = \sum_k \mu_k g^{(k)}(a,b), \tag{6}$$

where $\mu_k(a,b) = \left(B^{a,b}(\Delta_u \zeta_i)\right)_k$ *for all* $k \in \mathbb{N}$ *and* $1 \leq p < \infty$.

Proof : It is a clear that $B^{a,b}\left(\vartheta_p^{a,b}(\Delta_u)\right) = e^{(k)} \in l_p$, where $e^{(k)}$ is the sequence whose only non-zero term is a 1 at the kth place for each $k \in \mathbb{N}$. This implies that $g^{(k)}(a,b) \in \vartheta_p^{a,b}(\Delta_u)$ for each $k \in \mathbb{N}$.

Let $\zeta \in \vartheta_p^{a,b}(\Delta_u)$ be given. For every non-negative integer m, we define

$$\zeta^{[m]} = \sum_{k=0}^{m} \mu_k(a,b) g^{(k)}(a,b). \tag{7}$$

Then, we obtain by applying $B^{a,b}\left(\vartheta_p^{a,b}(\Delta_u)\right)$ to (7) with (6) that

$$B^{a,b}\left(\vartheta_p^{a,b}(\Delta_u\zeta_i^{[m]})\right) = \sum_{k=0}^{m} \mu_k(a,b) B^{a,b}\left(\Delta g_i^{(k)}(a,b)\right) = \sum_{k=0}^{m} \mu_k(a,b) e^{(k)}$$

and

$$\left\{B^{a,b}\left(\Delta_u\left(\zeta_i - \zeta_i^{[m]}\right)\right)\right\}_k = \begin{cases} 0, & \text{if } 0 \le k \le m, \\ \left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k, & \text{if } k > m, \end{cases}$$

for all $k \in \mathbb{N}$.

Given $\epsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{k=m_0+1}^{\infty} \left|\left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k\right|^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{2},$$

for all $m \ge m_0$. Hence,

$$f_{\vartheta_p^{a,b}(\Delta_u)}\left(\zeta - \zeta^{[m]}\right) = \left(\sum_{k=m+1}^{\infty} \left|\left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=m_0+1}^{\infty} \left|\left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k\right|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{2} < \epsilon,$$

for all $m \ge m_0$, which proves that $x \in \vartheta_p^{a,b}(\Delta_u)$ is represented as (6).

Let us show the uniqueness of this representation. Suppose, on the contrary; that there exists another representation $\zeta = \sum_k \lambda_k(a,b) g^k(a,b)$. Since the linear transformation T from $\vartheta_p^{a,b}(\Delta_u)$ to l_p used in the Theorem B above is continuous, we have

$$\left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k = \sum_k \lambda_k(a,b)\left[B^{a,b}\left(\Delta_u g_i^{(k)}(a,b)\right)\right]_k = \sum_k \lambda_k(a,b)\left(e^(k)\right)_k = \lambda_k(a,b).$$

which contradicts the fact that $\left[B^{a,b}\left(\Delta_u\zeta_i\right)\right]_k = \mu_k(a,b)$ for every $k \in \mathbb{N}$. Hence, the representation (6) is unique. This completes the proof.

4. Köthe duals

In this section, we compute the α-, β- and γ-duals of the spaces $\vartheta_p^{a,b}(\Delta_u)$ and $\vartheta_\infty^{a,b}(\Delta_u)$ for $1 \le p < \infty$.

For use in the lemma 4.1 below, we now give some properties:

$$\sup_{n\in\mathbb{N}}\sum_k |a_{n,k}|^q < \infty. \tag{8}$$

$$\sup_{n\in\mathbb{N}}\sum_n |a_{n,k}| < \infty. \tag{9}$$

$$\sup_{n,k\in\mathbb{N}} |a_{n,k}| < \infty. \tag{10}$$

$$\lim_{n\to\infty} a_{n,k} = a_k. \tag{11}$$

$$\sup_{K\in\mathbb{F}}\sum_k \left|\sum_{n\in K} a_{n,k}\right|^q < \infty. \tag{12}$$

$$\lim_{n\to\infty}\sum_k |a_{n,k}| = \sum_k \left|\lim_{n\to\infty} a_{n,k}\right|, \tag{13}$$

where $\mathbb{F}$ is the collection of all finite subsets of $\mathbb{N}$; $\frac{1}{p}+\frac{1}{q}=1$ and $1\le p<\infty$.

Lemma 4.1. *As in [7], for an infinite matrix $A=(a_{nk})$, we have following statements that are essential in further part of the text.*

(i) $A\in(l_1,\ l_1)$ if and only if (9) holds.

(ii) $A\in(l_1,\ c)$ if and only if (10) and (11) holds.

(iii) $A\in(l_1,\ l_\infty)$ if and only if (10) holds.

(iv) $A\in(l_p,\ l_1)$ if and only if (12) holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\le\infty$.

(v) $A\in(l_p,\ c)$ if and only if (8) and (11) holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.

(vi) $A\in(l_p,\ l_\infty)$ if and only if (8) holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.

(vii) $A\in(l_\infty,\ c)$ if and only if (11) and (13) holds with $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$.

(viii) $A\in(l_\infty,\ l_\infty)$ if and only if (8) holds with $q=1$.

Theorem 4.2. *Let $W_1^{a,b}$ and $W_2^{a,b}$ be defined as follows:*

$$W_1^{a,b}=\left\{t=(t_k)\in\Upsilon:\ \sup_{i\in\mathbb{N}}\sum_k\left|(a+b)^i\sum_{j=i}^{k}\begin{pmatrix} j\\ i\end{pmatrix}b^{-j}(-a)^{j-i}u_k^{-1}t_k\right|<\infty\right\}$$

and

$$W_2^{a,b}=\left\{t=(t_k)\in\Upsilon:\ \sup_{K\in\mathbb{F}}\sum_i\left|\sum_{k\in\mathbb{F}}(a+b)^i\sum_{j=i}^{k}\begin{pmatrix} j\\ i\end{pmatrix}b^{-j}(-a)^{j-i}u_k^{-1}t_k\right|^q<\infty\right\}.$$

Then $\left[\vartheta_1^{a,b}(\Delta_u)\right]^\alpha=W_1^{a,b}$ and $\left[\vartheta_p^{a,b}(\Delta_u)\right]^\alpha=W_2^{a,b}$, where $1<p\le\infty$.

Proof: Let $t=(t_n)\in\Upsilon$ be given and $\zeta=(\zeta_k)$ as defined by (5), so we can write

$$\begin{aligned} t_k\zeta_k &= \sum_{i=0}^{k}(a+b)^i\sum_{j=i}^{k}b^{-j}(-a)^{j-i}u_i^{-1}t_ky_j \\ &= \left(H^{a,b}y\right)_k \end{aligned}$$

for each $k\in\mathbb{N}$ and $H^{a,b}=\left(h_{k,i}^{a,b}\right)$ is given by

$$h_{k,i}^{a,b}=\begin{cases} (a+b)^i\sum_{j=i}^{k}\begin{pmatrix} j\\ i\end{pmatrix}b^{-j}(-a)^{j-i}u_i^{-1}t_k & \text{if } \ 0\le i\le k,\\ 0, & \text{if } \ i>k. \end{cases}$$

Therefore, we deduce that $tx=(t_k\zeta_k)\in l_1$ whenever $\zeta=(\zeta_k)\in b_1^{a,b}(\Delta_u)$ or $\zeta=(\zeta_k)\in b_p^{a,b}(\Delta_u)$ if and only if $H^{a,b}y\in l_1$ whenever $y=(y_k)\in l_1$ or $y=(y_k)\in l_p$, respectively. This shows that $t=(t_k)\in\left[b_1^{a,b}(\Delta_u)\right]^{\alpha}$ or $t=(t_k)\in\left[b_p^{a,b}(\Delta_u)\right]^{\alpha}$ if and only if $H^{a,b}\in(l_1,l_1)$ or $H^{a,b}\in(l_p,l_1)$, respectively, where $1<p\le\infty$. If we combine these two facts and utlizing (i) and (iv) of Lemma 4.1, we see

$$t=(t_k)=\left[b_1^{a,b}(\Delta_u)\right]^{\alpha} \ iff \ \sup_{i\in\mathbb{N}}\sum_k\left|(a+b)^i\sum_{j=i}^{k}\begin{pmatrix} j\\ i\end{pmatrix}b^{-j}(-a)^{j-i}u_k^{-1}t_k\right|<\infty$$

or

$$t=(t_k)=\left[b_p^{a,b}(\Delta_u)\right]^{\alpha}\Leftrightarrow\sup_{K\in\mathbb{F}}\sum_i\left|\sum_{k\in\mathbb{F}}(a+b)^i\sum_{j=i}^{k}\begin{pmatrix} j\\ i\end{pmatrix}b^{-j}(-a)^{j-i}u_k^{-1}t_k\right|^q<\infty,$$

respectively, where $1<p\le\infty$. Therefore, we have $\left[\vartheta_1^{a,b}(\Delta_u)\right]^{\alpha}=W_1^{a,b}$ and $\left[\vartheta_p^{a,b}(\Delta_u)\right]^{\alpha}=W_2^{a,b}$, where $1<p\le\infty$ and the proof is complete.$\diamond$

Theorem 4.3. *Let $W_3^{a,b}$, $W_4^{a,b}$, $W_5^{a,b}$, $W_6^{a,b}$ and $W_7^{a,b}$ be defined as follows:*

$$W_3^{a,b} = \{t = (t_k) \in \Upsilon : \lim_{n\to\infty} (a+b)^n \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-k} u_j^{-1} t_j \text{ exists for each } k \in \mathbb{N}\}$$

$$W_4^{a,b} = \left\{ t = (t_k) \in \Upsilon : \sup_{n,k\in\mathbb{N}} \left| (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_j^{-1} t_j \right| < \infty \right\}$$

$$W_5^{a,b} = \{t = (t_k) \in \Upsilon : \lim_{n\to\infty} \sum_k \left| (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_j^{-1} t_j \right| = \sum_k \left| \lim_{n\to\infty} (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_j^{-1} t_j \right| \}$$

$$W_6^{a,b} = \left\{ t = (t_k) \in \Upsilon : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_j^{-1} t_j \right|^q < \infty \right\}, \quad 1 < q < \infty$$

and

$$W_7^{a,b} = \left\{ t = (t_k) \in \Upsilon : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_j^{-1} t_j \right| < \infty \right\}.$$

Then

(i) $\left[\vartheta_1^{a,b}(\Delta_u)\right]^\beta = W_3^{a,b} \cap W_4^{a,b}$.

(ii) $\left[\vartheta_p^{a,b}(\Delta_u)\right]^\beta = W_3^{a,b} \cap W_6^{a,b},\ 1 < p < \infty$.

(iii) $\left[\vartheta_\infty^{a,b}(\Delta_u)\right]^\beta = W_3^{a,b} \cap W_5^{a,b}$.

(iv) $\left[\vartheta_1^{a,b}(\Delta_u)\right]^\gamma = W_4^{a,b}$.

(v) $\left[\vartheta_p^{a,b}(\Delta_u)\right]^\gamma = W_6^{a,b},\ 1 < p < \infty$.

(vi) $\left[\vartheta_\infty^{a,b}(\Delta_u)\right]^\gamma = W_7^{a,b}$.

Proof: Let $t = (t_n) \in \Upsilon$ be given and $\varsigma = (\varsigma_k)$ as defined by (5), so we can write

$$\begin{aligned}\sum_{k=0}^{n} t_k \varsigma_k &= \sum_{k=0}^{n} t_k \left(\sum_{i=0}^{k} (a+b)^i \sum_{j=i}^{k} \binom{j}{k} b^{-j}(-a)^{j-i} u_i^{-1} y_i \right) \\ &= \sum_{k=0}^{n} \left((a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} b^{-j}(-a)^{j-i} u_i^{-1} t_i \right) y_k \\ &= \left(Z^{a,b}\right)_n,\end{aligned}$$

where $Z^{a,b} = \left(z_{n,k}^{a,b}\right)$ is given by

$$z_{n,k}^{a,b} = \begin{cases} (a+b)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \begin{pmatrix} j \\ k \end{pmatrix} b^{-j}(-a)^{j-i} u_i^{-1} t_i & \text{if} \quad 0 \leq k \leq n, \\ 0, & \text{if} \quad k > n. \end{cases}$$

Therefore, we deduce that $t\varsigma = (t_k \varsigma_k) \in c$ if and only if $Z^{a,b} y \in c$ whenever $y \in l_1$, which implies that $t = (t_k) \in \left[\vartheta_1^{a,b}(\Delta_u)\right]^{\beta}$ if and only if $Z^{a,b} \in (l_1, c)$. Therefore, by lemma 4.1 (ii), we have $\left[\vartheta_1^{a,b}(\Delta_u)\right]^{\beta} = W_3^{a,b} \cap W_4^{a,b}$.

By similar fashion, instead of using (ii) of lemma 4.1, we employ (i) and (iii)-(viii), the proof follows.◇

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References

[1] F. Başar and B. Altay, *On the spaces of p-bounded variation and related matrix mappings*, Ukrainian Math. J.,1(2003), 136-147.

[2] M. Başarir, E. E. Kara, *On some difference sequence spaces of weighted means and compact operators,* Ann. Funct. Anal., 2(2011), 114-129.

[3] M. C. Bişgin, *The binomial sequence spaces of nonabsolute type,* J. Inequal. Appl. 2016, 309 (2016).

[4] A. H. Ganie, *Some new difference sequence space of non-absolute type,* Int. J. Math. & computational Methods, 1 (2016), 48-57.

[5] A. H. Ganie and N. A. Sheikh, *On some new sequence space of non-absolute type and matrix transformations,* J. Ejyp. Math. Soc., 21(2013), 34-40.

[6] Ab. Hamid Ganie, A. Mobin, N. A. Sheikh, T. Jalal and S. A. Gupkari, *Some new type of difference sequence space of non-absolute type*, Int. J. Modern Math. Sci., 14(1)(2016), 116-122.

[7] T. Hubert and S. Michael, *Matrixtransformationen von folgenräumen eine ergebnisübersicht*, Math. Z. 154(1977), 1-16.

[8] A. M. Jarrah and E. Malkowsky, *Ordinary, absolute and strong summability and matrix transformations,* Filomat, 17 (2003), 59-78.

[9] H Kizmaz, *On certain sequence spaces. Can. Math. Bull.*, 24(1981), 169-176.

[10] A. Manna, A. Maji and P.D.Srivastava, *Difference sequence spaces derived by using generalized means,* 23 (2015)-127-133.

[11] M. Mursaleen, *Infinite matrices and almost convergent sequences, Southeast Asian Bulletin of Math.*, 19(1)(1995), 45-48.

[12] M. Mursaleen and A. K. Noman, *On some new difference sequence spaces of non-absolute type,* Math. Comp. Model., 52(2010), 603-617.

[13] P.-N. Ng, . and P.-Y. Lee, *Cesàro sequences spaces of non-absolute type,* Comment Math. Prace Math., 2(1978), 429-433.

[14] G. M. Petersen, *Regular matrix transformations,* Mc Graw-Hill, London, (1966).

[15] H. Polat, V. Karakaya and N. Şimşek, *Difference sequence spaces derived by using a generalized weighted mean,* Appl. Math. Lett., 24 (5) (2011), 608-614.

[16] M. A. Sarigol, *On difference sequence spaces,* J. Karad. Tech. Univ. (ser. Math-Phy), X(1987), 63-71.

[17] N. A. Sheikh and A. H. Ganie, *A new paranormed sequence space and some matrix transformations,* Acta Math. Acad. Paed. Nyreg., 28 (2012), 47-58.

[18] N. A. Sheikh and A. H. Ganie, *A new type of sequence space of non-absolute type and matrix transformation,* WSEAS Transaction of Math., 8(12) (2013), 852-859.

[19] N. A. Sheikh and A. H. Ganie, infinite matrices and almost convergence, Filomat, 29(2015), 1183-1188.

[20] A. Wilansky, *Summability through Functional Analysis*, North Holland Mathematics Studies, Amsterdam - New York - Oxford, (1984).

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