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SYMMETRIC IDENTITIES OF HIGHER-ORDER CARLITZ'S TYPE q-CHANGHEE POLYNOMIALS UNDER S_3

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Abstract. In [1], Dolgy et al. introduced the Carlitz's type q-Changhee polynomials and investigated some identities and properties of those polynomials and numbers. In this paper, we find some interesting symmetric identities for the Carlitz's type q-Changhee polynomials under the symmetry group of order 3 arising from the multiple p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction

In the pure and applied mathematics, special functions and polynomials are very important roles. Newton and Leibniz gave the special functions as the solution of differential equations, and the special function theory and its applications have been studied by many researchers (see [2, 3, 4, 5, 6]). In the past years, several authors defined new special functions and investigated the relationships between those functions and the Euler polynomials, Bernoulli polynomials, Genocchi polynomials, Laguerre polynomials and Bell polynomials et al. which are important functions in the mathematics (see [1, 3, 4, 5, 7-19]). In addition, they obtained many new and interesting identities of these polynomials and numbers.

Throughout this paper p is a fixed odd prime number. We use the notations $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$.

For an indeterminate $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, the *$q$-analogue of number x* is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p)$ be the set of all continuous functions on $\mathbb{Z}_p$. A *fermionic p-adic q-integral* of $f \in C(\mathbb{Z}_p)$ is defined by Kim to be

$$\begin{aligned} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \lim_{N\to\infty} \frac{[2]_q}{2} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{(see [3, 4, 8, 14, 20, 31])}. \end{aligned} \tag{1.1}$$

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The *Stirling numbers of the first kind and the second kind* are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 0), \tag{1.2}$$

and

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \tag{1.3}$$

respectively, where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ (see [7, 9, 14, 21]).

By the equations (1.2) and (1.3), we can obtained easily the following equations

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}, \tag{1.4}$$

and

$$(\log(x+1))^n = n! \sum_{l=n}^{\infty} S_1(l,n) \frac{x^l}{l!}, \ (n \geq 0), \ (\text{see [7, 9, 14, 21]}). \tag{1.5}$$

The *Euler polynomial* $E_n(x)$ is given by the generating function to be

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \ (\text{see [4, 8, 9, 10, 11, 13, 14, 16, 20]}).$$

In the special case $x = 0$, $E_n = E_n(0)$ $(n \in \mathbb{N})$ is called the *Euler number.*

As a q-analogue of Euler numbers and polynomials, the *Carlitz's type q-Euler polynomial of order r* $\mathcal{E}_{n,q}^{(r)}(x)$ is defined by

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(r)}(x) \frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} e^{[x+y_1+\cdots+y_r]_q t} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r), \ (\text{see [1, 17, 18, 22-25]}). \tag{1.6}$$

In particular, if $x = 0$, $\mathcal{E}_{n,q}^{(r)} = \mathcal{E}_{n,q}^{(r)}(0)$ is called the *q-Euler number of order r.*

By (1.6), the Witt-type formula of Carlitz's type q-Euler polynomial of order r $\mathcal{E}_{n,q}^{(r)}(x)$ is

$$\mathcal{E}_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + y_1 + \cdots + y_r]_q^n d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r), \ (n \geq 0). \tag{1.7}$$

Since

$$\lim_{\lambda \to 0} (1 + \lambda t)^{\frac{1}{\lambda}} = e^t, \tag{1.8}$$

by the (1.6) and (1.8), the *degenerate q-Euler polynomial of order r* $\mathcal{E}_{n,\lambda,q}^{(r)}(x)$ is defined as [18, 27]:

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}^{(r)}(x) \frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (1 + \lambda t)^{\frac{[x+y_1+\cdots+y_r]_q}{\lambda}} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r). \tag{1.9}$$

By the gnenralized binomial expansion of $(1 + \lambda t)^{\frac{[x+y]_q}{\lambda}}$, we have

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda} \right)_n d\mu_{-q}(y) \frac{t^n}{n!}, \tag{1.10}$$

and by (1.9) and (1.10), we know that the Witt-type formula of degenerate q-Euler polynomial of order r is

$$\mathcal{E}_{n,\lambda,q}(x) = \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_{-q}(y), \ (n \in \mathbb{N}). \tag{1.11}$$

Furthermore, by (1.2) and (1.11), we can derive the following equation:

$$\begin{aligned}\mathcal{E}_{n,\lambda,q}(x) =& \lambda^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)^l d\mu_{-q}(y)\\ =& \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l) \mathcal{E}_{l,q}(x).\end{aligned}$$

From now on, we apply these polynomials to Changhee polynomials. In [7], Kim, Kim and Seo introduced the *Changhee polynomial* $Ch_n(x)$ which was defined by the generating function to be

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} =& \frac{2}{2+t}(1+t)^x\\ =& \int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y).\end{aligned} \tag{1.12}$$

In viewpoint of (1.6) and (1.12), we will define the *Carlitz's type q-Changhee polynomial of order r* $Ch_{n,q}^{(r)}(x)$ by the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\sum_{n=0}^{\infty} Ch_{n,q}^{(r)}(x) \frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (1+t)^{[x+y_1+\cdots+y_r]_q} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r). \tag{1.13}$$

Note that, by (1.2) and (1.13),

$$\begin{aligned}&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{[x+y_1+\cdots+y_r]_q} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r)\\ =& \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left([x+y_1+\cdots+y_r]_q\right)_n d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \right) \frac{t^n}{n!}\\ =& \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x+y_1+\cdots+y_r]_q^l d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \right) \frac{t^n}{n!},\end{aligned} \tag{1.14}$$

and hence, from (1.13) and (1.14), we know that, for each $n \in \mathbb{N}$,

$$\begin{aligned}Ch_{n,q}^{(r)}(x) =& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left([x+y_1+\cdots+y_r]_q\right)_n d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r)\\ =& \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x+y_1+\cdots+y_r]_q^l d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r).\end{aligned} \tag{1.15}$$

In the past twenty years, many generalizations of Changhee numbers and polynomials have been introduced, and discovered the relationships between several special functions and polynomials and those polynomials (see [1, 7, 17, 18, 19]).

These results shows that how these arise from several areas of mathematics and mathematical physics.

Symmetric properties of special polynomials are important and interesting things in number theory, pure and applied mathematics. Symmetric identities of many special polynomials were founded (see [22 - 30, 32 - 37]. In particular, Cesarano investigated some techniques regarding the generating functions used, and showed how can be applicable these identities to the porous materials theory (see [6]).

In this paper, we derive some interesting symmetric identities for the two cases of Carlitz's type q-Changhee polynomials under the symmetry group of order 3 arising from the fermionic p-adic q-integral on $\mathbb{Z}_p$.

2. Higher-order Carlitz's type q-Changhee polynomials

First of all we consider some equations. By (1.5), (1.6) and (1.13),

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{[x+y_1+\cdots+y_r]_q}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[x+y_1+\cdots+y_r]_q\log(1+t)}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\frac{\left([x+y_1+\cdots+y_r]_q^l\log(1+t)\right)^l}{l!}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\sum_{l=0}^{\infty}\mathcal{E}_{l,q}^{(r)}(x)\sum_{i=l}^{\infty}S_1(i,l)\frac{t^i}{i!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\mathcal{E}_{l,q}^{(r)}(x)S_1(n,l)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.1}$$

In addition, by replacing t as e^{t-1} in (1.13), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[x+y_1+\cdots+y_r]_q}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\sum_{n=0}^{\infty}Ch_{n,q}^{(r)}(x)\frac{(e^t-1)^n}{n!}\\
=&\sum_{n=0}^{\infty}Ch_{n,q}^{(r)}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}Ch_{m,q}^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.2}$$

By (2.1) and (2.2), we obtain the following lemma.

Lemma 2.1. *For each nonnegative integer n,*

$$Ch_{n,q}^{(r)}(x)=\sum_{l=0}^{n}\mathcal{E}_{l,q}^{(r)}(x)S_1(n,l),$$
$$\mathcal{E}_{n,q}^{(r)}(x)=\sum_{m=0}^{n}Ch_{m,q}^{(r)}(x)S_2(n,m).$$

Let $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, and let S_3 be the symmetry group of degree 3. For positive integers w_1,w_2,w_3 with $w_i\equiv 1 \pmod 2$ for each $i=1,2,3$, we consider the following integral equation for the fermionic p-adic q-integral on $\mathbb{Z}_p$;

$$\begin{aligned}
&\frac{1}{[w_1]_{-q}^r}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^r j_i+w_1\sum_{i=1}^r y_i\right]_q}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
=&\frac{1}{[w_1]_{-q}^r}\lim_{N\to\infty}\frac{1}{[p^N]_{-q^{w_1}}^r}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^r j_i+w_1\sum_{l=1}^r y_l\right]_q}(-q)^{w_1\sum_{l=1}^r y_l}\\
=&\frac{1}{[w_1]_{-q}^r}\lim_{N\to\infty}\frac{1}{[w_2p^N]_{-q^{w_1}}^r}\sum_{y_1,\ldots,y_r=0}^{w_2p^N-1}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^r j_i+w_1\sum_{i=1}^r y_l\right]_q}(-q)^{w_1\sum_{l=1}^r y_l}\\
=&\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_{-q}^r}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^r j_i+w_1\sum_{i=1}^r(i_i+w_2y_l)\right]_q}\\
&\times(-q)^{\sum_{l=1}^r w_1(i_l+w_2y_l)}\\
=&\sum_{i_1,\ldots,i_r=0}^{w_2-1}(-q)^{w_1\sum_{l=1}^r i_l}\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_{-q}^r}\\
&\times\sum_{y_1,\ldots,y_r=0}^{p^N-1}(-q)^{w_1w_2\sum_{i=1}^r y_i}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^r j_i+w_1\sum_{i=1}^r(i_i+w_2y_l)\right]_q}.
\end{aligned}\tag{2.3}$$

By the equation (2.3), if we put

$$\begin{aligned}
&F(w_1,w_2)=\frac{1}{[w_1]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_1-1}(-q)^{w_2\sum_{l=1}^r j_l}\\
&\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\left[w_1w_2x+w_2\sum_{l=1}^r i_l+w_1\sum_{l=1}^r y_l\right]_q}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\left(\frac{[2]_q}{2}\right)^r\lim_{N\to\infty}\sum_{j_1,\ldots,j_r=0}^{w_1-1}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}(-q)^{w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r i_l+w_1w_2\sum_{i=1}^r y_i}\\
&\quad\times(1+t)^{\left[w_1w_2x+w_1\sum_{i=1}^r i_i+w_2\sum_{l=1}^r j_l+w_1w_2\sum_{l=1}^r y_l\right]_q},
\end{aligned}\tag{2.4}$$

then we know that $F(w_1,w_2)=F(w_2,w_1)$. Hence, by (2.4), we obtain the following theorem.

Theorem 2.2. *For each positive odd integers* w_1 *and* w_2*, we have* $F(w_1, w_2) = F(w_2, w_1)$.

From now on, we consider another fermionic p-adic q-integral on $\mathbb{Z}_p$;

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r y_l+w_1w_3\sum_{l=1} i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)$$

$$=\lim_{N\to\infty}\left(\frac{[2]_{q^{w_2w_3}}}{2}\right)^r\sum_{y_1,\ldots,y_r=0}^{p^N-1}(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r x_l+w_1w_3\sum_{l=1} i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}$$
$$\times(-q)^{w_2w_3\sum_{i=1}^r y_i}$$

$$=\lim_{N\to\infty}\left(\frac{[2]_{q^{w_2w_3}}}{2}\right)^r\sum_{m_1,\ldots,m_r=0}^{w_1-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}$$
$$\times(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r y_l+w_1w_3\sum_{l=1} i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}(-q)^{w_2w_3\sum_{i=1}^r(m_i+w_1y_i)}. \tag{2.5}$$

By (2.4), we get

$$\frac{2^r}{[2]_{q^{w_2w_3}}^r}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,j_r=0}^{w_3-1}(-q)^{w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r y_l+w_1w_3\sum_{l=1} i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)$$
$$=\lim_{N\to\infty}\sum_{m_1,\ldots,m_r=0}^{w_1-1}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,i_r=0}^{w_3-1}\sum_{y_1,\ldots,y_r=0}^{p^N-1}$$
$$\times(-q)^{w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l+w_2w_3+\sum_{l=1}^r m_l+w_1w_2w_3\sum_{l=1}^r y_l}$$
$$\times(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r y_l+w_1w_3\sum_{l=1} i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}. \tag{2.6}$$

If we put

$$G(w_1,w_2,w_3)=\frac{2^r}{[2]_{q^{w_2w_3}}^r}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,i_r=0}^{w_3-1}(-q)^{w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r y_l+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l\right]_q}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r), \tag{2.7}$$

then, by (2.6), we know $G(w_1, w_2, w_3)$ is invariant for any permutation in symmetry group of degree 3 S_3.

Thus, by (2.5), (2.6) and (2.7), we obtain the following theorem.

Theorem 2.3. *For any positive odd integers* w_1, w_2, w_3 *and any* $\sigma \in S_3$, $G\left(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}\right)$ *have the same value.*

By the definition of $[x]_q$, we know that

$$\begin{aligned}
&\left[w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l\right]_q\\
=&\frac{1-q^{w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l}}{1-q}=\left(\frac{1-q^{w_1}}{1-q}\right)\left(\frac{1-q^{w_1w_2x+w_2\sum_{l=1}^{r}j_l+w_1\sum_{l=1}^{r}y_l}}{1-q^{w_1}}\right)\\
=&[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}.
\end{aligned} \tag{2.8}$$

The Lemma 2.1 and equation (2.8) yield the following equation;

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\left[w_1w_2x+w_2\sum_{i=1}^{r}j_i+w_1\sum_{i=1}^{r}y_i\right]_q}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}\log(1+t)}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
=&\sum_{n=0}^{\infty}[w_1]_q^n\frac{(\log(1+t))^n}{n!}\\
&\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}^n d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
=&\sum_{n=0}^{\infty}[w_1]_q^n\frac{1}{n!}n!\sum_{k=n}^{\infty}S_1(k,n)\frac{t^k}{k!}\mathcal{E}_{n,q^{w_1}}^{(n)}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}[w_1]_q^m S_1(n,m)\mathcal{E}_{m,q^{w_1}}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l\right)\right)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}[w_1]_q^m S_1(n,m)S_2(m,k)Ch_{k,q^{w_1}}^{(r)}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l\right)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

If we put

$$F_1(w_1,w_2)=\sum_{j_1,\ldots,j_r=0}^{w_1-1}\sum_{m=0}^{n}\sum_{k=0}^{m}(-q)^{w_2\sum_{l=1}^{r}i_l}\frac{[w_1]_q^m}{[w_1]_{-q}^r}S_1(n,m)S_2(m,k)Ch_{k,q^{w_1}}^{(r)}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l\right),$$

then by Theorem 2.2 and (2.9), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n and each positive odd integers w_1 and w_2, $F_1(w_1,w_2)=F_1(w_2,w_1)$.*

By the similar calculation of (2.8), it is not difficult to know that

$$\begin{aligned}&\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^{r}x_l+w_1w_3\sum_{l=1}^{r}i_l+w_1w_2\sum_{l=1}^{r}j_l\right]_q\\&=[w_2w_3]_q\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_2w_3}}.\end{aligned}\tag{2.10}$$

By the Lemma 2.1, (2.5) and (2.10), we obtain the following equation;

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^{r}x_l+w_1w_3\sum_{l=1}i_l+w_1w_2\sum_{l=1}^{r}j_l\right]_q}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{[w_2w_3]_q\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}x_l\right]_{q^{w_2w_3}}}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[w_2w_3]_q\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}x_l\right]_{q^{w_2w_3}}\log(1+t)}d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)\\
&=\sum_{n=0}^{\infty}[w_2w_3]_q^n\frac{(\log(1+t))^n}{n!}\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}x_l\right]_{q^{w_2w_3}}^n d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)\\
&=\sum_{n=0}^{\infty}[w_2w_3]_q^n\frac{1}{n!}n!\sum_{k=n}^{\infty}S_1(k,n)\frac{t^k}{k!}\mathcal{E}_{n,q^{w_2w_3}}^{(n)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}[w_2w_3]_q^mS_1(n,m)\mathcal{E}_{m,q^{w_2w_3}}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l\right)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}[w_2w_3]_q^mS_1(n,m)S_2(m,k)Ch_{k,q^{w_2w_3}}^{(r)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^{r}i_l+\frac{w_1}{w_3}\sum_{l=1}^{r}j_l\right)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.11}$$

By (2.5) and Theorem 2.3, we obtain the following theorem.

Theorem 2.5. *Let w_1, w_2, w_3 be positive odd integers. For each nonnegative integer n, the*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{i_1,\ldots,i_r=0}^{w_{\sigma(2)}-1}\sum_{j_1,\ldots,i_r=0}^{w_{\sigma(3)}-1}[w_{\sigma(2)}w_{\sigma(3)}]_q^mS_1(n,m)S_2(m,k)\\&\times(-q)^{w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^{r}i_l+w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^{r}j_l}Ch_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}}^{(r)}\left(w_{\sigma(1)}x+\frac{w_{\sigma(1)}}{w_{\sigma(2)}}\sum_{l=1}^{r}i_l+\frac{w_{\sigma(1)}}{w_{\sigma(3)}}\sum_{l=1}^{r}j_l\right)\end{aligned}$$

are invariant under any permutation $\sigma\in S_3$.

By the definition of q-analogue of number x, we note that

$$\begin{aligned}&\left[w_2x+\frac{w_2}{w_1}\sum_{i=1}^{r}j_i+\sum_{i=1}^{r}y_i\right]_{q^{w_1}}\\&=\frac{[w_2]_q}{[w_1]_q}\left[\sum_{i=1}^{r}j_i\right]_{q^{w_2}}+\left[w_2x+\sum_{i=1}^{r}y_i\right]_{q^{w_1}}q^{w_2\sum_{i=1}^{r}j_i}.\end{aligned}\tag{2.12}$$

The equations (1.2), (1.7), (1.15), Lemma 2.1 and (2.12) derive the following equation;

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}\right)_n d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\sum_{l=0}^{n}[w_1]_q^lS_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}^l d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\sum_{l=0}^{n}[w_1]_q^lS_1(n,l)\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{[w_2]_q}{[w_1]_q}\left[\sum_{i=1}^{r}j_i\right]_{q^{w_2}}+\left[w_2x+\sum_{i=1}^{r}y_i\right]_{q^{w_1}}q^{w_2\sum_{i=1}^{r}j_i}\right)^l d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}\frac{[w_2]_q^m}{[w_1]_q^{m-l}}S_1(n,l)\left[\sum_{i=1}^{r}j_i\right]_{q^{w_2}}^m q^{(l-m)w_2\sum_{i=1}^{r}j_i}\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_2x+\sum_{i=1}^{r}y_i\right]_{q^{w_1}}^{l-m}d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}\frac{[w_2]_q^m}{[w_1]_q^{m-l}}S_1(n,l)\left[\sum_{i=1}^{r}j_i\right]_{q^{w_2}}^m q^{(l-m)w_2\sum_{i=1}^{r}j_i}\mathcal{E}_{l-m,q^{w_1}}(w_2x)\\
&=\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{k=0}^{l-m}\binom{l}{m}\frac{[w_2]_q^m}{[w_1]_q^{m-l}}\left[\sum_{i=1}^{r}j_i\right]_{q^{w_2}}^m q^{(l-m)w_2\sum_{i=1}^{r}j_i}S_1(n,l)S_2(l-m,k)Ch_{k,q^{w_1}}^{(r)}(w_2x).\end{aligned}\tag{2.13}$$

By (2.13), we get

$$\begin{aligned}&\frac{1}{[w_1]_{-q}^r}\sum_{j_1,\ldots,j_r=0}^{w_1-1}(-q)^{w_2\sum_{l=1}^{r}j_l}\\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^{r}j_l+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}\right)_n d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r)\\
&=\sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{k=0}^{l-m}\binom{l}{m}\frac{[w_2]_q^m}{[w_1]_q^{m+r-l}}S_1(n,l)S_2(l-m,k)Ch_{k,q^{w_1}}^{(r)}(w_2x)R_{l+1,m}(w_1,w_2|q^{w_2}),\end{aligned}\tag{2.14}$$

where

$$R_{n,m}(u,v|q)=\sum_{j_1,\dots,j_r=0}^{u-1}(-1)^{v\sum_{l=1}^r j_l}q^{(l+1-m)w_2\sum_{i=1}^r j_i}\left[\sum_{i=1}^r j_i\right]_{q^{w_2}}^m.$$

If we put

$$F_2(w_1,w_2)=\frac{1}{[w_1]_{-q}^r}\sum_{j_1,\dots,j_r=0}^{w_1-1}(-q)^{w_2\sum_{l=1}^r j_l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_1}}\right)_n d\mu_{-q^{w_1}}(y_1)\cdots d\mu_{-q^{w_1}}(y_r),$$

then, by Theorem 2.2, (2.12) and (2.14), we cad derive the following theorem.

Theorem 2.6. *For each nonnegative integer n and each odd integers w_1,w_2, we have $F_2(w_1,w_2)=F_2(w_2,w_1)$.*

On the other hand, by the definition of $[x]_q$, we note that

$$\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^r i_l+\frac{w_1}{w_3}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_2w_3}}$$
$$=\frac{[w_1]_q}{[w_2w_3]_q}\left[w_2\sum_{l=1}^r j_l+w_3\sum_{l=1}^r i_l\right]_{q^{w_1}}+q^{w_1w_2\sum_{l=1}^r j_l+w_1w_3\sum_{l=1}^r i_l}\left[w_1x+\sum_{l=1}^r y_l\right]_{q^{w_2w_3}}. \tag{2.15}$$

Then, by the similar calculation of (2.13) and (2.15), we get

$$Ch_{n,q^{w_2w_3}}^{(r)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^r i_l+\frac{w_1}{w_3}\sum_{l=1}^r j_l\right)$$
$$=\sum_{m=0}^n S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^r i_l+\frac{w_1}{w_3}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_2w_3}}^m d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)$$
$$=\sum_{m=0}^n\sum_{k=1}^m\frac{[w_1]_q^{m-k}}{[w_2w_3]_q^{m-k}}\binom{m}{k}S_1(n,m)q^{kw_1w_2\sum_{l=1}^r j_l+kw_1w_3\sum_{l=1}^r i_l}\left[w_2\sum_{l=1}^r j_l+w_3\sum_{l=1}^r i_l\right]_{q^{w_1}}^{m-k}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_1x+\sum_{l=1}^r y_l\right]_{q^{w_2w_3}}^k d\mu_{-q^{w_2w_3}}(y_1)\cdots d\mu_{-q^{w_2w_3}}(y_r)$$
$$=\sum_{m=0}^n\sum_{k=0}^m\sum_{l=0}^k\frac{[w_1]_q^{m-k}}{[w_2w_3]_q^{m-k}}\binom{m}{k}S_1(n,m)S_2(k,l)q^{kw_1w_2\sum_{l=1}^r j_l+kw_1w_3\sum_{l=1}^r i_l}$$
$$\times\left[w_2\sum_{l=1}^r j_l+w_3\sum_{l=1}^r i_l\right]_{q^{w_1}}^{m-k}Ch_{l,q^{w_2w_3}}^{(r)}(w_1x). \tag{2.16}$$

If we put

$$T_{q^{w_1}}^{(r)}(u,v|m)=\sum_{i_1,\ldots,i_r=0}^{u-1}\sum_{j_1,\ldots,j_r=0}^{v-1}(-1)^{w_1\left(u\sum_{l=1}^{r}j_l+v\sum_{l=1}^{r}i_l\right)}$$
$$\times q^{w_1\left(mu\sum_{l=1}^{r}j_l+mv\sum_{l=1}^{r}i_l\right)}\left[u\sum_{l=1}^{r}j_l+v\sum_{l=1}^{r}i_l\right]_{q^{w_1}}^{s-t},$$

then, by Theorem 2.5 and (2.16), we obtain the following theorem.

Theorem 2.7. *For each positive odd integers w_1, w_2, w_3 and for each nonnegative integer n, the*

$$\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{s=0}^{k}\sum_{t=0}^{s}\sum_{l=0}^{t}\frac{\left[w_{\sigma(1)}\right]_q^{s-t}}{\left[w_{\sigma(2)}w_{\sigma(3)}\right]_q^{s+m-t}}S_1(n,m)S_1(k,s)S_2(m,k)S_2(t,l)$$
$$\times Ch_{l,q^{w_2w_3}}^{(r)}\left(w_{\sigma(1)}x\right)T_{q^{w_1}}^{(r)}(w_{\sigma(2)},w_{\sigma(3)}|t+1)$$

are invariant under any permutation $\sigma \in S_3$.

3. Conclusions

The Carlitz's type q-Changhee polynomials are closely related with the q-Euler polynomials, q-Genocchi polynomials, the Stirling numbers of the first kind and the second kind and the harmonic numbers, and so on. In this paper, we study the functions $G(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)})$ for the Carlitz's type q-Changhee polynomials is invariant under the symmetry group $\sigma \in S_3$. From the invariance of $G(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)})$ on the S_3, we construct symmetric identities of the Carlitz's type q-Changhee polynomials from the fermionic p-adic q-integral on $\mathbb{Z}_p$. In addition, we found two variable functions which were related Carlitz's type q-Changhee polynomials and commute those variables. As Bernoulli and Euler polynomials, our theorems on the Carlitz's type q-Changhee polynomials play important roles in finding some identities and properties for some numbers in algebraic number theory.

Authors contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Conflicts of interest

The authors declare no conflict of interest.

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