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SOME IDENTITIES OF THE TYPE 2 DEGENERATE BERNOULLI AND EULER NUMBERS

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ABSTRACT. Recently, several authors have studied the type 2 degenerate Bernoulli and Euler numbers and polynomials. In this paper, we deduce infinite families of differential equations satisfied by generating functions of the type 2 degenerate Bernoulli and Euler numbers and derive some identities from those differential equations.

1. Introduction

In [7], it was shown that odd integer power sums can be expressed in terms of some values of the type 2 Bernoulli polynomials and that alternating odd integer power sums in terms of certain values of the type 2 Euler polynomials. Further, two random variables were created from random variables having Laplace distributions and it was shown that their moments are closely connected with the type 2 Bernoulli and Euler numbers.

Quite recently, as degenerate versions of the type 2 Bernoulli and Euler numbers and polynomials, the type 2 degenerate Bernoulli and Euler numbers and polynomials were introduced in [5] and some of their properties and identities were investigated.

In this paper, first we deduce two infinite families of differential equations satisfied by the generating function of the type 2 degenerate Bernoulli numbers. To be precise, it is that generating function divided by t. Then, from those differential equations we derive some identities involving the type 2 degenerate Bernoulli numbers and those of higher-orders.

Next, we deduce two infinite families of differential equations satisfied by the generating function of the type 2 degenerate Euler numbers. Then, by virtue of those differential equations we derive some identities involving the type 2 degenerate Euler numbers and those of higher-orders.

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For the rest of this section, we will recall some of the necessary definitions that will be needed in the sequel.

The Bernoulli polynomials are defined by the generating function:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see } [2,3,4,6,15]). \tag{1.1}$$

When $x=0$, $B_n=B_n(0)$ are called the Bernoulli numbers.

As is well known, the type 2 Bernoulli polynomials are defined by the generating function:

$$\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\quad (\text{see } [1,7,8,10,14]). \tag{1.2}$$

When $x=0$, $b_n=b_n(0)$ are called the type 2 Bernoulli numbers.

The type 2 degenerate Bernoulli polynomials are given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}b_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see } [5]). \tag{1.3}$$

When $x=0$, $b_{n,\lambda}=b_{n,\lambda}(0)$ are called the type 2 degenerate Bernoulli numbers. For $r\in\mathbb{N}$, the type 2 degenerate Bernoulli polynomials of order r are given by

$$\Big(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}\Big)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x=0$, $b_{n,\lambda}^{(r)}=b_{n,\lambda}^{(r)}(0)$ are called the type 2 degenerate Bernoulli numbers of order r.

From (1.4), we readily see that

$$b_{n,\lambda}^{(r)}=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\cdots,l_r}b_{l_1,\lambda}\cdots b_{l_r,\lambda}. \tag{1.5}$$

As is well known that Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (\text{see } [2,3,4,6,15]). \tag{1.6}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers.

The type 2 Euler numbers are defined by the following generating function:

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}=\sum_{n=0}^{\infty}e_n\frac{t^n}{n!},\quad (\text{see } [1,5,7,8,13,14]). \tag{1.7}$$

In [1], Jang-Kim introduced the type 2 degenerate Euler numbers, which are given by the following generating function:

$$\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}=\sum_{n=0}^{\infty}e_{n,\lambda}\frac{t^n}{n!}. \tag{1.8}$$

When $\lambda\to 0$, the type 2 degenerate Euler numbers reduces to the type 2 Euler numbers.

$$\lim_{\lambda\to 0}e_{n,\lambda}=e_n.$$

For $r\in\mathbb{N}$, the type 2 degenerate Euler numbers of order r are defined by

$$\Big(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}\Big)^r=\sum_{n=0}^{\infty}e_{n,\lambda}^{(r)}\frac{t^n}{n!},\ \text{(see [5])}. \tag{1.9}$$

From (1.9), it is immediate to see that

$$e_{n,\lambda}^{(r)}=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\cdots,l_r}e_{l_1,\lambda}\cdots e_{l_r,\lambda}. \tag{1.10}$$

2. Some identities of the type 2 degenerate Bernoulli numbers arising from differential equations

Let

$$F=F(t;\lambda)=\frac{1}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}. \tag{2.1}$$

Let us take the derivative with respect to t of (2.1). Then we have

$$\begin{aligned}F^{(1)}=\frac{\partial}{\partial t}F(t;\lambda)&=\frac{1}{\big((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\big)^2}\Big(-\frac{1}{2}(1+\lambda t)^{\frac{1}{2\lambda}-1}\\&\quad-\frac{1}{2}(1+\lambda t)^{-\frac{1}{2\lambda}-1}\Big)\\&=-\frac{1}{2}(1+\lambda t)^{-1}F-(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2.\end{aligned} \tag{2.2}$$

From (2.2), we get

$$-2F^{(1)}=(1+\lambda t)^{-1}F+2(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2. \tag{2.3}$$

From (2.3), we have

$$\begin{aligned}-2F^{(2)}&=-\lambda(1+\lambda t)^{-2}F+(1+\lambda t)^{-1}F^{(1)}-(1+2\lambda)\\&\quad\times(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2+4(1+\lambda t)^{-\frac{1}{2\lambda}-1}FF^{(1)}.\end{aligned} \tag{2.4}$$

Multiplying -2 on both sides of (2.4) and by (2.3), we get

$$\begin{aligned}(-2)^2F^{(2)} &= 2\lambda(1+\lambda t)^{-2}F + (1+\lambda t)^{-2}F + 2(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 \\ &\quad + (2+4\lambda)(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 + 4(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 \\ &\quad + 8(1+\lambda t)^{-\frac{2}{2\lambda}-2}F^3 \\ &= (1+2\lambda)(1+\lambda t)^{-2}F + (8+4\lambda)(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 \\ &\quad + 8(1+\lambda t)^{-\frac{2}{2\lambda}-2}F^3.\end{aligned} \tag{2.5}$$

Continuing this process, we get

$$(-2)^N F^{(N)} = \sum_{k=0}^{N} a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N}F^{k+1}. \tag{2.6}$$

Let us take the derivative with respect to t of (2.6). Then we get

$$\begin{aligned}(-2)^N F^{(N+1)} &= \sum_{k=0}^{N}\Big(-\frac{k}{2}-N\lambda\Big)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1} \\ &\quad + \sum_{k=0}^{N}(k+1)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N}F^kF^{(1)}.\end{aligned} \tag{2.7}$$

Multiplying -2 on both sides of (2.7) and by (2.3), we have

$$\begin{aligned}(-2)^{N+1}F^{(N+1)} &= \sum_{k=0}^{N}(k+2N\lambda)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1} \\ &\quad + \sum_{k=0}^{N}(k+1)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1} \\ &\quad + \sum_{k=0}^{N}2(k+1)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k+1}{2\lambda}-N-1}F^{k+2} \\ &= \sum_{k=0}^{N}(2N\lambda+2k+1)a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1} \\ &\quad + \sum_{k=1}^{N+1}2ka_{k-1,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\end{aligned} \tag{2.8}$$

$$= (2N\lambda+1)a_{0,\lambda}(N)(1+\lambda t)^{-N-1}F + 2(N+1)a_{N,\lambda}(N) \times (1+\lambda t)^{-\frac{N+1}{2\lambda}-N-1}F^{N+2} + \sum_{k=1}^{N}\big((2N\lambda+2k+1) \times a_{k,\lambda}(N) + 2ka_{k-1,\lambda}(N)\big)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}.$$

Replacing N by $N+1$ in (2.6), we have

$$\begin{aligned}(-2)^{N+1}F^{(N+1)} &= \sum_{k=0}^{N+1} a_{k,\lambda}(N+1)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\ &= a_{0,\lambda}(N+1)(1+\lambda t)^{-N-1}F + a_{N+1,\lambda}(N+1)\\ &\quad\times (1+\lambda t)^{-\frac{N+1}{2\lambda}-N-1}F^{N+2} + \sum_{k=1}^{N} a_{k,\lambda}(N+1)\\ &\quad\times (1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we get

$$a_{0,\lambda}(N+1) = (2N\lambda+1)a_{0,\lambda}(N), \tag{2.10}$$

$$a_{N+1,\lambda}(N+1) = 2(N+1)a_{N,\lambda}(N), \tag{2.11}$$

$$a_{k,\lambda}(N+1) = (2N\lambda+2k+1)a_{k,\lambda}(N) + 2ka_{k-1,\lambda}(N), \tag{2.12}$$

where $1 \le k \le N$. From (2.3) and (2.6), we have

$$\begin{aligned}-2F^{(1)} &= \sum_{k=0}^{1} a_{k,\lambda}(1)(1+\lambda t)^{-\frac{k}{2\lambda}-1}F^{k+1}\\ &= a_{0,\lambda}(1)(1+\lambda t)^{-1}F + a_{1,\lambda}(1)(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2\\ &= (1+\lambda t)^{-1}F + 2(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2.\end{aligned} \tag{2.13}$$

Comparing the coefficients of (2.13), we have

$$a_{0,\lambda}(1) = 1, \quad a_{1,\lambda}(1) = 2. \tag{2.14}$$

Now, we observe that

$$\begin{aligned}a_{0,\lambda}(N+1) &= 2(N\lambda+\frac{1}{2})a_{0,\lambda}(N) = 2^2(N\lambda+\frac{1}{2})_{2,\lambda}a_{0,\lambda}(N-1)\\ &= \cdots = 2^N(N\lambda+\frac{1}{2})_{N,\lambda}a_{0,\lambda}(1) = 2^{N+1}(N\lambda+\frac{1}{2})_{N+1,\lambda},\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}a_{N+1,\lambda}(N+1) &= 2(N+1)a_{N,\lambda}(N) = 2^2(N+1)Na_{N-1,\lambda}(N-1)\\ &= \cdots = 2^N(N+1)_N a_{1,\lambda}(1) = 2^{N+1}(N+1)!,\end{aligned} \tag{2.16}$$

where $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda}=x\big(x-\lambda\big)\big(x-2\lambda\big)\cdots\big(x-(n-1)\lambda\big)$, for $n\geq 1$. Thus the proof for the next theorem is completed.

Theorem 2.1. *The following family of differential equations*

$$(-2)^N F^{(N)}=\sum_{k=0}^{N} a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N}F^{k+1},\ (N=0,1,2,\cdots)$$

have a solution $F=F(t;\lambda)=\frac{1}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}$,
where

$$\begin{aligned}
&a_{0,\lambda}(N)=2^N\big((N-1)\lambda+\frac{1}{2}\big)_{N,\lambda},\ \ a_{N,\lambda}(N)=2^N N!,\\
&a_{k,\lambda}(N)=(2(N-1)\lambda+2k+1)a_{k,\lambda}(N-1)+2ka_{k-1,\lambda}(N-1),\\
&(1\leq k\leq N-1).
\end{aligned}$$

Remark 1. From (2.12) and for $1\leq k\leq N-1$, we get

$$\begin{aligned}
a_{k,\lambda}(N)=&\sum_{i_1=0}^{N-k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}2^N\big((N-1)\lambda+k+\frac{1}{2}\big)_{N-k-i_1,\lambda}\\
&\times\big((k-2+i_1)\lambda+(k-1)+\frac{1}{2}\big)_{i_1-i_2,\lambda}\times\cdots\times\big((-1+i_k)\lambda+\frac{1}{2}\big)_{i_k,\lambda}k!,
\end{aligned}$$

From (1.3) and (2.1), we have

$$\begin{aligned}
F^{(N)}&=\frac{\partial^N}{\partial t^N}\Big(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}\cdot\frac{1}{t}\Big)\\
&=\frac{\partial^N}{\partial t^N}\Big(\sum_{n=0}^{\infty}b_{n,\lambda}\frac{t^n}{n!}\cdot\frac{1}{t}\Big)\\
&=\frac{\partial^N}{\partial t^N}\Big(\sum_{n=1}^{\infty}b_{n,\lambda}\frac{t^{n-1}}{n!}+\frac{1}{t}\Big)\\
&=\sum_{n=0}^{\infty}\frac{b_{n+N+1,\lambda}}{n+N+1}\frac{t^n}{n!}+(-1)^N N!\frac{1}{t^{N+1}}.
\end{aligned}\tag{2.17}$$

Multiplying $(-2)^N t^{N+1}$ on both sides of (2.17), we have

$$\begin{aligned}(-2)^N t^{N+1} F^{(N)} &= \sum_{n=0}^{\infty} (-2)^N \frac{b_{n+N+1,\lambda}}{n+N+1} \frac{t^{n+N+1}}{n!} + 2^N N! \\ &= \sum_{n=N+1}^{\infty} (-2)^N \frac{b_{n,\lambda}}{n} \frac{t^n}{(n-N-1)!} + 2^N N! \\ &= \sum_{n=N+1}^{\infty} (-2)^N b_{n,\lambda} \frac{(n-1)!}{(n-N-1)!} \frac{t^n}{n!} + 2^N N!.\end{aligned} \tag{2.18}$$

On the other hand, by (2.6) we have

$$\begin{aligned}(-2)^N t^{N+1} F^{(N)} &= \sum_{k=0}^{N} a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N} F^{k+1} t^{N+1} \\ &= \sum_{k=0}^{N} a_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N} \Big(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}} \Big)^{k+1} t^{N-k} \\ &= \sum_{k=0}^{N} a_{k,\lambda}(N) \Big(\sum_{l=0}^{\infty} \big(-\frac{k}{2\lambda} - N \big)_l \lambda^l \frac{t^l}{l!} \Big) \Big(\sum_{j=0}^{\infty} b_{j,\lambda}^{(k+1)} \frac{t^j}{j!} \Big) t^{N-k} \\ &= \sum_{k=0}^{N} a_{k,\lambda}(N) \Big(\sum_{l=0}^{\infty} \big\langle \frac{k}{2} + N\lambda \big\rangle_{l,\lambda} (-1)^l \frac{t^l}{l!} \Big) \Big(\sum_{j=0}^{\infty} b_{j,\lambda}^{(k+1)} \frac{t^j}{j!} \Big) t^{N-k} \\ &= \sum_{k=0}^{N} a_{k,\lambda}(N) \sum_{m=0}^{\infty} \Big(\sum_{j=0}^{m} \binom{m}{j} b_{j,\lambda}^{(k+1)} (-1)^{m-j} \big\langle \frac{k}{2} + N\lambda \big\rangle_{m-j,\lambda} \Big) \\ &\quad \times \frac{t^{m+N-k}}{m!} \\ &= \sum_{k=0}^{N} a_{N-k,\lambda}(N) \sum_{m=0}^{\infty} \Big(\sum_{j=0}^{m} \binom{m}{j} b_{j,\lambda}^{(N-k+1)} \big\langle \frac{N-k}{2} + N\lambda \big\rangle_{m-j,\lambda} \\ &\quad \times (-1)^{m-j} \Big) \frac{t^{m+k}}{m!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\min\{n,N\}} \sum_{j=0}^{n-k} \binom{n-k}{j} a_{N-k,\lambda}(N) b_{j,\lambda}^{(N-k+1)} (-1)^{n-k-j} \\ &\quad \times \big\langle \frac{N-k}{2} + N\lambda \big\rangle_{n-k-j,\lambda} (n)_k \Big) \frac{t^n}{n!},\end{aligned} \tag{2.19}$$

where $< x >_{0,\lambda} = 1$ and $< x >_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)$, for $n \geq 1$. From (2.18) and (2.19), we get the following theorem

Theorem 2.2. *Let $N, n \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}\sum_{k=0}^{\min\{n,N\}} \sum_{j=0}^{n-k} \binom{n-k}{j} a_{N-k,\lambda}(N) b_{j,\lambda}^{(N-k+1)} (-1)^{n-k-j} \\ \times \left\langle \frac{N-k}{2} + N\lambda \right\rangle_{n-k-j,\lambda} (n)_k \\ = \begin{cases} 2^N N!, & \text{if} \quad n = 0, \\ 0, & \text{if} \quad 1 \le n \le N, \\ (-2)^N b_{n,\lambda} \frac{(n-1)!}{(n-N-1)!}, & \text{if} \quad n \ge N+1. \end{cases},\end{aligned}$$

where $a_{k,\lambda}(N), (0 \le k \le n)$, are as in Theorem 2.1.

Now, we observe the inversion formula of **Theorem 2.1**. Multiplying $(1 + \lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (2.3), we have

$$(-2)F^2 = (1+\lambda t)^{\frac{1}{2\lambda}} F + 2(1+\lambda t)^{\frac{1}{2\lambda}+1} F^{(1)}. \tag{2.20}$$

Let us take the derivative with respect to t of (2.20). Then we get

$$\begin{aligned}(-2)2FF^{(1)} = \frac{1}{2}(1+\lambda t)^{\frac{1}{2\lambda}-1} F + (1+\lambda t)^{\frac{1}{2\lambda}} F^{(1)} + (1+2\lambda) \\ \times (1+\lambda t)^{\frac{1}{2\lambda}} F^{(1)} + 2(1+\lambda t)^{\frac{1}{2\lambda}+1} F^{(2)}.\end{aligned} \tag{2.21}$$

Multiplying $2(1+\lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (2.21) and by (2.20), we get

$$\begin{aligned}(-2)^2 2F^3 = 3(1+\lambda t)^{\frac{2}{2\lambda}} F + (8+4\lambda)(1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} \\ + 4(1+\lambda t)^{\frac{2}{2\lambda}+2} F^{(2)}.\end{aligned} \tag{2.22}$$

Continuing this process, we get the following equations

$$(-2)^N N! F^{N+1} = \sum_{k=0}^{N} b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k)}. \tag{2.23}$$

Taking the derivative with respect to t of (2.23), we have

$$\begin{aligned}(-2)^N (N+1)! F^N F^{(1)} = \sum_{k=0}^{N} \left(\frac{N}{2} + k\lambda\right) b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k-1} F^{(k)} \\ + \sum_{k=0}^{N} b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k+1)}.\end{aligned} \tag{2.24}$$

Multiplying $2(1+\lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (2.24) and by (2.20) and (2.23), we have

$$\begin{aligned}
(-2)^{N+1}(N+1)!F^{N+2} &= (-2)^{N}(N+1)!(1+\lambda t)^{\frac{1}{2\lambda}}F^{N+1} \\
&\quad + \sum_{k=0}^{N}(N+2k\lambda)b_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\
&\quad + \sum_{k=0}^{N}2b_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k+1}F^{(k+1)} \\
&= \sum_{k=0}^{N}(2N+2k\lambda+1)b_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\
&\quad + \sum_{k=1}^{N+1}2b_{k-1,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\
&= (2N+1)b_{0,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}}F + 2b_{N,\lambda}(N) \\
&\quad \times (1+\lambda t)^{\frac{N+1}{2\lambda}+N+1}F^{(N+1)} \\
&\quad + \sum_{k=1}^{N}\big((2N+2k\lambda+1)b_{k,\lambda}(N) \\
&\quad + 2b_{k-1,\lambda}(N)\big)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)}.
\end{aligned} \tag{2.25}$$

Replacing N by $N+1$ in (2.23), we have

$$\begin{aligned}
(-2)^{N+1}(N+1)!F^{N+2} &= \sum_{k=0}^{N+1}b_{k,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\
&= b_{0,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}}F \\
&\quad + b_{N+1,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+N+1}F^{(N+1)} \\
&\quad + \sum_{k=1}^{N}b_{k,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)}.
\end{aligned} \tag{2.26}$$

Comparing the coefficients of (2.25) and (2.26), we get

$$b_{0,\lambda}(N+1) = (2N+1)b_{0,\lambda}(N), \tag{2.27}$$

$$b_{N+1,\lambda}(N+1) = 2b_{N,\lambda}(N), \tag{2.28}$$

$$b_{k,\lambda}(N+1) = (2N+2k\lambda+1)b_{k,\lambda}(N) + 2b_{k-1,\lambda}(N), \tag{2.29}$$

where $1 \leq k \leq N$. From (2.20) and (2.23), we have

$$\begin{aligned}(-2)F^2 &= \sum_{k=0}^{1} b_{k,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}+k}F^{(k)}\\ &= b_{0,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}}F + b_{1,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}+1}F^{(1)}\\ &= (1+\lambda t)^{\frac{1}{2\lambda}}F + 2(1+\lambda t)^{\frac{1}{2\lambda}+1}F^{(1)}.\end{aligned} \tag{2.30}$$

From (2.30), we have

$$b_{0,\lambda}(1) = 1, \quad b_{1,\lambda}(1) = 2. \tag{2.31}$$

From (2.27), (2.28) and (2.31), we have

$$\begin{aligned}b_{0,\lambda}(N+1) &= 2\big(N+\frac{1}{2}\big)b_{0,\lambda}(N) = 2^2\big(N+\frac{1}{2}\big)_2 b_{0,\lambda}(N-1) = \cdots\\ &= 2^N\big(N+\frac{1}{2}\big)_N b_{0,\lambda}(1) = 2^{N+1}\big(N+\frac{1}{2}\big)_{N+1},\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}b_{N+1,\lambda}(N+1) &= 2b_{N,\lambda}(N) = 2^2 b_{N-1,\lambda}(N-1) = \cdots = 2^N b_{1,\lambda}(1)\\ &= 2^{N+1},\end{aligned} \tag{2.33}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ for $n \geq 1$. Thus, we can get the following theorem

Theorem 2.3. *The following family of differential equations*

$$(-2)^N N! F^{N+1} = \sum_{k=0}^{N} b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k}F^{(k)}, \ (N = 0,1,2,\cdots)$$

have a solution $F = F(t;\lambda) = \frac{1}{(1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}}$,
where

$$\begin{aligned}&b_{0,\lambda}(N) = 2^N\big(N-\frac{1}{2}\big)_N, \ \ b_{N,\lambda}(N) = 2^N,\\ &b_{k,\lambda}(N) = (2(N-1)+2k\lambda+1)b_{k,\lambda}(N-1) + 2b_{k-1,\lambda}(N-1),\\ &(1 \leq k \leq N-1).\end{aligned}$$

Remark 2. From (2.29) and for $1 \leq k \leq N-1$, we obtain the following expression.

$$\begin{aligned}b_{k,\lambda}(N) = &\sum_{i_1=0}^{N-k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} 2^N\big(N-1+k\lambda+\frac{1}{2}\big)_{N-k-i_1}\\ &\times\big((k-2+i_1)+(k-1)\lambda+\frac{1}{2}\big)_{i_1-i_2}\times\cdots\times\big(i_{k-1}+\lambda+\frac{1}{2}\big)_{i_{k-1}-i_k}\big(-1+i_k+\frac{1}{2}\big)_{i_k}.\end{aligned}$$

From (1.4) and (2.1), we have

$$
\begin{aligned}
(-2)^N N! t^{N+1} F^{N+1} &= (-2)^N N! \sum_{n=0}^{\infty} b_{n,\lambda}^{(N+1)} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (-2)^N N! b_{n,\lambda}^{(N+1)} \frac{t^n}{n!}.
\end{aligned} \tag{2.34}
$$

Also, by (2.23) and (2.17) we have

$$
\begin{aligned}
&(-2)^N N! t^{N+1} F^{N+1} = \sum_{k=0}^{N} b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k)} t^{N+1} \\
&= \sum_{k=0}^{N} b_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} \Big(\sum_{l=0}^{\infty} \frac{b_{l+k+1,\lambda}}{l+k+1} \frac{t^l}{l!} + (-1)^k k! \frac{1}{t^{k+1}} \Big) t^{N+1} \\
&= \sum_{k=0}^{N} b_{k,\lambda}(N) \Big(\sum_{m=0}^{\infty} \big(\frac{N}{2} + k\lambda\big)_{m,\lambda} \frac{t^m}{m!} \Big) \Big(\sum_{l=0}^{\infty} \frac{b_{l+k+1,\lambda}}{l+k+1} \frac{t^{l+N+1}}{l!} \Big) \\
&\quad + \sum_{k=0}^{N} b_{k,\lambda}(N) \Big(\sum_{m=0}^{\infty} \big(\frac{N}{2} + k\lambda\big)_{m,\lambda} \frac{t^m}{m!} \Big) (-1)^k k! t^{N-k}. \\
&= \sum_{k=0}^{N} b_{k,\lambda}(N) \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \big(\frac{N}{2} + k\lambda\big)_{n-l,\lambda} \frac{b_{l+k+1,\lambda}}{l+k+1} \frac{t^{n+N+1}}{n!} \\
&\quad + \sum_{k=0}^{N} b_{N-k,\lambda}(N) \sum_{m=0}^{\infty} \big(\frac{N}{2} + (N-k)\lambda\big)_{m,\lambda} (-1)^{N-k} (N-k)! \frac{t^{m+k}}{m!} \\
&= \sum_{n=N+1}^{\infty} \sum_{l=0}^{n-N-1} \sum_{k=0}^{N} b_{k,\lambda}(N) \binom{n-N-1}{l} \big(\frac{N}{2} + k\lambda\big)_{n-N-l-1,\lambda} \\
&\quad \times \frac{b_{l+k+1,\lambda}}{l+k+1} (n)_{N+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{k=0}^{\min\{n,N\}} b_{N-k,\lambda}(N) \\
&\quad \times \big(\frac{N}{2} + (N-k)\lambda\big)_{n-k,\lambda} (-1)^{N-k} (N-k)! (n)_k \frac{t^n}{n!}.
\end{aligned} \tag{2.35}
$$

From (2.34) and (2.35), we get the following theorem

Theorem 2.4. *For $N \in \mathbb{N} \cup \{0\}$, we have*
If $0 \le n \le N$,

$$
\begin{aligned}
(-2)^N N! b_{n,\lambda}^{(N+1)} = \sum_{k=0}^{n} b_{N-k,\lambda}(N) \big(\frac{N}{2} + (N-k)\lambda\big)_{n-k,\lambda} (-1)^{N-k} \\
\times (N-k)! (n)_k,
\end{aligned}
$$

If $n \geq N+1$,

$$\begin{aligned}(-2)^N N! b_{n,\lambda}^{(N+1)} = & \sum_{l=0}^{n-N-1}\sum_{k=0}^{N} b_{k,\lambda}(N)\binom{n-N-1}{l}\frac{b_{l+k+1,\lambda}}{l+k+1}(n)_{N+1} \\ & \times \Big(\frac{N}{2}+k\lambda\Big)_{n-N-l-1,\lambda} + \sum_{k=0}^{N} b_{N-k,\lambda}(N) \\ & \times \Big(\frac{N}{2}+(N-k)\lambda\Big)_{n-k,\lambda}(-1)^{N-k}(N-k)!(n)_k.\end{aligned}$$

Here $b_{k,\lambda}(N), (0 \leq k \leq N)$, *are as in Theorem 2.3.*

3. Some identities of type 2 degenerate Euler numbers arising from differential equations

Let

$$F = F(t;\lambda) = \frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}} + (1+\lambda t)^{-\frac{1}{2\lambda}}}. \tag{3.1}$$

Then, by taking the derivative with respect to t of (3.1), we get

$$\begin{aligned}F^{(1)} = \frac{\partial}{\partial t}F(t;x) = & \frac{-2}{\big((1+\lambda t)^{\frac{1}{2\lambda}} + (1+\lambda t)^{-\frac{1}{2\lambda}}\big)^2} \\ & \times \Big(\frac{1}{2}(1+\lambda t)^{\frac{1}{2\lambda}-1} - \frac{1}{2}(1+\lambda t)^{-\frac{1}{2\lambda}-1}\Big) \\ = & -\frac{1}{2}(1+\lambda t)^{-1}F + \frac{1}{2}(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2.\end{aligned} \tag{3.2}$$

Multiplying 2 on both sides of (3.2), we have

$$2F^{(1)} = -(1+\lambda t)^{-1}F + (1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2. \tag{3.3}$$

Let us take the derivative with respect to t of (3.3). Then we have

$$\begin{aligned}2F^{(2)} = & \lambda(1+\lambda t)^{-2}F - (1+\lambda t)^{-1}F^{(1)} + \Big(-\frac{1}{2}-\lambda\Big)(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 \\ & + 2(1+\lambda t)^{-\frac{1}{2\lambda}-1}FF^{(1)}.\end{aligned} \tag{3.4}$$

Multiplying 2 on both sides of (3.4) and by (3.3), we get

$$\begin{aligned}2^2F^{(2)} = & 2\lambda(1+\lambda t)^{-2}F + (1+\lambda t)^{-2}F - (1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 + (-1-2\lambda) \\ & \times (1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 - 2(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 + 2(1+\lambda)^{-\frac{2}{2\lambda}-2}F^3 \\ = & (1+2\lambda)(1+\lambda t)^{-2}F + (-4-2\lambda)(1+\lambda t)^{-\frac{1}{2\lambda}-2}F^2 \\ & + 2(1+\lambda t)^{-\frac{2}{2\lambda}-2}F^3.\end{aligned} \tag{3.5}$$

By proceeding inductively, we get

$$2^{N}F^{(N)}=\sum_{k=0}^{N}c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N}F^{k+1}. \tag{3.6}$$

Let us take the derivative with respect to t of (3.6). Then we get

$$\begin{aligned}2^{N}F^{(N+1)}&=\sum_{k=0}^{N}\big(-\frac{k}{2}-N\lambda\big)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\&\quad+\sum_{k=0}^{N}(k+1)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N}F^{k}F^{(1)}.\end{aligned} \tag{3.7}$$

Multiplying 2 on both sides of (3.7) and by (3.3), we get

$$\begin{aligned}2^{N+1}F^{(N+1)}&=\sum_{k=0}^{N}(-k-2N\lambda)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\&\quad+\sum_{k=0}^{N}(-k-1)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\&\quad+\sum_{k=0}^{N}(k+1)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k+1}{2\lambda}-N-1}F^{k+2}\\&=\sum_{k=0}^{N}(-2N\lambda-2k-1)c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\&\quad+\sum_{k=1}^{N+1}kc_{k-1,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}\\&=(-2N\lambda-1)c_{0,\lambda}(N)(1+\lambda t)^{-N-1}F+(N+1)c_{N,\lambda}(N)\\&\quad\times(1+\lambda t)^{-\frac{N+1}{2\lambda}-N-1}F^{N+2}+\sum_{k=1}^{N}\big((-2N\lambda-2k-1)\\&\quad\times c_{k,\lambda}(N)+kc_{k-1,\lambda}(N)\big)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}.\end{aligned} \tag{3.8}$$

Replacing N by $N+1$ in (3.6), we have

$$\begin{aligned}2^{N+1}F^{(N+1)} &= \sum_{k=0}^{N+1} c_{k,\lambda}(N+1)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1} \\ &= c_{0,\lambda}(N+1)(1+\lambda t)^{-N-1}F \\ &\quad + c_{N+1,\lambda}(N+1)(1+\lambda t)^{-\frac{N+1}{2\lambda}-N-1}F^{N+2} \\ &\quad + \sum_{k=1}^{N} c_{k,\lambda}(N+1)(1+\lambda t)^{-\frac{k}{2\lambda}-N-1}F^{k+1}.\end{aligned} \tag{3.9}$$

Comparing the coefficients of (3.8) and (3.9), we get

$$c_{0,\lambda}(N+1) = (-2N\lambda - 1)c_{0,\lambda}(N), \tag{3.10}$$

$$c_{N+1,\lambda}(N+1) = (N+1)c_{N,\lambda}(N), \tag{3.11}$$

$$c_{k,\lambda}(N+1) = (-2N\lambda - 2k - 1)c_{k,\lambda}(N) + kc_{k-1,\lambda}(N), \tag{3.12}$$

where $1 \leq k \leq N$.

From (3.3) and (3.6), we get

$$\begin{aligned}2F^{(1)} &= \sum_{k=0}^{1} c_{k,\lambda}(1)(1+\lambda t)^{-\frac{k}{2\lambda}-1}F^{k+1} \\ &= c_{0,\lambda}(1)(1+\lambda t)^{-1}F + c_{1,\lambda}(1)(1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2 \\ &= -(1+\lambda t)^{-1}F + (1+\lambda t)^{-\frac{1}{2\lambda}-1}F^2.\end{aligned} \tag{3.13}$$

From (3.13), we have

$$c_{0,\lambda}(1) = -1 \quad c_{1,\lambda}(1) = 1. \tag{3.14}$$

From (3.10),(3.11) and (3.14), we get

$$\begin{aligned}c_{0,\lambda}(N+1) &= (-2)\big(N\lambda + \frac{1}{2}\big)c_{0,\lambda}(N) = (-2)^2\big(N\lambda + \frac{1}{2}\big)_{2,\lambda}c_{0,\lambda}(N-1) \\ &= \cdots = (-2)^N\big(N\lambda + \frac{1}{2}\big)_{N,\lambda}c_{0,\lambda}(1) \\ &= (-2)^{N+1}\big(N\lambda + \frac{1}{2}\big)_{N+1,\lambda},\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}c_{N+1,\lambda}(N+1) &= (N+1)c_{N,\lambda}(N) = (N+1)_2 c_{N-1,\lambda}(N-1) = \cdots \\ &= (N+1)_N c_{1,\lambda}(1) = (N+1)!.\end{aligned} \tag{3.16}$$

Theorem 3.1. *The following family of differential equations*

$$2^N F^{(N)} = \sum_{k=0}^{N} c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N} F^{k+1},\ (N=0,1,2,\cdots)$$

have a solution $F = F(t;\lambda) = \frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}} + (1+\lambda t)^{-\frac{1}{2\lambda}}}$,
where

$$c_{0,\lambda}(N) = (-2)^N\big((N-1)\lambda + \frac{1}{2}\big)_{N,\lambda},\ \ c_{N,\lambda}(N) = N!,$$
$$c_{k,\lambda}(N+1) = (-2N\lambda - 2k - 1)c_{k,\lambda}(N) + kc_{k-1,\lambda}(N),$$
$$(1 \le k \le N-1).$$

Remark 3. From (3.12) and for $1 \le k \le N-1$, we have

$$c_{k,\lambda}(N) = k! \sum_{i_1=0}^{N-k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} (-2)^{N-k}\big((N-1)\lambda + k + \frac{1}{2}\big)_{N-k-i_1,\lambda}$$
$$\times \big((k-2+i_1)\lambda + k - 1 + \frac{1}{2}\big)_{i_1-i_2,\lambda} \times \cdots \times \big((-1+i_k)\lambda + \frac{1}{2}\big)_{i_k,\lambda}.$$

Recalling that $F = F(t;\lambda)$ is the generating function of type 2 degenerate Euler numbers, the left hand side of (3.6) is equal to

$$\begin{aligned} 2^N F^{(N)} &= 2^N \frac{\partial^N}{\partial t^N} F(t;\lambda) \\ &= 2^N \frac{\partial^N}{\partial t^N}\Big(\sum_{n=0}^{\infty} e_{n,\lambda}\frac{t^n}{n!}\Big) \\ &= \sum_{n=0}^{\infty} 2^N e_{n+N,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{3.17}$$

The right hand side of (3.6) is equal to

$$\begin{aligned} &\sum_{k=0}^{N} c_{k,\lambda}(N)(1+\lambda t)^{-\frac{k}{2\lambda}-N} F^{k+1} \\ &= \sum_{k=0}^{N} c_{k,\lambda}(N)\Big(\sum_{l=0}^{\infty}\big(-\frac{k}{2\lambda}-N\big)_l \lambda^l \frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} e_{m,\lambda}^{(k+1)}\frac{t^m}{m!}\Big) \\ &= \sum_{k=0}^{N} c_{k,\lambda}(N)\Big(\sum_{l=0}^{\infty}\langle\frac{k}{2} + N\lambda\rangle_{l,\lambda}(-1)^l \frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} e_{m,\lambda}^{(k+1)}\frac{t^m}{m!}\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\sum_{k=0}^{N} c_{k,\lambda}(N)\binom{n}{m}(-1)^{n-m}\langle\frac{k}{2} + N\lambda\rangle_{n-m,\lambda} e_{m,\lambda}^{(k+1)}\Big)\frac{t^n}{n!}. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we get the following theorem.

Theorem 3.2. *Let $N \in \mathbb{N}$, and let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$e_{n+N,\lambda} = 2^{-N} \sum_{m=0}^{n} \sum_{k=0}^{N} c_{k,\lambda}(N) \binom{n}{m} (-1)^{n-m} \big\langle \frac{k}{2} + N\lambda \big\rangle_{n-m,\lambda} e_{m,\lambda}^{(k+1)},$$

where $c_{k,\lambda}(N), (0 \leq k \leq N)$, are as in Theorem 3.1.

Now, we investigate the inversion formula of **Theorem 3.1.**
Multiplying $(1+\lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (3.3), we get

$$2(1+\lambda t)^{\frac{1}{2\lambda}+1} F^{(1)} = -(1+\lambda t)^{\frac{1}{2\lambda}} F + F^2. \tag{3.19}$$

From (3.19), we have

$$F^2 = (1+\lambda t)^{\frac{1}{2\lambda}} F + 2(1+\lambda t)^{\frac{1}{2\lambda}+1} F^{(1)}. \tag{3.20}$$

Let us take the derivative with respect to t of (3.20). Then we get

$$\begin{aligned} 2FF^{(1)} = & \frac{1}{2}(1+\lambda t)^{\frac{1}{2\lambda}-1} F + (1+\lambda t)^{\frac{1}{2\lambda}} F^{(1)} + (1+2\lambda)(1+\lambda t)^{\frac{1}{2\lambda}} F^{(1)} \\ & + 2(1+\lambda t)^{\frac{1}{2\lambda}+1} F^{(2)}. \end{aligned} \tag{3.21}$$

Multiplying $2(1+\lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (3.21), and by (3.19) and (3.20), we get

$$\begin{aligned} 2F^3 &= 2(1+\lambda t)^{\frac{1}{2\lambda}} F^2 + (1+\lambda t)^{\frac{2}{2\lambda}} F + 2(1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} + (2+4\lambda) \\ &\quad \times (1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} + 4(1+\lambda t)^{\frac{2}{2\lambda}+2} F^{(2)} \\ &= 2(1+\lambda t)^{\frac{2}{2\lambda}} F + 4(1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} + (1+\lambda t)^{\frac{2}{2\lambda}} F + (4+4\lambda) \\ &\quad \times (1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} + 4(1+\lambda t)^{\frac{2}{2\lambda}+2} F^{(2)} \\ &= 3(1+\lambda t)^{\frac{2}{2\lambda}} F + (8+4\lambda)(1+\lambda t)^{\frac{2}{2\lambda}+1} F^{(1)} + 4(1+\lambda t)^{\frac{2}{2\lambda}+2} F^{(2)}. \end{aligned} \tag{3.22}$$

By proceeding inductively, we get

$$N!F^{N+1} = \sum_{k=0}^{N} d_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k)}. \tag{3.23}$$

From (3.23), we have

$$\begin{aligned} (N+1)!F^N F^{(1)} = & \sum_{k=0}^{N} \big(\frac{N}{2} + k\lambda\big) d_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k-1} F^{(k)} \\ & + \sum_{k=0}^{N} d_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k+1)}. \end{aligned} \tag{3.24}$$

Multiplying $2(1+\lambda t)^{\frac{1}{2\lambda}+1}$ on both sides of (3.24), and by (3.19) and (3.23), we have

$$\begin{aligned}(N+1)!F^{N+2} &= (N+1)!(1+\lambda t)^{\frac{1}{2\lambda}}F^{N+1} \\ &\quad+\sum_{k=0}^{N}(N+2k\lambda)d_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\ &\quad+\sum_{k=0}^{N}2d_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k+1}F^{(k+1)} \\ &=\sum_{k=0}^{N}(2N+2k\lambda+1)d_{k,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\ &\quad+\sum_{k=1}^{N+1}2d_{k-1,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\ &= (2N+1)d_{0,\lambda}(N)(1+\lambda t)^{\frac{N+1}{2\lambda}}F+2d_{N,\lambda}(N) \\ &\quad\times(1+\lambda t)^{\frac{N+1}{2\lambda}+N+1}F^{(N+1)}+\sum_{k=1}^{N}\big((2N+2k\lambda+1) \\ &\quad\times d_{k,\lambda}(N)+2d_{k-1,\lambda}(N)\big)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)}.\end{aligned}\tag{3.25}$$

By replacing N by $N+1$ in (3.23), we get

$$\begin{aligned}(N+1)!F^{N+2} &= \sum_{k=0}^{N+1}d_{k,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)} \\ &= d_{0,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}}F \\ &\quad+d_{N+1,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+N+1}F^{(N+1)} \\ &\quad+\sum_{k=1}^{N}d_{k,\lambda}(N+1)(1+\lambda t)^{\frac{N+1}{2\lambda}+k}F^{(k)}.\end{aligned}\tag{3.26}$$

By (3.25) and (3.26), we have

$$d_{0,\lambda}(N+1)=(2N+1)d_{0,\lambda}(N),\tag{3.27}$$

$$d_{N+1,\lambda}(N+1)=2d_{N,\lambda}(N),\tag{3.28}$$

$$d_{k,\lambda}(N+1)=(2N+2k\lambda+1)d_{k,\lambda}(N)+2d_{k-1,\lambda}(N).\tag{3.29}$$

From (3.20) and (3.23), we get

$$\begin{aligned} F^2 &= \sum_{k=0}^{1} d_{k,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}+k}F^{(k)} \\ &= d_{0,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}}F + d_{1,\lambda}(1)(1+\lambda t)^{\frac{1}{2\lambda}+1}F^{(1)} \\ &= (1+\lambda t)^{\frac{1}{2\lambda}}F + 2(1+\lambda t)^{\frac{1}{2\lambda}+1}F^{(1)}. \end{aligned} \tag{3.30}$$

Thus we have

$$d_{0,\lambda}(1) = 1, \quad d_{1,\lambda}(1) = 2. \tag{3.31}$$

Form (3.27) and (3.28), we get

$$\begin{aligned} d_{0,\lambda}(N+1) &= 2\big(N+\frac{1}{2}\big)d_{0,\lambda}(N) = 2^2\big(N+\frac{1}{2}\big)_2 d_{0,\lambda}(N-1) \\ &= \cdots = 2^N\big(N+\frac{1}{2}\big)_N d_{0,\lambda}(1) = 2^{N+1}\big(N+\frac{1}{2}\big)_{N+1}, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} d_{N+1,\lambda}(N+1) &= 2d_{N,\lambda}(N) = 2^2 d_{N-1,\lambda}(N-1) = \cdots = 2^N d_{1,\lambda}(1) \\ &= 2^{N+1}. \end{aligned} \tag{3.33}$$

Thus we get the following theorem.

Theorem 3.3. *The following family of differential equations*

$$N!F^{N+1} = \sum_{k=0}^{N} d_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k}F^{(k)}, \ (N=1,2,3,\cdots)$$

have a solution $F = F(t;\lambda) = \frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}$,
where

$$\begin{aligned} &d_{0,\lambda}(N) = 2^N\big(N-\frac{1}{2}\big)_N, \ \ d_{N,\lambda}(N) = 2^N, \\ &d_{k,\lambda}(N) = (2(N-1)+2k\lambda+1)d_{k,\lambda}(N-1) + 2d_{k-1,\lambda}(N-1), \\ &(1 \le k \le N-1). \end{aligned}$$

Remark 4. From (3.29), we obtain the following expression.

$$\begin{aligned} d_{k,\lambda}(N) = &\sum_{j_1=0}^{N-k}\sum_{j_2=0}^{j_1}\cdots\sum_{j_k=0}^{j_{k-1}} 2^N\big(N-1+k\lambda+\frac{1}{2}\big)_{N-k-j_1} \\ &\times \big((k-2+j_1)+(k-1)\lambda+\frac{1}{2}\big)_{j_1-j_2} \times \cdots \times \big(j_{k-1}+\lambda+\frac{1}{2}\big)_{j_{k-1}-j_k} \\ &\times \big(-1+j_k+\frac{1}{2}\big)_{j_k}, \ \ (1 \le k \le N-1). \end{aligned}$$

From the right hand side of **Theorem 3.3**, we get

$$\begin{aligned}
&\sum_{k=0}^{N} d_{k,\lambda}(N)(1+\lambda t)^{\frac{N}{2\lambda}+k} F^{(k)} \\
&= \sum_{k=0}^{N} d_{k,\lambda}(N)\Big(\sum_{l=0}^{\infty}\big(\frac{N}{2\lambda}+k\big)_l \lambda^l \frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} e_{m+k,\lambda}\frac{t^m}{m!}\Big) \\
&= \sum_{k=0}^{N} d_{k,\lambda}(N)\Big(\sum_{l=0}^{\infty}\big(\frac{N}{2}+k\lambda\big)_{l,\lambda} \frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} e_{m+k,\lambda}\frac{t^m}{m!}\Big) \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\sum_{k=0}^{N} d_{k,\lambda}(N)\binom{n}{m}\big(\frac{N}{2}+k\lambda\big)_{n-m,\lambda} e_{m+k,\lambda}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{3.34}$$

From (3.34), we get the following theorem

Theorem 3.4. *Let $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$e_{n,\lambda}^{(N+1)} = \frac{1}{N!}\sum_{m=0}^{n}\sum_{k=0}^{N} d_{k,\lambda}(N)\binom{n}{m}\big(\frac{N}{2}+k\lambda\big)_{n-m,\lambda} e_{m+k,\lambda},$$

where $d_{k,\lambda}(N), (0 \le k \le N)$, are as in Theorem 3.3.

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