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CERTAIN CURVATURE CONDITIONS ON 3-DIMENSIONAL f-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study certain curvature conditions on three-dimensional f-Kenmotsu manifolds.

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1. Introduction

A Riemannian manifold $(M^n, g),\ n > 2$ is called generalized recurrent if its curvature tensor R of type $(1, 3)$ satisfies the condition [3]

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{1.1}$$

where α and β are two 1-forms and these are defined by

$$\alpha(X) = g(X,A),\ \beta(X) = g(X,B) \tag{1.2}$$

and A, B are vector fields associated with 1-forms α and β respectively.

The generalized recurrent manifolds have been studied by several authors such as R. S. D. Dubey [6], H. Singh and Q. Khan [13], C. Özgur [10], K. Arslan et al. [1] and many others. Specially, if the 1-form β vanishes, then (1.1) turns into the notion of a recurrent manifold introduced by A. G. Walker [16].

A Riemannian manifold $(M^n, g),\ n > 2$ is called generalized Ricci-recurrent [4] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + (n-1)\beta(X)g(Y,Z), \tag{1.3}$$

where α and β are defined in (1.2).

In 1972, K. Kenmotsu [8] introduced and studied a new class of almost contact metric manifolds, later known as Kenmotsu manifolds. Z. Olszak and R. Rosca [9] studied f-Kenmotsu manifolds, an almost contact metric manifold which is normal and locally conformal almost cosymplectic. Further, they gave a geometric interpretation of f-Kenmotsu manifold and proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold. Recently, f-Kenmotsu manifolds have been studied by many authors in several ways to a different extent such as ([2], [15], [17], [18]).

In 1977, T. Takahasi [14] introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. S. S. Shukla and M. K. Shukla have studied ϕ-Ricci symmetric Kenmotsu manifolds [11] and ϕ-symmetric para-Sasakian manifolds [12]. U. C. De, A. Yildiz and A. F. Yaliniz have studied ϕ-recurrent Kenmotsu manifolds and proved that a locally ϕ-recurrent Kenmotsu spacetime is the Robertson-Walker spacetime [5].

Motivated by the above studies, in this paper we study certain curvature conditions on 3-dimensional f-Kenmotsu manifolds. The paper is organized as follows: In Section 2, we give a brief account of a 3-dimensional f-Kenmotsu manifold and an example of 3-dimensional f-Kenmotsu manifold is also given in this section. In Section 3, we study ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifolds. Section 4 deals with the study of recurrent and ϕ-recurrent 3-dimensional f-Kenmotsu manifolds. Sections 5 and 6 are devoted to study generalized recurrent and generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifolds respectively. In Section 7, we have shown that a generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifold admitting a cyclic Ricci-tensor is an Einstein manifold.

2. f-Kenmotsu manifolds

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(X) = g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$, where I is the identity map on the tangent bundle TM, ϕ is a tensor field of $(1,1)$ type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the Levi-Civita connection of g satisfy

$$(\nabla_X \phi)(Y) = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$. If $f = 0$, then the manifold is cosymplectic [7]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.
In an f-Kenmotsu manifold, from (2.4) we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{2.5}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. This does not hold in general if dim $M = 3$ [9].

$$(\nabla_X \eta)Y = f[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.6}$$

In a 3-dimensional Riemannian manifold, we have

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \tag{2.7}$$

In a 3-dimensional f-Kenmotsu manifold, we have

$$R(X, Y)Z = (\frac{r}{2} + 2f^2 + 2f')[g(Y, Z)X - g(X, Z)Y] \tag{2.8}$$

$$-(\frac{r}{2}+3f^2+3f')[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y],$$

$$S(X,Y)=(\frac{r}{2}+f^2+f')g(X,Y)-(\frac{r}{2}+3f^2+3f')\eta(X)\eta(Y), \tag{2.9}$$

$$QX=(\frac{r}{2}+f^2+f')X-(\frac{r}{2}+3f^2+3f')\eta(X)\xi, \tag{2.10}$$

where R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.
Now from (2.8), we have

$$R(X,Y)\xi=-(f^2+f')[\eta(Y)X-\eta(X)Y], \tag{2.11}$$

$$R(\xi,X)Y=-(f^2+f')[g(X,Y)\xi-\eta(Y)X], \tag{2.12}$$

$$R(\xi,X)\xi=-(f^2+f')[\eta(X)\xi-X], \tag{2.13}$$

$$\eta(R(X,Y)Z)=-(f^2+f')[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \tag{2.14}$$

$$S(\phi X,\phi Y)=S(X,Y)+2(f^2+f')\eta(X)\eta(Y). \tag{2.15}$$

And from (2.9), we get

$$S(X,\xi)=-2(f^2+f')\eta(X), \tag{2.16}$$

$$Q\xi=-2(f^2+f')\xi. \tag{2.17}$$

Definition 2.1. *A 3-dimensional f-Kenmotsu manifold is said to be an η-Einstein manifold if the Ricci tensor S of type $(0,2)$ is of the form*

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.18}$$

where a and b are smooth functions on M. In particular, if $b=0$, then the manifold is said to be an Einstein manifold.

Example of a 3-dimensional f-Kenmotsu manifold. We consider the 3-dimensional manifold $M=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1=e^z\frac{\partial}{\partial x},\ e_2=e^z\frac{\partial}{\partial y},\ e_3=-e^z\frac{\partial}{\partial z}=\xi,$$

which are linearly independent at each point of M^3 and hence form a basis of T_pM.
Define the Riemannian metric g on M as

$$g(e_1,e_1)=\ g(e_2,e_2)=\ g(e_3,e_3)=1,\ g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

Let η be the 1-form on M defined as $\eta(X)=g(X,e_3)=g(X,\xi)$ for all $X\in\chi(M)$, and let ϕ be the $(1,1)$ tensor field on M defined as

$$\phi e_1=-e_2,\ \phi e_2=e_1,\ \phi e_3=0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi)=g(\xi,\xi)=1,\ \phi^2X=-X+\eta(X)\xi,\ \eta(\phi X)=0,$$

$$g(X,\xi)=\eta(X),\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for all $X,Y\in\chi(M)$. Now by direct computations, we obtain

$$[e_1,e_2]=0,\ [e_1,e_3]=e^ze_1,\ [e_2,e_3]=e^ze_2.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z)+Yg(Z, X)-Zg(X, Y)-g(X, [Y, Z])+g(Y, [Z, X])+g(Z, [X, Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -e^z e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = e^z e_1,\ \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_2} e_2 = -e^z e_3,\ \nabla_{e_2} e_3 = e^z e_2,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

$$\text{Let} \qquad X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M).$$

It can be easily verified that the manifold satisfies

$\nabla_X \xi = f[X - \eta(X)\xi]$ and $(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X]$ for $\xi = e_3$, where $f = e^z$.
Hence, we conclude that M is a 3-dimensional f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular 3-dimensional f-Kenmotsu manifold.

3. ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifolds

Definition 3.1. *A 3-dimensional f-Kenmotsu manifold is said to be ϕ-Ricci symmetric if the Ricci operator Q satisfies* [11]

$$\phi^2((\nabla_X Q)Y) = 0 \tag{3.1}$$

for all vector fields X, Y on M and $S(X, Y) = g(QX, Y)$. If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

In this section, we assume that a 3-dimensional f-Kenmotsu manifold is ϕ-Ricci symmetric. Then by virtue (2.1) it follows from (3.1) that

$$-(\nabla_X Q)Y + \eta((\nabla_X Q)Y)\xi = 0. \tag{3.2}$$

Taking inner product of (3.2) with Z and using (2.2), we find

$$-g[(\nabla_X Q)Y, Z] + \eta((\nabla_X Q)Y)\eta(Z) = 0$$

which can be written as

$$-g(\nabla_X QY, Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)Y)\eta(Z) = 0. \tag{3.3}$$

Now putting $Y = \xi$ in (3.3) and using (2.5), (2.17), we obtain

$$2(f^2 + f')g(\nabla_X \xi, Z) + fS(X, Z) - f\eta(X)S(\xi, Z) + \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{3.4}$$

In view of (2.5) and (2.16), (3.4) takes the form

$$2(f^2 + f')fg(X, Z) + fS(X, Z) + \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{3.5}$$

Replacing X by ϕX and Z by ϕZ in (3.5) yields

$$S(\phi X, \phi Z) + 2(f^2 + f')g(\phi X, \phi Z) = 0, \qquad f \neq 0. \tag{3.6}$$

In view of (2.3) and (2.14), (3.6) turns to

$$S(X, Z) = -2(f^2 + f')g(X, Z). \tag{3.7}$$

Contarcting X and Z in (3.7), we get

$$r = -6(f^2 + f'). \tag{3.8}$$

Thus we can state the following:

Theorem 3.2. *A ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifold is an Einstein manifold with the scalar curvature $-6(f^2+f')$.*

Now by taking covariant differentiation of (2.10) with respect to W, we have

$$(\nabla_W Q)X = [\frac{dr(W)}{2} + 2f(Wf) + W(f')]X - f(\frac{r}{2} + 3f^2 + 3f')[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(X)W]. \tag{3.9}$$

Operating ϕ^2 to the both sides of (3.9) and using (2.2), we find

$$\phi^2((\nabla_W Q)X) = [\frac{dr(W)}{2} + 2f(Wf) + W(f')](-X + \eta(X)\xi) - f(\frac{r}{2} + 3f^2 + 3f')\eta(X)(-W + \eta(W)\xi). \tag{3.10}$$

Taking X orthogonal to ξ, we get from (3.10)

$$\phi^2((\nabla_W Q)X) = -[\frac{dr(W)}{2} + 2f(Wf) + W(f')]X. \tag{3.11}$$

Thus we can state the following:

Theorem 3.3. *A 3-dimensional f-Kenmotsu manifold is locally ϕ-Ricci symmetric if and only if the scalar curvature r is constant, provided f is a constant.*

4. Recurrent and ϕ-recurrent 3-dimensional f-Kenmotsu manifolds

Definition 4.1. *A 3-dimensional f-Kenmotsu manifold is said to be recurrent if there exists a non-zero 1-form A such that*

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W \tag{4.1}$$

for all vector fields X, Y, Z and W on M.
If the 1-form A vanishes, then the manifold reduces to a symmetric manifold.

In this section, first we consider a recurrent 3-dimensional f-Kenmotsu manifold. Therefore the curvature tensor of the manifold satisfies (4.1). By a suitable contraction of (4.1), we get

$$(\nabla_X S)(Z,W) = A(X)S(Z,W). \tag{4.2}$$

This implies that

$$\nabla_X S(Z,W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = A(X)S(Z,W) \tag{4.3}$$

which by taking $W=\xi$ and then using (2.5) and (2.16) takes the form

$$-2(f^2+f')(\nabla_X \eta)Z - fS(Z,X) + f\eta(X)S(Z,\xi) = -2(f^2+f')A(X)\eta(Z). \tag{4.4}$$

By using (2.6) and (2.16) in (4.4), we find

$$S(X,Z) = -2(f^2+f')g(X,Z) - \frac{2(f^2+f')}{f}A(X)\eta(Z). \tag{4.5}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (4.5), we have

$$S(X,Z) = -2(f^2+f')g(X,Z) - \frac{2(f^2+f')}{f}\eta(X)\eta(Z), \quad f \neq 0. \tag{4.6}$$

Thus we can state the following:

Theorem 4.2. *A recurrent 3-dimensional f-Kenmotsu manifold is an η-Einstein manifold.*

If $A = 0$, then (4.5) reduces to $S(X,Z) = -2(f^2 + f')g(X,Z)$. Thus we have

Corollary 4.3. *A symmetric 3-dimensional f-Kenmotsu manifold is an Einstein manifold.*

Next we consider a ϕ-recurrent 3-dimensional f-Kenmotsu manifold.

Definition 4.4. *A 3-dimensional f-Kenmotsu manifold is said to be a ϕ-recurrent, if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z \tag{4.7}$$

for all vector fields X, Y, Z and W on M. If X, Y, Z and W are orthogonal to ξ, then the manifold is called locally ϕ-recurrent manifold. If the 1-form A vanishes, then the manifold reduces to a locally ϕ-symmetric manifold.

By taking the covariant differentation of (2.8), we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z = {} & [\frac{dr(W)}{2} + 4f(Wf) + 2W(f')][g(Y,Z)X - g(X,Z)Y] \\ & -f(\frac{r}{2} + 3f^2 + 3f')[g(Y,Z)g(X,W)\xi - 2g(Y,Z)\eta(X)\eta(W)\xi + g(Y,Z)\eta(X)W \\ & -g(X,Z)g(Y,W)\xi + 2g(X,Z)\eta(Y)\eta(W)\xi - g(X,Z)\eta(Y)W + g(Y,W)\eta(Z)X \\ & -2\eta(Y)\eta(Z)\eta(W)X + g(W,Z)\eta(Y)X - g(X,W)\eta(Z)Y - g(Z,W)\eta(X)Y \\ & +2\eta(X)\eta(W)\eta(Z)Y] - [\frac{dr(W)}{2} + 6f(Wf) + 3W(f')][g(Y,Z)\eta(X)\xi \\ & -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].\end{aligned} \tag{4.8}$$

Operating ϕ^2 to the both sides of (4.8) and using (2.2) yields

$$\begin{aligned}\phi^2((\nabla_W R)(X,Y)Z) = {} & [\frac{dr(W)}{2} + 4f(Wf) + 2W(f')][g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y] \\ & -f(\frac{r}{2} + 3f^2 + 3f')[g(Y,Z)\eta(X)\phi^2 W - g(X,Z)\eta(Y)\phi^2 W + g(Y,W)\eta(Z)\phi^2 X \\ & -2\eta(Y)\eta(Z)\eta(W)\phi^2 X + g(W,Z)\eta(Y)\phi^2 X - g(X,W)\eta(Z)\phi^2 Y - g(Z,W)\eta(X)\phi^2 Y \\ & +2\eta(X)\eta(W)\eta(Z)\phi^2 Y] - [\frac{dr(W)}{2} + 6f(Wf) + 3W(f')][\eta(Y)\eta(Z)\phi^2 X - \eta(X)\eta(Z)\phi^2 Y]\end{aligned} \tag{4.9}$$

which by using (2.1) and then considering X, Y, Z and W orthogonal to ξ reduces to

$$\phi^2((\nabla_W R)(X,Y)Z) = -[\frac{dr(W)}{2} + 4f(Wf) + 2W(f')][g(Y,Z)X - g(X,Z)Y]. \tag{4.10}$$

Thus we can state the following:

Theorem 4.5. *A 3-dimensional f-Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant, provided f is a constant.*

Now by virtue of (4.7), (4.10) can be written as

$$(4.11)\quad A(W)R(X,Y)Z = -[\frac{dr(W)}{2} + 4f(Wf) + 2W(f')][g(Y,Z)X - g(X,Z)Y].$$

Putting $W = e_i$ in (4.11), where $\{e_i\}$, $i = 1,2,3$ is an orthonomal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

where $\lambda = -\frac{1}{2A(e_i)}[dr(e_i)+8f(e_i f)+2(e_i f')]$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Thus, the manifold is of constant curvature λ. Hence, we can state the following:

Theorem 4.6. *A locally ϕ-recurrent 3-dimensional f-Kenmotsu manifold is of constant curvature.*

If r is constant, then $dr(e_i) = 0$. Thus we have

Corollary 4.7. *The curvature tensor of a locally ϕ-recurrent 3-dimensional f-Kenmotsu manifold vanishes if and only if the scalar curvature r is constant, provided f is a constant.*

5. Generalized recurrent 3-dimensional f-Kenmotsu manifolds

Let M be a generalized recurrent 3-dimensional f-Kenmotsu manifold. Then the curvature tensor of the manifold satisfies the condition (1.1). By taking $Y = W = \xi$ in (1.1), we have

$$(5.1)\qquad (\nabla_X R)(\xi,Z)\xi = \alpha(X)R(\xi,Z)\xi + \beta(X)[g(Z,\xi)\xi - g(\xi,\xi)Z].$$

The left hand side of (5.1) can be written as

$$(5.2)\quad (\nabla_X R)(\xi,Z)\xi = \nabla_X R(\xi,Z)\xi - R(\nabla_X\xi,Z)\xi - R(\xi,\nabla_X Z)\xi - R(\xi,Z)\nabla_X\xi$$

which in view of (2.5) and (2.13) takes the form

$$(5.3)\qquad (\nabla_X R)(\xi,Z)\xi = -(f^2+f')\eta(Z)\nabla_X\xi - f[R(X,Z)\xi - \eta(X)R(\xi,Z)\xi].$$

By using (2.5), (2.11) and (2.13) in the above equation, we get

$$(5.4)\qquad (\nabla_X R)(\xi,Z)\xi = 0.$$

Thus in view of (5.4), (5.1) turns to

$$(5.5)\qquad \alpha(X)R(\xi,Z)\xi + \beta(X)[g(Z,\xi)\xi - g(\xi,\xi)Z] = 0$$

which by using (2.2) and (2.12) yields

$$(5.6)\qquad \beta(X) = (f^2+f')\alpha(X).$$

Thus we can state the following:

Theorem 5.1. *In a generalized recurrent 3-dimensional f-Kenmotsu manifold, the 1-forms are related by $\beta(X) = (f^2+f')\alpha(X)$.*

6. Generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifolds

Let M be a generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifold. Then the Ricci tensor of the manifold satisfies the condition (1.3). Putting $Z = \xi$ in (1.3), we have

$$(\nabla_X S)(Y,\xi) = \alpha(X)S(Y,\xi) + 2\beta(X)g(Y,\xi). \tag{6.1}$$

The left hand side of (6.1) can be written as

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi). \tag{6.2}$$

Therefore from (6.1) and (6.2), we have

$$\nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi) = \alpha(X)S(Y,\xi) + 2\beta(X)g(Y,\xi). \tag{6.3}$$

By virtue of (2.5) and (2.16), (6.3) becomes

$$\begin{aligned} &-2(f^2+f')(\nabla_X\eta)Y - fS(X,Y) + f\eta(X)S(Y,\xi) \\ &= -2(f^2+f')\alpha(X)\eta(Y) + 2\beta(X)\eta(Y). \end{aligned} \tag{6.4}$$

By using (2.6) and (2.16), (6.4) takes the form

$$-2(f^2+f')fg(X,Y) - fS(X,Y) = -2(f^2+f')\alpha(X)\eta(Y) + 2\beta(X)\eta(Y) \tag{6.5}$$

which by taking $Y = \xi$ and then using (2.1) and (2.16) yields

$$\beta(X) = (f^2+f')\alpha(X). \tag{6.6}$$

Thus we can state the following:

Theorem 6.1. *In a generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifold, the 1-forms are related by $\beta(X) = (f^2+f')\alpha(X)$.*

7. Generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifolds admitting cyclic Ricci-tensor

In this section, we suppose that a generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifold admits a cyclic Ricci tensor S, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{7.1}$$

By virtue of (1.3) it follows from (7.1) that

$$\begin{aligned} &\alpha(X)S(Y,Z) + 2\beta(X)g(Y,Z) + \alpha(Y)S(Z,X) + 2\beta(Y)g(Z,X) \\ &+\alpha(Z)S(X,Y) + 2\beta(Z)g(X,Y) = 0. \end{aligned} \tag{7.2}$$

Putting $Z = \xi$ in (7.2) and using (2.1), (2.2) and (2.16), we have

$$\begin{aligned} &[-2(f^2+f')\alpha(X) + 2\beta(X)]\eta(Y) + [-2(f^2+f')\alpha(Y) + 2\beta(Y)]\eta(X) \\ &+\alpha(\xi)S(X,Y) + 2\beta(\xi)g(X,Y) = 0 \end{aligned} \tag{7.3}$$

which in view of (6.6), reduces to

$$\alpha(\xi)S(X,Y) + 2\beta(\xi)g(X,Y) = 0. \tag{7.4}$$

This implies that

$$S(X,Y) = Kg(X,Y), \quad \text{where } K = -\frac{2\beta(\xi)}{\alpha(\xi)}. \tag{7.5}$$

Thus we can state the following:

Theorem 7.1. *A generalized Ricci-recurrent 3-dimensional f-Kenmotsu manifold admitting a cyclic Ricci-tensor is an Einstein manifold.*

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