Advanced Studies in Contemporary Mathematics
29 (2019), No. 4, pp. 591 - 601

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.4.591

APPROXIMATE SOLUTION OF NONLINEAR OPTIMAL CONTROL PROBLEM WITH FIXED ENDS

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ABSTRACT. We consider nonlinear general problem of optimal control with fixed ends and apply two approaches for its solving: Pontryagin maximum principle and Galerkin method. We developed the new method of obtaining approximate solution for this problem and explored its basic properties.

1. Introduction

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists: Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems differ from variation problems by the additional requirements imposed on sought solution, and these requirements are sometimes difficult and even impossible to fit applying for solving the methods of the calculus of variations. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content. The central results of the theory the Pontryagin Maximum Principle and the dynamic programming method of Bellman became widely known in the scientific and engineering community, and these are now widely used in various academic fields.

Optimal control problems are classified on several types: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1,2,4,7] Our interest is related with the

2010 *Mathematics Subject Classification.* 34K35, 47N70, 93-XX.

Key words and phrases. Optimal control problem, Pontryagin maximum principle, Galerkin method, residual function, trial function.

[2] 1.7219.2017/6.7.

general problem with fixed ends and special form of objective function. We consider two approaches for solving this problem: Pontryagin maximum principle and Galerkin method and compare obtained results.

2. Statement of the problem

We consider the problem of optimal control in the form

$$J = \int_{t_0}^{t_1} (c_1x_1 + c_2x_2 + \cdots + c_nx_n)u^2 dt \to min,$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ldots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ \ldots \\ B_n \end{pmatrix} u, \quad (2.1)$$

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \ldots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \ldots \\ x_n^0 \end{pmatrix}, \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \\ \ldots \\ x_n(t_1) \end{pmatrix} = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \ldots \\ x_n^1 \end{pmatrix}, u \in U \subseteq R,$$

where $u(t)$ is control variable, $x(t) \in R^n$ is state variable, $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \ldots \\ x_n^0 \end{pmatrix}$, $x^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \ldots \\ x_n^1 \end{pmatrix}$ are fixed ends of trajectory, c_1, c_2 , $\cdots$, c_n are constants, t_0, t_1 are fixed moments of time.

Or in a brief form

$$J = \int_{t_0}^{t_1} (c^T x)u^2 dt \to \min, \dot{x} = Ax + Bu, x(t_0) = x^0, x(t_1) = x^1, u \in U \subseteq R, \quad (2.2)$$

where A and B are $n \times n$ and $n \times 1$ matrices respectively.

Note that problem (2.1) (or (2.2)) is important generalization of the minimum energy problem

$$J = \int_{t_0}^{t_1} u^2 dt \to \min, \dot{x} = Ax + Bu, x(t_0) = x^0, x(t_1) = x^1, u \in U \subseteq R$$

that arises in many applications in mechanics, economy or other areas. Objective function in problem (2.1) (or (2.2)) contains additional conditions imposed on state variables extending the variety of solving real problems. We introduce the following definition.

Definition 1. *Pair functions $x(t)$, $u(t)$ satisfying all conditions of the problem* (2.2) *and minimizing the objective function J is called its solution.*

The purpose of this research is in obtaining the efficient methods of solution of the problem (2.2). For general case of a mathematical model the problem of obtaining the exact solutions is very complicated. The most numerical methods using for solution of the problem (2.2) Newton's method, Gradient method and so on [10,11] can give an approximate solution with some acceptable or not acceptable accuracy. And the problem of convergence of applied methods has very important role. Numerical solution of the problem (2.2) was developed by a number of researchers, for instance, L.T. Aschepkov [2,6], F.P. Vasiliev [5,11], R.P. Fedorenko [8]. In general case, when we solve optimal control problem, there isn't a formula for getting unknown initial values for conjugate variables. It's worth to mention that there is a bad convergence of initial approximation for the values of conjugate variables to the values that put zeros for residual functions because of permanent getting by them their local minimum [9,10,11]. The latter means that neither Newton's method nor Gradient method don't give a good result.

In this paper, applying Pontryagin maximum principle we obtain the form of optimal control and utilizing Galerkin method with the proper choice of trial functions we get approximate solution of the problem (2.2). We show that under some conditions Galerkin method allows to obtain an exact solution of the problem (2.2). This advantage of Galerkin method can be generalized on the other types of optimal control problems.

3. Main results

By classification, the problem (2.2) is the general problem of optimal control [1,2,6,7], that is, the problem that has mobile ends of an integral curve. It has the form

$$J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min$$

$$J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1) \begin{cases} \leq 0, i = 1, 2, ..., m_0, \\ = 0, i = m_0 + 1, ..., m, \end{cases} \tag{3.1}$$

$$\dot{x} = f(x, u, t), u \in U, t_1 \geq t_0.$$

Here $\Phi_0, \Phi_1, ..., \Phi_m$are the given functions of the class $C_1(R^n \times R^n \times R \times R \to R)$, m_0 is an integer nonnegative number, and m is a natural number. If $m_0 = 0$ or $m_0 = m$, then the general problem only has constraints-equalities $J_i = 0, i = 0, 1, ..., m$, or only constraints-inequalities $J_i \leq 0, i = 0, 1, ..., m$, respectively. The *process* is said to be a quaternion $x(t), u(t), t_0, t_1$ that satisfies all conditions of the general problem except, possibly, the first condition. A process $x(t), u(t), t_0, t_1$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t), \tilde{t}_0, \tilde{t}_1$, the following inequality is true

$$\Phi_0(x(t_0), x(t_1), t_0, t_1) \leq \Phi_0(\tilde{x}(t_0), \tilde{x}(t_1), \tilde{t}_0, \tilde{t}_1).$$

The general problem consists of determining the optimal process.
The necessary conditions of optimality are defined by the Pontryagin maximum principle [1,2,7].

Theorem 3.1. ***(maximum principle for the general problem)***
Let $x(t), u(t), t_0, t_1$ be an optimal process of the general problem. Then there exists a vector $\lambda = (\lambda_0, \lambda_1, ..., \lambda_m)$ and a continuous solution $\psi(t)$ of a conjugate system of differential equations $\dot{\psi} = -H(\psi, x(t), u(t), t)$, satisfying conditions:
1) non-triviality, non-negativity, and complementary slackness
$$\lambda \neq 0, \lambda_i \geq 0, i = 0, 1, ..., m_0, \lambda_i \Phi_i(x(t_0), x(t_1), t_0, t_1) = 0, i = 1, 2, ..., m_0;$$
2) transversality
$$\psi(t_0) = L_{x(t_0)}(\lambda, x(t_0), x(t_1), t_0, t_1), \psi(t_1) = -L_{x(t_1)}(\lambda, x(t_0), x(t_1), t_0, t_1),$$
$$\frac{d}{dt_0} L(\lambda, x(t_0), x(t_1), t_0, t_1) = 0, \frac{d}{dt_1} L(\lambda, x(t_0), x(t_1), t_0, t_1) = 0;$$
3) maximum of Hamiltonian
$$H(\psi(t), x(t), u(t), t) = \max_{v \in U} H(\psi(t), x(t), v, t), t_0 \leq t \leq t_1$$
with Lagrange and Hamilton functions
$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \sum_{i=0}^{m} \lambda_i \Phi_i(x(t_0), x(t_1), t_0, t_1), H(\psi, x, u, t) = \sum_{j=1}^{n} \psi_j f_j(x, u, t).$$

Transforming the problem (2.2) to standard form (3.1) yields
$$\begin{gathered} x_{n+1}(t_1) \to \min, \\ \dot{x} = Ax + Bu, \dot{x}_{n+1} = (c^T x)u^2, \\ x(t_0) = x^0, x_{n+1}(t_0) = 0 \\ x(t_1) = x^1, u \in U \subseteq R. \end{gathered} \tag{3.2}$$

We form Lagrange and Hamilton functions
$$H(\psi, x, u, t) = \psi^T(Ax + Bu) + \psi_{n+1}(c^T x)u^2,$$
$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_{n+1}(t_1) + \sum_{i=1}^{n} \lambda_i(x_i(t_0) - x_i^0) + \sum_{i=n+1}^{2n} \lambda_i(x_i(t_1) - x_i^1).$$
Since the left end of a trajectory and times moments $t_0$and t_1 are fixed, corresponding transversality conditions are satisfied automatically (see [1,2]) and we can simplify second, in above, function omitting the terms related with equalities $x(0) = x^0$and $x_{n+1}(0) = 0$. That is,
the Lagrange function can be rewritten
$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_{n+1}(t_1) + \sum_{i=1}^{n} \lambda_i(x_i(t_1) - x_i^1).$$
We get the conjugate system
$$\left\{ \begin{array}{c} \dot{\psi} = -A^T\psi - c\psi_{n+1}u^2 \\ \dot{\psi}_{n+1} = 0 \end{array} \right. .$$
Its solution is $\psi(t, \tilde{C}_1, \tilde{C}_2, \cdots, \tilde{C}_n, \mu)$, $\psi_{n+1} = \mu$, where $\tilde{C}_1, \tilde{C}_2, \cdots, \tilde{C}_n, \mu$ are constants of integration.

Maximum of the Hamiltonian condition is reduced to solution of the extreme problem

$$H(\psi, x, u, t) = ... + \psi^T Bu + \mu(c^T x)u^2 \to \max_{u \in R}, t_0 \le t \le t_1,$$

where we denote the terms that don't depend on control variables by three dot. Analysis of the latter problem arrive us at the following optimal control function:

$u^{opt}(t) = \frac{\psi^T B}{2\mu(c^T x)}$, for $\mu(c^T x) < 0$,

or in coordinate form

$$u^{opt}(t) = \frac{1}{2\mu(c_1 x_1 + c_2 x_2 + ... + c_n x_n)} \sum_{j=1}^{n} B_j \psi_j(t). \tag{3.3}$$

If $\mu(c^T x) \ge 0$we obtain unusable control $u(t) = \pm\infty$.

To obtain solution of the problem (3.2), we solve the system 2n differential equations

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j + \frac{1}{2\mu(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)} B_i \sum_{j=1}^{n} B_j \psi_j$$

$$\dot{\psi}_i = \frac{d\psi_i}{dt} = -\sum_{j=1}^{n} A_{ji} \psi_j - c_i \frac{1}{4\mu^2(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)^2} \left(\sum_{j=1}^{n} B_j \psi_j \right)^2, i = 1, \cdots, n \tag{3.4}$$

with 2n boundary conditions

$$x(t_0) = x^0, x(t_1) = x^1.$$

Differential equation $\dot{x}_{n+1} = (c^T x)u^2$and initial condition $x_{n+1}(t_0) = 0$ can be omitted here since they define an objective function of the original problem.

Nonlinearity and complexity of the obtained system (3.4) don't unable us to get its solution analytically. One of the effective methods for its solving is Galerkin one.

According to Galerkin method [3], we construct approximate solution in the form

$$x_j \approx \hat{x}_j = x_j^0 + \sum_{k=1}^{M} a_{j,k} \varphi_{j,k}(t),$$

$$\psi_j \approx \hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k} \varphi_{n+j,k}(t), j = 1, \cdots, n. \tag{3.5}$$

Here $\varphi_{j,k}(t), j = 1, \cdots, 2n, k = 1, \cdots, M$ are trial functions satisfying the following conditions

$$\varphi_{j,k}(t_0) = 0, j = 1, \cdots, n, k = 1, \cdots, M, \tag{3.6}$$

and

$$\forall j = 1, \cdots, n, \exists k, l, m : \varphi_{j,k}(t_1) \ne 0, \varphi_{n+j,l}(t_0) \ne 0, \varphi_{n+j,m}(t_1) \ne 0.$$

First condition guarantees that $x(t_0) = x^0$ and second one ensures non-triviality of a conjugate function $\psi(t)$.

The choice of a set of trial functions is critical for realization of Galerkin method. The basic requirement is: the functions $\varphi_{j,k}(t), j = 1, \cdots, 2n, k = 1, \cdots, M$ must be linearly independent on given interval $[t_0, t_1]$.

Assuming that trial functions are continuous and differentiable, we have

$$\frac{d}{dt}x_j \approx \frac{d}{dt}\hat{x}_j = \sum_{k=1}^{M} a_{j,k}\frac{d}{dt}\varphi_{j,k}(t), \tag{3.7}$$

$$\frac{d}{dt}\psi_j \approx \frac{d}{dt}\hat{\psi}_j = \sum_{k=1}^{M} a_{n+j,k}\frac{d}{dt}\varphi_{n+j,k}(t), j = 1, \cdots, n.$$

Substituting (3.5) into (3.4) and equating corresponding derivatives in (3.4) and (3.7) yields

$$\begin{aligned}
&\sum_{k=1}^{M} a_{i,k}\frac{d}{dt}\varphi_{i,k}(t) \approx \sum_{j=1}^{n} A_{ij}(x_j^0 + \sum_{k=1}^{M} a_{j,k}\phi_{j,k}(t)) \\
&+ \frac{1}{2\mu\left(c_1(x_1^0+\sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t))+c_2(x_2^0+\sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t))+\cdots+c_n(x_n^0+\sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t))\right)} \\
&\times B_i \sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t),
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{M} a_{n+i,k}\frac{d}{dt}\varphi_{n+i,k}(t) \approx -\sum_{j=1}^{n} A_{ji} \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t)) \\
&- c_j \frac{1}{4\mu^2\left(c_1(x_1^0+\sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t))+c_2(x_2^0+\sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t))+\cdots+c_n(x_n^0+\sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t))\right)^2} \\
&\times \left(\sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t)\right)^2, \ (i = 1, \cdots, n).
\end{aligned}$$

Residual functions for variables x_i and ψ_i are accordingly

$$\begin{aligned}
&R_{[t_0,t_1]}^{x_i} = \sum_{k=1}^{M} a_{i,k}\frac{d}{dt}\varphi_{i,k}(t) - \sum_{j=1}^{n} A_{ij}(x_j^0 + \sum_{k=1}^{M} a_{j,k}\phi_{j,k}(t)) \\
&- \frac{1}{2\mu\left(c_1(x_1^0+\sum_{k=1}^{M} a_{1,k}\phi_{1,k}(t))+c_2(x_2^0+\sum_{k=1}^{M} a_{2,k}\phi_{2,k}(t))+\cdots+c_n(x_n^0+\sum_{k=1}^{M} a_{n,k}\phi_{n,k}(t))\right)} \\
&\times B_i \sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t),
\end{aligned}$$

$$R^{\psi_i}_{[t_0,t_1]} = \sum_{k=1}^{M} a_{n+i,k}\frac{d}{dt}\varphi_{n+i,k}(t) + \sum_{j=1}^{n} A_{ji}\sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t)$$

$$c_j \frac{1}{4\mu^2\left(c_1(x_1^0+\sum_{k=1}^M a_{1,k}\phi_{1,k}(t))+c_2(x_2^0+\sum_{k=1}^M a_{2,k}\phi_{2,k}(t))+\cdots+c_n(x_n^0+\sum_{k=1}^M a_{n,k}\phi_{n,k}(t))\right)^2}$$

$$\times\left(\sum_{j=1}^{n} B_j \sum_{k=1}^{M} a_{n+j,k}\phi_{n+j,k}(t)\right)^2,\ (i=1,\cdots,n).$$

By Galerkin method [3], we have

$$\int_{t_0}^{t_1} R^{x_s}_{[t_0,t_1]} W_{s,k}dt = 0, s=1,\cdots,n; k=1,\cdots,M,$$

$$\int_{t_0}^{t_1} R^{\psi_{s-n}}_{[t_0,t_1]} W_{s,k}dt = 0, s=n+1,\cdots,2n; k=1,\cdots,M,$$

where $W_{s,k}$ are weight functions. For simplicity, we choose weight functions $W_{s,k}$as

$$W_{s,k}=\varphi_{s,k}, s=1,\cdots,2n; k=1,\cdots,M.$$

To satisfy the condition $x(t_1)=x^1$, we add the term

$$(\hat{x}_{s-n}-x^1_{s-n})\left.\tilde{W}_{s,k}\right|_{t=t_1}, s=n+1,\cdots,2n; k=1,\cdots,M$$

into the second equation. It is reasonable, for convenience, to take

$$\tilde{W}_{s,k}=-\varphi_{s,k}, s=n+1,\cdots,2n; k=1,\cdots,M.$$

Finally, we get the following linear system of 2nM equations in 2nM variables $a_{j,k}$:

$$\begin{aligned}&\int_{t_0}^{t_1} R^{x_s}_{[t_0,t_1]}\varphi_{s,k}dt = 0, s=1,\cdots,n; k=1,\cdots,M,\\ &\int_{t_0}^{t_1} R^{\psi_{s-n}}_{[t_0,t_1]}\varphi_{s,k}dt + (\hat{x}^0_{s-n}+\sum_{k=1}^{M} a_{s-n,k}\varphi_{s-n,k}(t_1)-x^1_{s-n})\tilde{W}_{s,k}(t_1)=0,\\ &s=n+1,\cdots,2n; k=1,\cdots,M.\end{aligned} \tag{3.8}$$

Solution of the linear system (3.8) gives an approximate solution (3.5) of the problem (3.2). The accuracy of approximation depends on the choice of trial functions and the exact solution of optimal control problem. Convergence of the method is related with the solvability of the system (3.8). Proper choice of weight and trial functions allows us to get an exact solution.

4. Illustrating example

We solve general optimal control problem

$$J=\int_0^1 (x_1+x_2)u^2dt \to \min,$$

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = u\end{cases} \tag{4.1}$$

$$x_1(0) = 1, x_2(0) = 0,$$
$$x_1(1) = 1, x_2(1) = 1,$$
$$u \in R.$$

by means of two methods – using maximum principle and Galerkin method.
Approach 1. We use Pontryagin maximum principle.
Solution. At first, we transform problem (4.1) to standard form

$$J = x_3(1) \to \min,$$
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ \dot{x}_3 = (x_1 + x_2)u^2 \end{cases} \tag{4.2}$$
$$x_1(0) = 1, x_2(0) = 0, x_3(0) = 0,$$
$$x_1(1) = 1, x_2(1) = 1, u \in R.$$

We form Lagrange and Hamilton functions

$$H(\psi, x, u, t) = \psi_1 x_2 + \psi_2 u + \psi_3(x_1 + x_2)u^2,$$
$$L(\lambda, x^0, x^1, t_0, t_1) = \lambda_0 x_3(1) + \lambda_1(x_1(1) - 1) + \lambda_2(x_2(1) - 1),$$

conjugate system $\begin{cases} \dot{\psi}_1 = -\psi_3 u^2 \\ \dot{\psi}_2 = -\psi_1 - \psi_3 u^2 \\ \dot{\psi}_3 = 0 \end{cases}$.

Note that initial conditions $x_1(0) = 1, x_2(0) = 0, x_3(0) = 0$ are not included in Lagrange function since left side of trajectory is fixed and, as a result, transversality conditions at initial time $t_0 = 0$ hold automatically (see [1]).

From conjugate system, we determine

$$\psi_3(t) = C_3. \tag{4.3}$$

Conditions 1 and 2 of the theorem are:

$$1)\lambda_0 \geq 0, \lambda_0 + |\lambda_1| + |\lambda_2| \neq 0;$$
$$2) \begin{cases} \psi_1(1) = -\lambda_1 \\ \psi_2(1) = -\lambda_2 \\ \psi_3(1) = -\lambda_0 \end{cases}. \tag{4.4}$$

From (4.3) and (4.4), we get

$$\psi_3(t) = -\lambda_0. \tag{4.5}$$

Case (a): $\lambda_0 = 0$, $\lambda_1 = 0$, $\lambda_2 \neq 0$. Then solution of conjugate system is

$$\begin{cases} \psi_1(t) = 0 \\ \psi_2(t) = -\lambda_2 \\ \psi_3(t) = 0 \end{cases}$$

And condition 3 of the theorem becomes

$$H(\psi, x, u, t) = -\lambda_2 u \to \max_{u \in R}, 0 \leq t \leq 1.$$

Solution of the last extreme problem is $u(t) = \pm\infty$. Unusable solution.

Case (b): $\lambda_0 = 0$, $\lambda_1 \neq 0$, $\lambda_2 = 0$. And conjugate system becomes $\begin{cases} \psi_1(t) = -\lambda_1 \\ \psi_2(t) = \lambda_1(t-1) \\ \psi_3(t) = 0 \end{cases}$.

Solution of the extreme problem

$$H(\psi, x, u, t) = \lambda_1(t-1)u \to \max_{u\in R}, 0 \le t \le 1$$

is $u(t) = \pm\infty$. Again, unusable solution.

Case (c): $\lambda_0 > 0$ $(\lambda_0 = 1)$,$\lambda_1 - \forall$, $\lambda_2 - \forall$. Form conjugate system we determine

$$\psi_3(t) = -1.$$

Condition 3 of the theorem has the form

$$H(\psi, x, u, t) = \psi_2 u - (x_1 + x_2)u^2 \to \max_{u\in R}, 0 \le t \le 1.$$

Its solution (optimal control) is $u^{opt}(t) = \frac{\psi_2}{2(x_1+x_2)}$ in case if $x_1(t) + x_2(t) > 0, 0 \le t \le 1$. If $x_1(t) + x_2(t) \le 0$ for some $t \in [0,1]$ we get unusable solution $u(t) = \pm\infty$.

We cannot use directly this control for getting optimal trajectory since functions ψ_2, $x_1$and x_2 are not defined. Nevertheless, this form of optimal control is the basis for obtaining approximate solution of the problem (4.2) by Galerkin method.

Approach 2. Using of Galerkin method.

We take the following trial functions:

$$\varphi_{1,1}(t) = \varphi_{2,1} = t, \varphi_{1,2}(t) = \varphi_{2,2} = t^2, \varphi_{1,3}(t) = \varphi_{2,3} = t^3;$$

$$\varphi_{3,1}(t) = \varphi_{4,1} = 1, \varphi_{3,2}(t) = \varphi_{4,2} = t, \varphi_{3,3}(t) = \varphi_{4,3} = t^2,$$

satisfying the conditions (3.6). We form approximate solution (3.5) of the system (3.4)

$$\begin{aligned} \hat{x}_1 &= 1 + a_{1,1}t + a_{1,2}t^2 + a_{1,3}t^3, \\ \hat{x}_2 &= 0 + a_{2,1}t + a_{2,2}t^2 + a_{2,3}t^3, \\ \hat{\psi}_1 &= a_{3,1} + a_{3,2}t + a_{3,3}t^2, \\ \hat{\psi}_2 &= a_{4,1} + a_{4,2}t + a_{4,3}t^2. \end{aligned} \tag{4.6}$$

According to (3.3), optimal control is $u^{opt}(t) = \frac{\psi_2}{2(x_1+x_2)}$.

Substituting (4.6) into (3.4), forming residual functions and integrating expressions (3.8), yield the following nonlinear system:

$$\int_0^1 (a_{1,1} + (2a_{1,2} - a_{2,1})t + (3a_{1,3} - a_{2,2})t^2 - a_{2,3}t^3)tdt = 0,$$

$$\int_0^1 (a_{1,1} + (2a_{1,2} - a_{2,1})t + (3a_{1,3} - a_{2,2})t^2 - a_{2,3}t^3)t^2dt = 0,$$

$$\int_0^1 (a_{1,1} + (2a_{1,2} - a_{2,1})t + (3a_{1,3} - a_{2,2})t^2 - a_{2,3}t^3)t^3dt = 0,$$

$$\int_0^1 \left(a_{2,1} + 2a_{2,2}t + 3a_{2,3}t^2 - \frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right) tdt = 0,$$

$$\int_0^1 \left(a_{2,1} + 2a_{2,2}t + 3a_{2,3}t^2 - \frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right) t^2dt = 0,$$

$$\int_0^1 \left(a_{2,1} + 2a_{2,2}t + 3a_{2,3}t^2 - \frac{a_{4,1}+a_{4,2}t+a_{4,3}t^2}{2(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)}\right) t^3dt = 0,$$

$$\int_0^1 \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)dt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$
$$\int_0^1 \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)tdt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$
$$\int_0^1 \left(a_{3,2}+2a_{3,3}t-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)t^2dt-a_{1,1}-a_{1,2}-a_{1,3}=0,$$
$$\int_0^1 \left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)dt$$
$$-a_{2,1}-a_{2,2}-a_{2,3}=0,$$
$$\int_0^1 \left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)tdt$$
$$-a_{2,1}-a_{2,2}-a_{2,3}=0,$$
$$\int_0^1 \left((a_{3,1}+a_{4,2})+(a_{3,2}+2a_{4,3})t+a_{3,3}t^2-\frac{(a_{4,1}+a_{4,2}t+a_{4,3}t^2)^2}{4(1+(a_{1,1}+a_{2,1})t+(a_{1,2}+a_{2,2})t^2+(a_{1,3}+a_{2,3})t^3)^2}\right)t^2dt$$
$$-a_{2,1}-a_{2,2}-a_{2,3}=0.$$

Its solution is

$$a_{1,1}=-0.3949, a_{1,2}=-0.309, a_{1,3}=0.704,$$
$$a_{2,1}=-3.584, a_{2,2}=8.036, a_{2,3}=-3.456,$$
$$a_{3,1}=-13.539, a_{3,2}=0, a_{3,3}=0,$$
$$a_{4,1}=-1.98, a_{4,2}=13.539, a_{4,3}=0.$$

And, finally, we get approximate solution of the problem (4.2)

$$\hat{x}_1(t)=1-0.3949t-0.309t^2+0.704t^3,$$
$$\hat{x}_2(t)=-3.584t+8.036t^2-3.456t^3,$$
$$\hat{\psi}_1(t)=-13.539,$$
$$\hat{\psi}_2(t)=-1.98+13.539t,$$

$$u^{opt}(t)=\frac{-1.98+13.539t}{2-7.9506t+15.4547t^2-5.504t^3}, 0\le t\le 1.$$

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Funding

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

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