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LIE ALGEBRA AND FORMULAS FOR SOLUTIONS FOR SOME SYSTEMS OF DIFFERENCE EQUATIONS

M. FOLLY-GBETOULA

ABSTRACT. We construct infinitesimal generators of some systems of difference equations. We use these generators to construct similarity variables that enable us to obtain exact solutions. Periodicity of the solutions are studied for some specific cases. The work in this paper is a generalization of a recent result by Elsayed [E. M. Elsayed, The expressions of solutions and periodicity for some nonlinear systems of rational difference equations, Advanced Studies in Contemporary Mathematics **25:3** (2015), 341–367].

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1. INTRODUCTION

Over a century ago, Sophus Lie analyzed the invariance of equations under a continuous group of transformations later known as Lie symmetry method. Symmetry analysis has been utilized to solve differential equations for many decades and has been applied to difference equations recently. Even though Maeda studied difference equations from Lie symmetry approach in twentieth century [14, 15], Hydon has rekindled interests for solving difference equations via symmetry after his powerful work in [12]. It is now known that most (if not all) of the roles of important tools in the study of differential equations, such as symmetry, Noether's theorems, conservation laws, multipliers, etc, are retained when discrete analogues are considered (see [11, 18, 17]). One of the roles of symmetry methods is the reduction of order and/or number of variables of differential equations via compatible canonical coordinates or similarity variables and this applies to difference equations also.

Very often, these difference equations arise from the discretization of differential equations, especially with time dependent phenomena, and several methods for discretizing differential equations exist in the literature, for example, [4]. On this note, difference equations have applications in numerous fields that deal with differential equations, for example, biology, study of population (growth and movement), economy, physics [1, 5, 2]. Although difference equations look simple, their solutions can be very difficult to find. Various methods for solving difference equations exist; but to the best of our knowledge, the symmetry approach is somewhat new. The reader can refer to a book by Hydon [12] and some of the recent articles [7, 8, 9, 10] for further knowledge on solving difference equations via the symmetry approach.

In this paper, motivated by the findings in [6] where the author investigated the expressions of solutions and periodic nature of the systems

$$(1)\qquad x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-3}y_{n-2}x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-3}}{\pm1+\pm x_n y_{n-1}x_{n-2}y_{n-3}},$$

we extend the results in [6] by considering the system of difference equations

$$(2)\qquad x_{n+1}=\frac{x_{n-3}}{a_n+b_n x_{n-3}y_{n-2}x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-3}}{c_n+d_n x_n y_{n-1}x_{n-2}y_{n-3}},$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. For the systems of the form (2), we derive all Lie point symmetries and we give formulas for closed form solutions. We will explain how all the theorems in [6] are obtained using the generalized solutions of (2) that we will present in this paper. Furthermore, we will discuss the periodicity character of some special cases.

1.1. **Preliminaries.** We start with some background of symmetry methods for difference equations. Note that throughout this paper, we utilize definitions and notation in [12, 16, 3].
Consider the two dimensional system of fourth order ordinary difference equations of the form

$$(3)\qquad x_{n+4}=\Omega_1(x_n,x_{n+2},y_{n+1},y_{n+3}),\quad y_{n+4}=\Omega_2(x_{n+1},x_{n+3},y_n,y_{n+2}),$$

where n denotes the independent variable; x_n and y_n the dependent variables. Note that u_{n+i} denotes the 'i-th shift' of u_n.
Consider the group of transformations

$$(4)\qquad (n,x_n,y_n)\mapsto(n,\tilde{x}_n=x_n+\varepsilon Q_1(n,x_n)+O(\varepsilon^2),\tilde{y}_n=y_n+\varepsilon Q_2(n,y_n)+O(\varepsilon^2)),$$

where $Q=(Q_1,Q_2)$ is the characteristic of the group of point transformations. Let

$$(5)\qquad X=Q_1\frac{\partial}{\partial x_n}+Q_2\frac{\partial}{\partial y_n}$$

be the corresponding infinitesimal generator. The group of transformations (4) is a symmetry group if and only if

$$(6)\qquad Q_i(n+4,\Omega_i)-\mathcal{X}(\Omega_i)=0,\quad i=1,2$$

whenever (3) holds. Here, $\mathcal{X}$ denotes the prolongation of X to all shifts of x_n and y_n appearing in the right hand sides of equations in (3). Equations in (6), known as the linearized symmetry condition, can be solved for Q_i by applying the appropriate differential operators. The Q_i's, together with the canonical coordinates [13]

$$(7)\qquad s^k=\int\frac{du^k}{Q_i(n,u^k)},\quad i=1,2,$$

are necessary in the reduction of order of (3). One can also use the following definition to verify whether or not a given function is invariant under a given group of transformations.

Definition 1. [16] *Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \text{for all} \qquad x \in M.$$

2. Main results

As mentioned earlier, the equation under study is the extension of equation (1) to (2). We first consider the forward form

$$x_{n+4} = \frac{x_n}{A_n + B_n x_n y_{n+1} x_{n+2} y_{n+3}},\ y_{n+4} = \frac{y_n}{C_n + D_n x_{n+3} y_n x_{n+1} y_{n+2}} \tag{8}$$

of (2), where $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, $(C_n)_{n\in\mathbb{N}_0}$ and $(D_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers.

2.1. **Symmetries and difference invariants.** The criterion that gives the Lie point symmetries is obtained by forcing (6) on (8). This leads to

(9)
$$S^4 Q_1 + \frac{B_n x_n^2 (x_{n+2} y_{n+3} S Q_2 + x_{n+2} y_{n+1} S^3 Q_2 + y_{n+1} y_{n+3} S^2 Q_1) - A_n Q_1}{(A_n + B_n x_n y_{n+1} x_{n+2} y_{n+3})^2} = 0$$

(10)
$$S^4 Q_2 + \frac{D_n y_n^2 (x_{n+1} x_{n+3} S^2 Q_2 + x_{n+3} y_{n+2} S Q_1 + x_{n+1} y_{n+2} S^3 Q_1) - C_n Q_2}{(C_n + D_n x_{n+3} y_n x_{n+1} y_{n+2})^2} = 0.$$

Note that $S^i : n \to n+i$ denotes the forward shift operator. The procedure for solving these functional equations is as follows:

- Firstly, apply the differential operator $\frac{\partial}{\partial x_n} + \frac{A_n}{B_n x_n^2 + x_{n+2} y_{n+3}} \frac{\partial}{\partial y_{n+1}}$ (respectively $\frac{\partial}{\partial y_n} + \frac{C_n}{D_n x_{n+3} y_n^2 y_{n+2}} \frac{\partial}{\partial x_{n+1}}$) on the first (respectively second) equation in (9).
- Secondly, clear fractions in the resulting equations and differentiate the first (resp. second) equation with respect to x_n (resp. y_n) twice.
- Lastly, solve the differential equations for Q_i, $i = 1, 2$, and substitute the latter in (9) to eliminate any dependency among the arbitrary functions that appear in Q.

The computation has been omitted because of tediousness. Following the above procedure, we have that

$$Q = (Q_1, Q_2) = (\alpha(n) x_n, \beta(n) y_n), \tag{11}$$

where α and β are functions of n satisfying

$$\alpha(n) + \alpha(n+2) + \beta(n+1) + \beta(n+3), \tag{12a}$$

$$\alpha(n+1) + \alpha(n+3) + \beta(n+2) + \beta(n), \tag{12b}$$

$$\alpha(n) = \alpha(n+4),\ \beta(n) = \beta(n+4). \tag{12c}$$

Therefore, the system of equations (8) admits a six dimensional Lie point symmetries, namely

$$X_1 = x_n\partial x_n - y_n\partial y_n, \tag{13}$$

$$X_2 = i^n x_n\partial x_n, \tag{14}$$

$$X_3 = (-i)^n x_n\partial x_n, \tag{15}$$

$$X_4 = (-1)^n x_n\partial x_n + (-1)^n y_n\partial y_n, \tag{16}$$

$$X_5 = (-i)^n y_n\partial y_n, \tag{17}$$

$$X_6 = i^n y_n\partial y_n. \tag{18}$$

The canonical coordinates, using X_1, are given by

$$s^1(n) = \int \frac{dx_n}{x_n} = \ln|x_n| \quad \text{and} \quad s^2(n) = \int \frac{dy_n}{-y_n} = -\ln|y_n| \tag{19}$$

and the difference invariants which are inspired by the form of the final constraints (12) are given by

(20a)

$$\mathbf{u}_n = s^1(n) + s^1(n+2) - s^2(n+1) - s^2(n+3) = \ln|x_n x_{n+2} y_{n+1} y_{n+3}|$$

(20b)

$$\mathbf{v}_n = s^1(n+1) + s^1(n+3) - s^2(n) - s^2(n+2) = \ln|x_{n+1} x_{n+3} y_n y_{n+2}|.$$

Note that we are utilizing X_1 so $\alpha(n) = 1$ and $\beta(n) = -1$. It is easy to check, using the definition 1 together with (13), that (20) are indeed invariant under the group of transformations of (8). However, for simplicity, we prefer using the compatible variables

$$|u_n| = \exp(-\mathbf{u}_n) \quad \text{and} \quad |v_n| = \exp(-\mathbf{v}_n) \tag{21}$$

which are also invariant functions. This gives a convenient choice of the change variables which need not any lucky guesses.

2.2. **Reduction and exact solutions.** Using the new variables given in (21), equation (8) can be reduced to first-order difference equations $u_{n+1} = C_n v_n + D_n$ and $v_{n+1} = A_n u_n + B_n$. The latter can further be iterated to get

$$u_{n+2} = \hat{\alpha}_n u_n + \hat{\beta}_n, \; v_{n+2} = \hat{\gamma}_n v_n + \hat{\lambda}_n, \tag{22}$$

where $\hat{\alpha}_n = A_n C_{n+1},\ \hat{\beta}_n = B_n C_{n+1} + D_{n+1},\ \hat{\gamma}_n = A_{n+1} C_n,\ \hat{\lambda}_n = A_{n+1} D_n + B_{n+1}$. On the one hand, the iteration of (22) permits one to write its closed form solutions as follows:

$$u_n = u_j \prod_{k=0}^{n-1} \hat{\alpha}_{2k+j} + \sum_{l=0}^{n-1} \left(\hat{\beta}_{2l+j} \prod_{k=l+1}^{n-1} \hat{\alpha}_{2k+j} \right), \tag{23a}$$

$$v_n = v_j \prod_{k=0}^{n-1} \hat{\gamma}_{2k+j} + \sum_{l=0}^{n-1} \left(\hat{\lambda}_{2l+j} \prod_{k=l+1}^{n-1} \hat{\gamma}_{2k+j} \right), \tag{23b}$$

for $j = 0, 1$.

On the other hand, it follows from (8) that

$$x_{n+4} = \frac{u_n}{v_{n+1}} x_n, \quad y_{n+4} = \frac{v_n}{u_{n+1}} y_n \tag{24}$$

and thus (after straightforward iterations)

$$x_{4n+j} = x_j \prod_{k=0}^{n-1} \frac{u_{4k+j}}{v_{4k+1+j}}, \quad y_{4n+j} = y_j \prod_{k=0}^{n-1} \frac{v_{4k+j}}{u_{4k+1+j}}, \quad j = 0, 1, 2, 3. \tag{25}$$

So, thanks to (23), we have

$$x_{4n} = x_0 \prod_{k=0}^{n-1} \frac{u_0 \prod\limits_{k_1=0}^{2k-1} \hat{\alpha}_{2k_1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\beta}_{2l} \prod\limits_{k_2=l+1}^{2k-1} \alpha_{2k_2} \right)}{v_1 \prod\limits_{k_1=0}^{2k-1} \hat{\gamma}_{2k_1+1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\lambda}_{2l+1} \prod\limits_{k_2=l+1}^{2k-1} \hat{\gamma}_{2k_2+1} \right)}, \tag{26a}$$

$$x_{4n+1} = x_1 \prod_{k=0}^{n-1} \frac{u_1 \prod\limits_{k_1=0}^{2k-1} \hat{\alpha}_{2k_1+1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\beta}_{2l+1} \prod\limits_{k_2=l+1}^{2k-1} \hat{\alpha}_{2k_2+1} \right)}{v_0 \prod\limits_{k_1=0}^{2k} \hat{\gamma}_{2k_1} + \sum\limits_{l=0}^{2k} \left(\hat{\lambda}_{2l} \prod\limits_{k_2=l+1}^{2k} \hat{\gamma}_{2k_2} \right)}, \tag{26b}$$

$$x_{4n+2} = x_2 \prod_{k=0}^{n-1} \frac{u_0 \prod\limits_{k_1=0}^{2k} \hat{\alpha}_{2k_1} + \sum\limits_{l=0}^{2k} \left(\hat{\beta}_{2l} \prod\limits_{k_2=l+1}^{2k} \hat{\alpha}_{2k_2} \right)}{v_1 \prod\limits_{k_1=0}^{2k} \hat{\gamma}_{2k_1+1} + \sum\limits_{l=0}^{2k} \left(\hat{\lambda}_{2l+1} \prod\limits_{k_2=l+1}^{2k} \hat{\gamma}_{2k_2+1} \right)}, \tag{26c}$$

$$x_{4n+3} = x_3 \prod_{k=0}^{n-1} \frac{u_1 \prod\limits_{k_1=0}^{2k} \hat{\alpha}_{2k_1+1} + \sum\limits_{l=0}^{2k} \left(\hat{\beta}_{2l+1} \prod\limits_{k_2=l+1}^{2k} \hat{\alpha}_{2k_2+1} \right)}{v_0 \prod\limits_{k_1=0}^{2k+1} \hat{\gamma}_{2k_1} + \sum\limits_{l=0}^{2k+1} \left(\hat{\lambda}_{2l} \prod\limits_{k_2=l+1}^{2k+1} \hat{\gamma}_{2k_2} \right)}, \tag{26d}$$

$$y_{4n} = y_0 \prod_{k=0}^{n-1} \frac{v_0 \prod\limits_{k_1=0}^{2k-1} \hat{\gamma}_{2k_1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\lambda}_{2l} \prod\limits_{k_2=l+1}^{2k-1} \hat{\gamma}_{2k_2} \right)}{u_1 \prod\limits_{k_1=0}^{2k-1} \hat{\alpha}_{2k_1+1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\beta}_{2l+1} \prod\limits_{k_2=l+1}^{2k-1} \hat{\alpha}_{2k_2+1} \right)}, \tag{26e}$$

$$y_{4n+1} = y_1 \prod_{k=0}^{n-1} \frac{v_1 \prod\limits_{k_1=0}^{2k-1} \hat{\gamma}_{2k_1+1} + \sum\limits_{l=0}^{2k-1} \left(\hat{\lambda}_{2l+1} \prod\limits_{k_2=l+1}^{2k-1} \hat{\gamma}_{2k_2+1} \right)}{u_0 \prod\limits_{k_1=0}^{2k} \hat{\alpha}_{2k_1} + \sum\limits_{l=0}^{2k} \left(\hat{\beta}_{2l} \prod\limits_{k_2=l+1}^{2k} \hat{\alpha}_{2k_2} \right)}, \tag{26f}$$

$$y_{4n+2} = y_2 \prod_{k=0}^{n-1} \frac{v_0 \prod\limits_{k_1=0}^{2k} \hat{\gamma}_{2k_1} + \sum\limits_{l=0}^{2k} \left(\hat{\lambda}_{2l} \prod\limits_{k_2=l+1}^{2k} \hat{\gamma}_{2k_2} \right)}{u_1 \prod\limits_{k_1=0}^{2k} \hat{\alpha}_{2k_1+1} + \sum\limits_{l=0}^{2k} \left(\hat{\beta}_{2l+1} \prod\limits_{k_2=l+1}^{2k} \hat{\alpha}_{2k_2+1} \right)}, \tag{26g}$$

(26h)
$$y_{4n+3} = y_3 \prod_{k=0}^{n-1} \frac{v_1 \prod_{k_1=0}^{2k} \hat{\gamma}_{2k_1+1} + \sum_{l=0}^{2k} \left(\hat{\lambda}_{2l+1} \prod_{k_2=l+1}^{2k} \hat{\gamma}_{2k_2+1} \right)}{u_0 \prod_{k_1=0}^{2k+1} \hat{\alpha}_{2k_1} + \sum_{l=0}^{2k+1} \left(\hat{\beta}_{2l} \prod_{k_2=l+1}^{2k+1} \hat{\alpha}_{2k_2} \right)}.$$

We obtain solutions to (2) from those of (8) by back-shifting equations in (26) thrice and by replacing u_i and v_i by $1/(x_i x_{i+2} y_{i+1} y_{i+3})$and $1/(y_i y_{i+2} x_{i+1} x_{i+3})$, respectively. This yields the explicit closed form solutions of (2) given by

(27a)
$$x_{4n-3} = x_{-3} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{2k-1} \alpha_{2k_1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k-1} \left(\beta_{2l} \prod_{k_2=l+1}^{2k-1} \alpha_{2k_2} \right)}{(a_0 + b_0 x_{-3}x_{-1}y_{-2}y_0) \prod_{k_1=0}^{2k-1} \gamma_{2k_1+1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k-1} \left(\lambda_{2l+1} \prod_{k_2=l+1}^{2k-1} \gamma_{2k_2+1} \right)},$$

(27b)
$$x_{4n-2} = x_{-2} \prod_{k=0}^{n-1} \frac{(c_0 + d_0 x_{-2}x_0y_{-1}y_{-3}) \prod_{k_1=0}^{2k-1} \alpha_{2k_1+1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k-1} \left(\beta_{2l+1} \prod_{k_2=l+1}^{2k-1} \alpha_{2k_2+1} \right)}{\prod_{k_1=0}^{2k} \gamma_{2k_1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k} \left(\lambda_{2l} \prod_{k_2=l+1}^{2k} \gamma_{2k_2} \right)},$$

(27c)
$$x_{4n-1} = x_{-1} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{2k} \alpha_{2k_1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k} \left(\beta_{2l} \prod_{k_2=l+1}^{2k} \alpha_{2k_2} \right)}{(a_0 + b_0 x_{-3}x_{-1}y_{-2}y_0) \prod_{k_1=0}^{2k} \gamma_{2k_1+1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k} \left(\lambda_{2l+1} \prod_{k_2=l+1}^{2k} \gamma_{2k_2+1} \right)},$$

(27d)
$$x_{4n} = x_0 \prod_{k=0}^{n-1} \frac{(c_0 + d_0 x_{-2}x_0y_{-1}y_{-3}) \prod_{k_1=0}^{2k} \alpha_{2k_1+1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k} \left(\beta_{2l+1} \prod_{k_2=l+1}^{2k} \alpha_{2k_2+1} \right)}{\prod_{k_1=0}^{2k+1} \gamma_{2k_1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k+1} \left(\lambda_{2l} \prod_{k_2=l+1}^{2k+1} \gamma_{2k_2} \right)},$$

(27e)
$$y_{4n-3} = y_{-3} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{2k-1} \gamma_{2k_1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k-1} \left(\lambda_{2l} \prod_{k_2=l+1}^{2k-1} \gamma_{2k_2} \right)}{(c_0 + d_0 x_{-2}x_0y_{-1}y_{-3}) \prod_{k_1=0}^{2k-1} \alpha_{2k_1+1} + x_{-2}x_0y_{-1}y_{-3} \sum_{l=0}^{2k-1} \left(\beta_{2l+1} \prod_{k_2=l+1}^{2k-1} \alpha_{2k_2+1} \right)},$$

(27f)
$$y_{4n-2} = y_{-2} \prod_{k=0}^{n-1} \frac{(a_0 + b_0 x_{-3}x_{-1}y_{-2}y_0) \prod_{k_1=0}^{2k-1} \gamma_{2k_1+1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k-1} \left(\lambda_{2l+1} \prod_{k_2=l+1}^{2k-1} \gamma_{2k_2+1} \right)}{\prod_{k_1=0}^{2k} \alpha_{2k_1} + x_{-3}x_{-1}y_{-2}y_0 \sum_{l=0}^{2k} \left(\beta_{2l} \prod_{k_2=l+1}^{2k} \alpha_{2k_2} \right)},$$

(27g)

$$y_{4n-1}=y_{-1}\prod_{k=0}^{n-1}\frac{\prod_{k_1=0}^{2k}\gamma_{2k_1}+x_{-2}x_0y_{-1}y_{-3}\sum_{l=0}^{2k}\left(\lambda_{2l}\prod_{k_2=l+1}^{2k}\gamma_{2k_2}\right)}{(c_0+d_0x_{-2}x_0y_{-1}y_{-3})\prod_{k_1=0}^{2k}\alpha_{2k_1+1}+x_{-2}x_0y_{-1}y_{-3}\sum_{l=0}^{2k}\left(\beta_{2l+1}\prod_{k_2=l+1}^{2k}\alpha_{2k_2+1}\right)},$$

(27h)

$$y_{4n}=y_0\prod_{k=0}^{n-1}\frac{(a_0+b_0x_{-3}x_{-1}y_{-2}y_0)\prod_{k_1=0}^{2k}\gamma_{2k_1+1}+x_{-3}x_{-1}y_{-2}y_0\sum_{l=0}^{2k}\left(\lambda_{2l+1}\prod_{k_2=l+1}^{2k}\gamma_{2k_2+1}\right)}{\prod_{k_1=0}^{2k+1}\alpha_{2k_1}+x_{-3}x_{-1}y_{-2}y_0\sum_{l=0}^{2k+1}\left(\beta_{2l}\prod_{k_2=l+1}^{2k+1}\alpha_{2k_2}\right)},$$

where $\alpha_n=a_nc_{n+1},\ \beta_n=b_nc_{n+1}+d_{n+1},\ \gamma_n=a_{n+1}c_n,\ \lambda_n=a_{n+1}d_n+b_{n+1}$.

2.3. **Case where $a_n,\ b_n,\ c_n,$ and d_n are two periodic.** Let $(a_n)_{n\in\mathbb{N}}=(a_0,\ a_1,\ a_0,\dots)$, $(b_n)_{n\in\mathbb{N}}=(b_0,\ b_1,\ b_0,\dots)$, $(c_n)_{n\in\mathbb{N}}=(c_0,\ c_1,\ c_0,\dots)$ and $(d_n)_{n\in\mathbb{N}}=(d_0,\ d_1,\ d_0,\dots)$. Then, $\tilde{\alpha}_{2n}=a_0c_1,\ \tilde{\beta}_{2n}=b_0c_1+d_1,\ \tilde{\gamma}_{2n}=a_1c_0,\ \tilde{\lambda}_{2n}=a_1d_0+b_1$ and $\tilde{\alpha}_{2n+1}=a_1c_0,\ \tilde{\beta}_{2n+1}=b_1c_0+d_0,\ \tilde{\gamma}_{2n+1}=a_0c_1,\ \tilde{\lambda}_{2n+1}=a_0d_1+b_0$. Equations in (27) simplify to

(28a)

$$x_{4n-3}=x_{-3}\prod_{k=0}^{n-1}\frac{(a_0c_1)^{2k}+x_{-3}x_{-1}y_{-2}y_0(b_0c_1+d_1)\sum_{l=0}^{2k-1}(a_0c_1)^l}{(a_0+b_0x_{-3}x_{-1}y_{-2}y_0)(a_0c_1)^{2k}+x_{-3}x_{-1}y_{-2}y_0(a_0d_1+b_0)\sum_{l=0}^{2k-1}(a_0c_1)^l},$$

(28b)

$$x_{4n-2}=x_{-2}\prod_{k=0}^{n-1}\frac{(c_0+d_0x_{-2}x_0y_{-1}y_{-3})(a_1c_0)^{2k}+x_{-2}x_0y_{-1}y_{-3}(b_1c_0+d_0)\sum_{l=0}^{2k-1}(a_1c_0)^l}{(a_1c_0)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(a_1d_0+b_1)\sum_{l=0}^{2k}(a_1c_0)^l},$$

(28c)

$$x_{4n-1}=x_{-1}\prod_{k=0}^{n-1}\frac{(a_0c_1)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(b_0c_1+d_1)\sum_{l=0}^{2k}(a_0c_1)^l}{(a_0+b_0x_{-3}x_{-1}y_{-2}y_0)(a_0c_1)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(a_0d_1+b_0)\sum_{l=0}^{2k}(a_0c_1)^l},$$

(28d)

$$x_{4n}=x_0\prod_{k=0}^{n-1}\frac{(c_0+d_0x_{-2}x_0y_{-1}y_{-3})(a_1c_0)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(b_1c_0+d_0)\sum_{l=0}^{2k}(a_1c_0)^l}{(a_1c_0)^{2k+2}+x_{-2}x_0y_{-1}y_{-3}(a_1d_0+b_1)\sum_{l=0}^{2k+1}(a_1c_0)^l},$$

(28e)

$$y_{4n-3}=y_{-3}\prod_{k=0}^{n-1}\frac{(a_1c_0)^{2k}+x_{-2}x_0y_{-1}y_{-3}(a_1d_0+b_1)\sum_{l=0}^{2k-1}(a_1c_0)^l}{(c_0+d_0x_{-2}x_0y_{-1}y_{-3})(a_1c_0)^{2k}+x_{-2}x_0y_{-1}y_{-3}(b_1c_0+d_0)\sum_{l=0}^{2k-1}(a_1c_0)^l},$$

(28f)

$$y_{4n-2} = y_{-2}\prod_{k=0}^{n-1}\frac{(a_0+b_0x_{-3}x_{-1}y_{-2}y_0)(a_0c_1)^{2k}+x_{-3}x_{-1}y_{-2}y_0(a_0d_1+b_0)\sum_{l=0}^{2k-1}(a_0c_1)^l}{(a_0c_1)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(b_0c_1+d_1)\sum_{l=0}^{2k}(a_0c_1)^l},$$

(28g)

$$y_{4n-1} = y_{-1}\prod_{k=0}^{n-1}\frac{(a_1c_0)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(a_1d_0+b_1)\sum_{l=0}^{2k}(a_1c_0)^l}{(c_0+d_0x_{-2}x_0y_{-1}y_{-3})(a_1c_0)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(b_1c_0+d_0)\sum_{l=0}^{2k}(a_1c_0)^l},$$

(28h)

$$y_{4n} = y_0\prod_{k=0}^{n-1}\frac{(a_0+b_0x_{-3}x_{-1}y_{-2}y_0)(a_0c_1)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(a_0d_1+b_0)\sum_{l=0}^{2k}(a_0c_1)^l}{(a_0c_1)^{2k+2}+x_{-3}x_{-1}y_{-2}y_0(b_0c_1+d_1)\sum_{l=0}^{2k+1}(a_0c_1)^l},$$

2.4. **Case where a_n, b_n, c_n, and d_n are constant.** Let $a_n = a$, $b_n = b$, $c_n = c$ and $d_n = d$. Then, $\tilde{\alpha}_{2n} = ac$, $\tilde{\beta}_{2n} = bc+d$, $\tilde{\gamma}_{2n} = ac$, $\tilde{\lambda}_{2n} = ad+b$ and $\tilde{\alpha}_{2n+1} = ac$, $\tilde{\beta}_{2n+1} = bc+d$, $\tilde{\gamma}_{2n+1} = ac$, $\tilde{\lambda}_{2n+1} = ad+b$. Equations in (27) simplify to

(29a)

$$x_{4n-3} = x_{-3}\prod_{k=0}^{n-1}\frac{(ac)^{2k}+x_{-3}x_{-1}y_{-2}y_0(bc+d)\sum_{l=0}^{2k-1}(ac)^l}{(a+bx_{-3}x_{-1}y_{-2}y_0)(ac)^{2k}+x_{-3}x_{-1}y_{-2}y_0(ad+b)\sum_{l=0}^{2k-1}(ac)^l},$$

(29b)

$$x_{4n-2} = x_{-2}\prod_{k=0}^{n-1}\frac{(c+dx_{-2}x_0y_{-1}y_{-3})(ac)^{2k}+x_{-2}x_0y_{-1}y_{-3}(bc+d)\sum_{l=0}^{2k-1}(ac)^l}{(ac)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(ad+b)\sum_{l=0}^{2k}(ac)^l},$$

(29c)

$$x_{4n-1} = x_{-1}\prod_{k=0}^{n-1}\frac{(ac)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(bc+d)\sum_{l=0}^{2k}(ac)^l}{(a+bx_{-3}x_{-1}y_{-2}y_0)(ac)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(ad+b)\sum_{l=0}^{2k}(ac)^l},$$

(29d)

$$x_{4n} = x_0\prod_{k=0}^{n-1}\frac{(c+dx_{-2}x_0y_{-1}y_{-3})(ac)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(bc+d)\sum_{l=0}^{2k}(ac)^l}{(ac)^{2k+2}+x_{-2}x_0y_{-1}y_{-3}(ad+b)\sum_{l=0}^{2k+1}(ac)^l},$$

(29e)

$$y_{4n-3}=y_{-3}\prod_{k=0}^{n-1}\frac{(ac)^{2k}+x_{-2}x_0y_{-1}y_{-3}(ad+b)\sum\limits_{l=0}^{2k-1}(ac)^l}{(c+dx_{-2}x_0y_{-1}y_{-3})(ac)^{2k}+x_{-2}x_0y_{-1}y_{-3}(bc+d)\sum\limits_{l=0}^{2k-1}(ac)^l},$$

(29f)

$$y_{4n-2}=y_{-2}\prod_{k=0}^{n-1}\frac{(a+bx_{-3}x_{-1}y_{-2}y_0)(ac)^{2k}+x_{-3}x_{-1}y_{-2}y_0(ad+b)\sum\limits_{l=0}^{2k-1}(ac)^l}{(ac)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(bc+d)\sum\limits_{l=0}^{2k}(ac)^l},$$

(29g)

$$y_{4n-1}=y_{-1}\prod_{k=0}^{n-1}\frac{(ac)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(ad+b)\sum\limits_{l=0}^{2k}(ac)^l}{(c+dx_{-2}x_0y_{-1}y_{-3})(ac)^{2k+1}+x_{-2}x_0y_{-1}y_{-3}(bc+d)\sum\limits_{l=0}^{2k}(ac)^l},$$

(29h)

$$y_{4n}=y_0\prod_{k=0}^{n-1}\frac{(a+bx_{-3}x_{-1}y_{-2}y_0)(ac)^{2k+1}+x_{-3}x_{-1}y_{-2}y_0(ad+b)\sum\limits_{l=0}^{2k}(ac)^l}{(ac)^{2k+2}+x_{-3}x_{-1}y_{-2}y_0(bc+d)\sum\limits_{l=0}^{2k+1}(ac)^l}.$$

Remark 1. *Setting $a=b=c=d=1$, equations in* (29) *reduces to Theorem 1 in* [6].
Setting $a=1$, $b=-1$, $c=1$ and $d=-1$, equations in (29) *reduces to Theorem 2 in* [6].
Setting $a=-1$, $b=1$, $c=-1$ and $d=-1$, equations in (29) *reduces to Theorem 3 in* [6].
Setting $a=-1$, $b=-1$, $c=-1$ and $d=1$, equations in (29) *reduces to Theorem 4 in* [6].
Setting $a=1$, $b=-1$, $c=1$ and $d=1$, equations in (29) *reduces to Theorem 5 in* [6].
Setting $a=1$, $b=1$, $c=1$ and $d=-1$, equations in (29) *reduces to Theorem 8 in* [6].
Setting $a=-1$, $b=1$, $c=-1$ and $d=1$, equations in (29) *reduces to Theorem 9 in* [6].
Setting $a=-1$, $b=-1$, $c=-1$, $d=-1$, equations in (29) *reduces to Theorem 10 in* [6].
Setting $a=-1$, $b=-1$, $c=1$ and $d=1$, equations in (29) *reduces to Theorem 14 in* [6].
Setting $a=1$, $b=1$, $c=-1$ and $d=1$, equations (29) *reduces to Theorem 15 in* [6].
Setting $a=1$, $b=-1$, $c=-1$ and $d=-1$, equations in (29) *reduces to Theorem 16 in* [6].
Setting $a=-1$, $b=1$, $c=1$ and $d=-1$, equations in (29) *reduces to Theorem 17 in* [6].

Setting $a = -1,\ b = 1,\ c = 1$ *and* $d = 1$*, equations in* (29) *reduces to Theorem 18 in* [6].
Setting $a = 1,\ b = 1,\ c = -1$ *and* $d = -1$*, equations in* (29) *reduces to Theorem 19 in* [6].
Setting $a = 1,\ b = -1,\ c = -1$ *and* $d = 1$*, equations in* (29) *reduces to Theorem 20 in* [6].
Setting $a = -1,\ b = -1,\ c = 1$ *and* $d = -1$*, equations in* (29) *reduces to Theorem 21 in* [6].

2.5. **Existence of four periodic solutions.** Letting

$$x_{-3}x_{-1}y_{-2}y_0 = x_{-2}x_0y_{-3}y_{-1} = \frac{1-a}{b} = \frac{1-c}{d} \tag{30}$$

in Equations (29), we obtain that

$$\begin{aligned} &x_{4n-3} = x_{-3}, x_{4n-2} = x_{-2}, x_{4n-1} = x_{-1}, x_{4n} = x_0, \\ &y_{4n-3} = y_{-3}, y_{4n-2} = y_{-2}, y_{4n-1} = y_{-1}, y_{4n} = y_0. \end{aligned} \tag{31}$$

This means that when imposing conditions in (30), all solutions of (2) are periodic with period four.

Remark 2. *Theorem 7 in* [6] *is a special case of* (31) *obtained by setting* $x_{-3} = y_{-3} = 0$ *and* $a = c = 1$.
The first part of Theorem 12 in [6] *is a special case of* (31) *obtained when* $x_{-3}x_{-1}y_{-2}y_0 = x_{-2}x_0y_{-3}y_{-1} = 2$*, that is, when* $a = c = -1$ *and* $b = d = 1$.
The second part of Theorem 13 in [6] *is a special case of* (31) *obtained when* $x_{-3}x_{-1}y_{-2}y_0 = x_{-2}x_0y_{-3}y_{-1} = -2$*, that is, when* $a = b = c = d = -1$.

The two figures below are some examples where the solutions of (2) are four periodic.

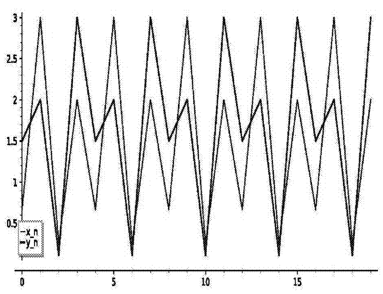

FIGURE 1. $a = 2, b = -1, c = 3, d = -2, x_{-3} = 3, x_{-2} = 1/4, x_{-1} = 2, x_0 = 2/3, y_{-3} = 2, y_{-2} = 1/9, y_{-1} = 3, y_0 = 3/2$.

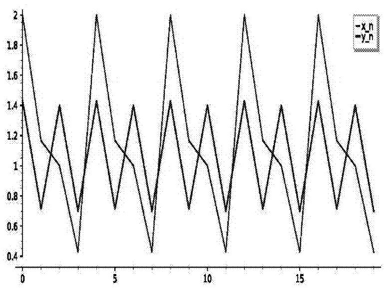

FIGURE 2. $a = 0.3, b = 0.7, c = 0.4, d = 0.6, x_{-3} = 7/6, x_{-2} = 1, x_{-1} = 3/7, x_0 = 2, y_{-3} = 5/7, y_{-2} = 7/5, y_{-1} = 0.7, y_0 = 10/7$.

2.6. **Existence of eight periodic solutions.** Letting

$$x_{-3}x_{-1}y_{-2}y_0 = -x_{-2}x_0y_{-3}y_{-1} = -\frac{1+a}{b} = \frac{1+c}{d} \tag{32}$$

in Equations (29), we obtain that

$$\begin{aligned} & x_{8n-7} = -x_{-3}, x_{8n-6} = -x_{-2}, x_{8n-5} = -x_{-1}, x_{8n-4} = -x_0, \\ & x_{8n-3} = x_{-3}, x_{8n-2} = x_{-2}, x_{8n-1} = x_{-1}, x_{8n} = x_0, y_{8n-7} = -y_{-3}, \\ & y_{8n-6} = -y_{-2}, y_{8n-5} = -y_{-1} y_{8n-4} = -y_0. y_{8n-3} = y_{-3}, \\ & y_{8n-2} = y_{-2}, y_{8n-1} = y_{-1},\ y_{8n} = y_0. \end{aligned} \tag{33}$$

This means that when imposing conditions in (32), all solutions of (2) are periodic with period eight.

Remark 3. *Theorem 6 in* [6] *is a special case of* (33) *obtained by setting* $x_{-3}x_{-1}y_{-2}y_0 = -x_{-2}x_0y_{-3}y_{-1} = 2$, *that is,* $a = c = d = 1$ *and* $b = -1$.
Theorem 11 in [6] *is a special case of* (33) *obtained by setting* $x_{-3}x_{-1}y_{-2}y_0 = -x_{-2}x_0y_{-3}y_{-1} = -2$, *that is,* $a = b = c = 1$ *and* $d = -1$.
The second part of Theorem 12 in [6] *is a special case of* (33) *obtained when* $x_{-3}x_{-1}y_{-2}y_0 = -x_{-2}x_0y_{-3}y_{-1} = 0$, *that is, when* $a = c = -1$ *and* $b = d = 1$ *together with* $x_{-3} = y_{-3} = 0$.
The first part of Theorem 13 in [6] *is a special case of* (33) *obtained when* $-x_{-3}x_{-1}y_{-2}y_0 = x_{-2}x_0y_{-3}y_{-1} = 0$, *that is, when* $a = b = c = d = -1$ *together with* $x_{-3} = y_{-3} = 0$.

The two figures below are some examples where the solutions of (8) are eight periodic.

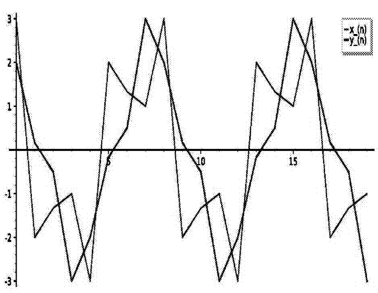

FIGURE 3. $a = 0.1, b = -0.55, c = 0.2, d = 0.6, x_{-3} = 2, x_{-2} = 4/3, x_{-1} = 1, x_0 = 3, y_{-3} = -1/6,\ y_{-2} = 1/2, y_{-1} = 3, y_0 = 2$.

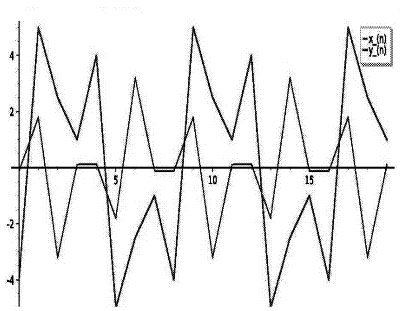

FIGURE 4. $a = -0.5, b = -0.25, c = 2, d = 1.5, x_{-3} = 1.8, x_{-2} = -3.2, x_{-1} = 1/9, x_0 = 1/8, y_{-3} = 5,\ y_{-2} = 2.5, y_{-1} = 1, y_0 = 4$.

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School of Mathematics, University of the Witwatersrand, 2050, Johannesburg, South Africa.

E-mail address: Mensah.Folly-Gbetoula@wits.ac.za