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SOME IDENTITIES OF THE PARTIALLY DEGENERATE CHANGHEE-GENOCCHI POLYNOMIALS AND NUMBERS

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ABSTRACT. In this paper, we introduce the partially degenerate Changhee-Genocchi polynomials and numbers and investigated some identities of these polynomials. Furthermore, we investigate some explicit identities and properties of the partially degenerate Changhee-Genocchi arising from the nonlinear differential equations.

1. Introduction

As is well known, the Genocchi polynomials $G_n(x)$ are defined by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \text{ (see [1, 3, 6, 17, 19]).} \tag{1.1}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers.

The Changhee polynomials $Ch_n(x)$ are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \text{ (see [5, 8, 13, 16, 18]).} \tag{1.2}$$

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers.

By replacing t by $e^t - 1$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \frac{2}{e^t+1}e^{xt}\\ &= \sum_{m=0}^{\infty} \frac{Ch_m(x)}{m!}(e^t-1)^m\\ &= \sum_{m=0}^{\infty} Ch_m(x)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_m(x)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{1.3}$$

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where $E_m(x)$ are ordinary Euler polynomials and $S_2(n,m)$ are the Stirling numbers of second kind.

Thus, we have

$$E_n(x) = \sum_{m=0}^{n} Ch_m(x) S_2(n,m). \tag{1.4}$$

Now, we define the degenerate exponential function as follow:

$$e_\lambda^t = (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \frac{(1)_{n,\lambda}}{n!} t^n, \text{ (see [9])}. \tag{1.5}$$

Where $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$.

Note that

$$\lim_{\lambda\to 0} e_\lambda^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t. \tag{1.6}$$

In [2], L. Carlitz consider the degenerate Euler polynomials which are given by the generating function to be

$$\frac{2}{e_\lambda^t + 1} e_\lambda^{tx} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.7}$$

The Bernoulli polynomials of the second kind $b_n(x)$ are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \text{ (see [17])}. \tag{1.8}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.

The Changhee-Genocchi polynomials $CG_n(x)$ are defined by the generating function to be

$$\frac{2\log(1+t)}{2+t}(1+t)^x = \sum_{n=0}^{\infty} CG_n(x) \frac{t^n}{n!} \text{ (see [3])}. \tag{1.9}$$

When $x = 0$, $CG_n = CG_n(0)$ are called the Changhee-Genocchi numbers.

The Genocchi-Changhee polynomials $GCh_n(x)$ are defined by the generating function to be

$$\frac{2t}{2+t}(1+t)^x = \sum_{n=0}^{\infty} GCh_n(x) \frac{t^n}{n!}. \tag{1.10}$$

When $x = 0$, $GCh_n = GCh_n(0)$ are called the Genocchi-Changhee numbers.

The degenerate Changhee-Genocchi polynomials $CG_n(x \mid \lambda)$ are defined by the generating function to be

$$\frac{2\log(1+\log e_\lambda^t)}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x = \sum_{n=0}^{\infty} CG_n(x \mid \lambda) \frac{t^n}{n!} \text{ (see [11])}. \tag{1.11}$$

When $x = 0$, $CG_n = CG_n(0 \mid \lambda)$ are called the degenerate Changhee-Genocchi numbers.

We recall the Stirling numbers of the first kind $S_1(n,m)$ and $S_2(n,m)$ are defined by

$$\frac{1}{m!}(\log(1+t))^m = \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \text{ (see [4, 7, 14, 20])}. \tag{1.12}$$

and

$$\frac{1}{m!}(e^t-1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \text{ (see [10, 12, 15])}. \tag{1.13}$$

Recently, B-M. Kim et al. studied Changhee-Genocchi polynomials and some identities of these polynomials. They also introduced Changhee-Genocchi polynomials and investigated some identities of these polynomials ([3]). Also, H. -I. Kwon et al. introduced degenerate Changhee-Genocchi polynomials and some identities of these polynomials and investigated some identities of these polynomials ([11]). In this paper, we introduce the partially degenerate Changhee-Genocchi polynomials and numbers and investigated some identities of these polynomials. Furthermore, we investigate some explicit identities and properties of the partially degenerate Changhee-Genocchi arising from the nonlinear differential equations.

2. The partially degenerate Changhee-Genocchi polynomials and numbers

In this section, we define the partially degenerate Changhee-Genocchi polynomials and numbers and investigate some identities of the partially degenerate Changhee-Genocchi polynomials.

Now, we consider the degenerate Genocchi polynomials which are given by the generating function to be

$$\frac{2t}{e_\lambda^t+1}e_\lambda^{tx} = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $G_{n,\lambda} = G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

It is not difficult to show that $G_{0,\lambda}(0) = 0$.
So,

$$\frac{2t}{e_\lambda^t+1}e_\lambda^{tx} = \sum_{n=0}^{\infty} \frac{G_{n+1,\lambda}(x)}{n+1}\frac{t^{n+1}}{n!}. \tag{2.2}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{G_{n+1,\lambda}(x)}{n+1}\frac{t^{n+1}}{n!} &= t\frac{2}{e_\lambda^t+1}e_\lambda^{tx} \\ &= \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^{n+1}}{n!}.\end{aligned} \tag{2.3}$$

Comparing the coefficients on the both sides in (2.3), we have the following result.

Theorem 2.1. *Let* $\lambda \in \mathbb{C}_p$ *with* $0 < |\lambda|_p < 1$. *Then*

$$\frac{G_{n+1,\lambda}(x)}{n+1} = E_{n,\lambda}(x), \ (n \geq 0). \tag{2.4}$$

In [4], the degenerate Changhee polynomials which are given by

$$\frac{2}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.5}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.5), we get

$$\begin{aligned}\frac{2}{2+t}(1+t)^x &= \sum_{m=0}^{\infty} Ch_{m,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^m \\ &= \sum_{m=0}^{\infty} Ch_{m,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n}{n!}t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_{m,\lambda}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Thus, we obtain the following result.

Theorem 2.2. *Let* $\lambda \in \mathbb{C}_p$ *with* $0 < |\lambda|_p < 1$. *Then*

$$Ch_n(x) = \sum_{m=0}^{n} Ch_{m,\lambda}(x)\lambda^{n-m}S_2(n,m). \tag{2.7}$$

Now, we define the partially degenerate Changhee-Genocchi polynomials which are given by

$$\frac{2\log(1+t)}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x = \sum_{n=0}^{\infty} \widehat{CG}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.8}$$

When $x = 0$, $\widehat{CG}_{n,\lambda} = \widehat{CG}_{n,\lambda}(0)$ are called the partially degenerate Changhee-Genocchi numbers.

Also, we define the higher-order partially degenerate Changhee-Genocchi numbers which are given by the generating function to be

$$\left(\frac{2\log(1+t)}{2+\log e_\lambda^t}\right)^k = \sum_{n=0}^{\infty} \widehat{CG}_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{2.9}$$

Now, we observe that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{2\log(1+t)}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x &= \frac{2\log(1+t)}{2+t}(1+t)^x\\ &= \frac{2t}{2+t}(1+t)^x\frac{\log(1+t)}{t}\\ &= \left(\sum_{l=0}^{\infty} GCh_l(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}GCh_l(x)D_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Where D_m are the Daehee numbers.

Comparing the coefficients on the both sides in (2.10), we have the following result.

Theorem 2.3. *Let $\lambda\in\mathbb{C}_p$ with $0<|\lambda|_p<1$ and $\lambda\to 0$. Then*

$$\widehat{CG}_{n,0}(x)=\sum_{l=0}^{n}\binom{n}{l}GCh_l(x)D_{n-l}. \tag{2.11}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{CG}_{n,\lambda}\frac{t^n}{n!} &= \frac{2\log(1+t)}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x\\ &= \frac{2t}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x\frac{\log(1+t)}{t}\\ &= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda}D_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Comparing the coefficients on the both sides in (2.12), we have the following result.

Theorem 2.4. *Let $\lambda\in\mathbb{C}_p$ with $0<|\lambda|_p<1$. Then*

$$\widehat{CG}_{n,\lambda}=\sum_{l=0}^{n}\binom{n}{l}Ch_{l,\lambda}D_{n-l}. \tag{2.13}$$

We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{CG}_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2\log(1+t)}{2+\log e_\lambda^t}(1+\log e_\lambda^t)^x \\ &= \left(\sum_{l=0}^{\infty}\widehat{CG}_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}(x)_k\frac{1}{k!}(\log e_\lambda^t)^k\right) \\ &= \left(\sum_{l=0}^{\infty}\widehat{CG}_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}(x)_k\lambda^{m-k}S_1(m,k)\right)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(x)_k\lambda^{m-k}S_1(m,k)\widehat{CG}_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

Theorem 2.5. *Let $\lambda \in \mathbb{C}_p$ with $0 < |\lambda|_p < 1$. Then*

$$\widehat{CG}_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(x)_k\lambda^{m-k}S_1(m,k)\widehat{CG}_{n-m,\lambda}. \tag{2.15}$$

3. The partially degenerate Changhee-Genocchi numbers arising from differential equations

In this section, we investigate some identities of the partially degenerate Changhee-Genocchi numbers arising from differential equations.

Let

$$F = F(t) = \frac{1}{2\lambda + \log(1+\lambda t)}. \tag{3.1}$$

Then, by taking the derivative with respect to t of (3.1), we obtain

$$\begin{aligned}F^{(1)} = \frac{d}{dt}F(t) &= \frac{1}{(2\lambda+\log(1+\lambda t))^2}\left(-\frac{\lambda}{1+\lambda t}\right) \\ &= \left(-\frac{\lambda}{1+\lambda t}\right)F^2.\end{aligned} \tag{3.2}$$

From (3.2), we have

$$\lambda F^2 = -(1+\lambda t)F^{(1)}. \tag{3.3}$$

By taking the derivative with respect to t in (3.3), we note that

$$2\lambda F F^{(1)} = -\lambda F^{(1)} - (1+\lambda t)F^{(2)}. \tag{3.4}$$

Thus, by multiple $(1+\lambda t)$ on the both sides of (3.4), we obtain

$$2\lambda(1+\lambda t)FF^{(1)} = -\lambda(1+\lambda t)F^{(1)} - (1+\lambda t)^2F^{(2)}. \tag{3.5}$$

From (3.3) and (3.5), we get

$$2\lambda^2F^3 = \lambda(1+\lambda t)F^{(1)} + (1+\lambda t)^2F^{(2)}. \tag{3.6}$$

From the above equation, we have

$$\begin{aligned} 3!\lambda^2F^2F^{(1)} &= \lambda^2F^{(1)} + \lambda(1+\lambda t)F^{(2)} + 2\lambda(1+\lambda t)F^{(2)} + (1+\lambda t)^2F^{(3)} \\ &= \lambda^2F^{(1)} + 3\lambda(1+\lambda t)F^{(2)} + (1+\lambda t)^2F^{(3)}. \end{aligned} \tag{3.7}$$

Multiply $(1+\lambda t)$ on the both sides of (3.7), we get

$$3!\lambda^2(1+\lambda t)F^2F^{(1)} = \lambda^2(1+\lambda t)F^{(1)} + 3\lambda(1+\lambda t)^2F^{(2)} + (1+\lambda t)^3F^{(3)}. \tag{3.8}$$

From (3.3) and (3.8), we obtain

$$3!\lambda^3F^4 = -\lambda^2(1+\lambda t)F^{(1)} - 3\lambda(1+\lambda t)^2F^{(2)} - (1+\lambda t)^3F^{(3)}. \tag{3.9}$$

Continuing this process, we get

$$N!\lambda^NF^{N+1} = (-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k}(1+\lambda t)^kF^{(k)}. \tag{3.10}$$

Let us take the derivative on the both sides of (3.10) with respect to t. Then we obtain

$$\begin{aligned} (N+1)!\lambda^NF^NF^{(1)} &= (-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}k(1+\lambda t)^{k-1}F^{(k)} \\ &\quad + (-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k}(1+\lambda t)^kF^{(k+1)}. \end{aligned} \tag{3.11}$$

Multiply $(1+\lambda t)$ on the both sides of (3.11), we have

$$\begin{aligned} (N+1)!\lambda^N(1+\lambda t)F^NF^{(1)} &= (-1)^N\sum_{k=1}^{N}ka_k(N)\lambda^{N-k+1}(1+\lambda t)^kF^{(k)} \\ &\quad + (-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k}(1+\lambda t)^{k+1}F^{(k+1)}. \end{aligned} \tag{3.12}$$

Then, by (3.3) and (3.12), we obtain

$$\begin{aligned}(N+1)!\lambda^{N+1}F^{N+2} &= (-1)^{N+1}\sum_{k=1}^{N} ka_k(N)\lambda^{N-k+1}(1+\lambda t)^k F^{(k)} \\ &+ (-1)^{N+1}\sum_{k=1}^{N} a_k(N)\lambda^{N-k}(1+\lambda t)^{k+1} F^{(k+1)} \\ &= (-1)^{N+1}\sum_{k=1}^{N} ka_k(N)\lambda^{N-k+1}(1+\lambda t)^k F^{(k)} \\ &+ (-1)^{N+1}\sum_{k=2}^{N+1} a_{k-1}(N)\lambda^{N-k+1}(1+\lambda t)^{k} F^{(k)} \\ &= (-1)^{N+1}a_1(N)\lambda^N(1+\lambda t)F^{(1)} \\ &+ (-1)^{N+1}a_N(N)(1+\lambda t)^{N+1}F^{(N+1)} \\ &+ (-1)^{N+1}\sum_{k=2}^{N}\left(ka_k(N)+a_{k-1}(N)\right)\lambda^{N-k+1}(1+\lambda t)^k F^{(k)}.\end{aligned} \tag{3.13}$$

By substituting N by $N+1$ given in (3.10), we have another equation.

$$\begin{aligned}(N+1)!\lambda^{N+1}F^{N+2} &= (-1)^{N+1}\sum_{k=1}^{N+1} a_k(N+1)\lambda^{N-k+1}(1+\lambda t)^k F^{(k)} \\ &= (-1)^{N+1}a_1(N+1)\lambda^N(1+\lambda t)F^{(1)} \\ &+ (-1)^{N+1}a_{N+1}(N+1)(1+\lambda t)^{N+1}F^{(N+1)} \\ &+ (-1)^{N+1}\sum_{k=2}^{N} a_k(N+1)\lambda^{N-k+1}(1+\lambda t)^k F^{(k)}.\end{aligned} \tag{3.14}$$

Comparing the coefficients on the both sides of (3.13) and (3.14), we have

$$a_1(N+1) = a_1(N), \quad a_{N+1}(N+1) = a_N(N), \tag{3.15}$$

and

$$a_k(N+1) = ka_k(N) + a_{k-1}(N), \quad for \ \ 2 \le k \le N. \tag{3.16}$$

From (3.3) and (3.10), for $N=1$, we obtain

$$\begin{aligned}\lambda^2F^2 &= -\sum_{k=1}^{1} a_k(1)\lambda^{1-k}(1+\lambda t)^k F^{(k)} \\ &= -a_1(1)(1+\lambda t)F^{(1)} \\ &= -(1+\lambda t)F^{(1)}.\end{aligned} \tag{3.17}$$

From (3.17), we get

$$a_1(1) = 1. \tag{3.18}$$

From (3.15), we have the following result using (3.18).

$$a_1(N+1) = a_1(N) = a_1(N-1) = \cdots = a_1(1) = 1. \tag{3.19}$$

and

$$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1. \tag{3.20}$$

From (3.16), for $2 \le k \le N$, we have

$$\begin{aligned} a_k(N+1) &= ka_k(N) + a_{k-1}(N) \\ &= k(ka_k(N-1) + a_{k-1}(N-1)) + a_{k-1}(N) \\ &= k^2 a_k(N-1) + ka_{k-1}(N-1) + a_{k-1}(N) \\ &= \cdots \\ &= k^{N-k+1}a_k(k) + k^{N-k}a_{k-1}(k) + \cdots + a_{k-1}(N) \end{aligned} \tag{3.21}$$

Therefore by (3.15) and (3.21), we get

$$\begin{aligned} a_k(N+1) &= k^{N-k+1}a_k(k) + k^{N-k}a_{k-1}(k) + \cdots + a_{k-1}(N) \\ &= \sum_{i_1=0}^{N-k+1} k^{N-k+1-i_1} a_{k-1}(k-1+i_1) \\ &= \sum_{i_1=0}^{N-k+1} k^{N-k+1-i_1} \sum_{i_2=0}^{i_1} (k-1)^{i_1-i_2} a_{k-2}(k-2+i_2) \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} k^{N-k+1-i_1} (k-1)^{i_1-i_2} a_{k-2}(k-2+i_2) \\ &= \cdots \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k+1-i_1}(k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}} a_1(1+i_{k-1}). \end{aligned} \tag{3.22}$$

From (3.19) and (3.22), we obtain

$$a_k(N+1) = \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k+1-i_1}(k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}}. \tag{3.23}$$

Thus, we have the following theorem.

Theorem 3.1. *Let $N \in \mathbb{N}$. Then the following differential equation,*

$$N!\lambda^N F^{N+1} = (-1)^N \sum_{k=1}^{N} a_k(N)\lambda^{N-k}(1+\lambda t)^k F^{(k)}$$

have a solution $F = F(t) = \frac{1}{2\lambda + \log(1+\lambda t)}$, where

$$a_N(N) = 1, \quad a_1(N) = 1$$

and

$$a_k(N) = \sum_{i_1=0}^{N-k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k-i_1}(k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}}, \quad for \ \ 2 \le k \le N.$$

From (3.1), we get

$$\begin{aligned} F &= \frac{1}{2\lambda + \log(1+\lambda t)} \\ &= \frac{t}{\log(1+t)} \times \frac{1}{2\lambda t} \times \frac{2\log(1+t)}{2+\log e_\lambda^t} \\ &= \left(\sum_{l_1=0}^{\infty} b_{l_1}\frac{t^{l_1}}{l_1!}\right) \times \left(\sum_{l_2=1}^{\infty} CG_{l_2,\lambda}\frac{1}{2\lambda}\frac{t^{l_2-1}}{l_2!}\right) \\ &= \left(\sum_{l_1=0}^{\infty} b_{l_1}\frac{t^{l_1}}{l_1!}\right) \times \left(\sum_{l_2=0}^{\infty} CG_{l_2+1,\lambda}\frac{1}{2\lambda(l_2+1)}\frac{t^{l_2}}{l_2!}\right) \\ &= \sum_{l_3=0}^{\infty}\left(\sum_{l_2=0}^{l_3}\binom{l_3}{l_2} b_{l_3-l_2} CG_{l_2+1,\lambda}\frac{1}{2\lambda(l_2+1)}\right)\frac{t^{l_3}}{l_3!}. \end{aligned} \tag{3.24}$$

From the above equation, we get

$$\begin{aligned} F^{(k)} &= \left(\frac{d}{dt}\right)^k F(t) \\ &= \left(\frac{d}{dt}\right)^k \left(\sum_{l_3=0}^{\infty}\left(\sum_{l_2=0}^{l_3}\binom{l_3}{l_2} b_{l_3-l_2} CG_{l_2+1,\lambda}\frac{1}{2\lambda(l_2+1)}\right)\frac{t^{l_3}}{l_3!}\right) \\ &= \sum_{l_3=0}^{\infty}\left(\sum_{l_2=0}^{l_3+k}\binom{l_3+k}{l_2} b_{l_3-l_2+k} CG_{l_2+1,\lambda}\frac{1}{2\lambda(l_2+1)}\right)\frac{t^{l_3}}{l_3!}. \end{aligned} \tag{3.25}$$

Multiply $2^{N+1}\lambda(\log(1+t))^{N+1}$ on the right sides of (3.10), we get

$$\begin{aligned}
&(-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}2^{N+1}(\log(1+t))^{N+1}(1+\lambda t)^kF^{(k)}\\
&=(-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}2^{N+1}\left(\sum_{M_1=N+1}^{\infty}S_1(M_1,N+1)\frac{t^{M_1}}{M_1!}\right)\\
&\times\left(\sum_{M_2=0}^{k}(k)_{M_2}\lambda^{M_2}\frac{t^{M_2}}{M_2!}\right)F^{(k)} \qquad (3.26)\\
&=(-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}2^{N+1}\sum_{M_3=N+1}^{\infty}\left(\sum_{M_1=N+1}^{M_3}\binom{M_3}{M_1}\right.\\
&\left.\times S_1(M_1,N+1)(k)_{M_3-M_1}\lambda^{M_3-M_1}\right)\frac{t^{M_3}}{M_3!}F^{(k)}.
\end{aligned}$$

Where $S_1(n,k)$ is the Stirling number of the first kind.

Thus, by (3.25) and (3.26), we get

$$\begin{aligned}
&(-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}2^{N+1}(\log(1+t))^{N+1}(1+\lambda t)^kF^{(k)}\\
&=(-1)^N\sum_{k=1}^{N}a_k(N)\lambda^{N-k+1}2^{N+1}\sum_{M_3=N+1}^{\infty}\left(\sum_{M_1=N+1}^{M_3}\binom{M_3}{M_1}\right.\\
&\left.\times S_1(M_1,N+1)(k)_{M_3-M_1}\lambda^{M_3-M_1}\right)\frac{t^{M_3}}{M_3!}\\
&\times\left(\sum_{l_3=0}^{\infty}\left(\sum_{l_2=0}^{l_3+k}\binom{l_3+k}{l_2}b_{l_3-l_2+k}CG_{l_2+1,\lambda}\frac{1}{2\lambda(l_2+1)}\right)\frac{t^{l_3}}{l_3!}\right) \qquad (3.27)\\
&=(-1)^N\sum_{k=1}^{N}\sum_{n=N+1}^{\infty}\sum_{M_3=N+1}^{n}\sum_{M_1=N+1}^{M_3}\sum_{l_2=0}^{n-M_3+k}\binom{n}{M_3}\binom{M_3}{M_1}\binom{n-M_3+k}{l_2}\\
&\times a_k(N)\lambda^{N-k+M_3-M_1}2^NS_1(M_1,N+1)(k)_{M_3-M_1}b_{n-M_3-l_2+k}\\
&\times CG_{l_2+1,\lambda}\frac{1}{l_2+1}\times\frac{t^n}{n!}
\end{aligned}$$

$$= (-1)^N \sum_{n=N+1}^{\infty} \left(\sum_{k=1}^{N} \sum_{M_3=N+1}^{n} \sum_{M_1=N+1}^{M_3} \sum_{l_2=0}^{n-M_3+k} \binom{n}{M_3} \binom{M_3}{M_1} \binom{n-M_3+k}{l_2} \right.$$
$$\times a_k(N) \lambda^{N-k+M_3-M_1} 2^N S_1(M_1, N+1) (k)_{M_3-M_1} b_{n-M_3-l_2+k}$$
$$\left. \times CG_{l_2+1,\lambda} \frac{1}{l_2+1} \right) \times \frac{t^n}{n!}.$$

Also, multiply $2^{N+1}\lambda(\log(1+t))^{N+1}$ on the left sides of (3.10), we get

$$\begin{aligned} N! 2^{N+1} \lambda^{N+1} (\log(1+t))^{N+1} F^{N+1} &= N! \left(\frac{2\log(1+t)}{2+\log e_\lambda^t} \right)^{N+1} \\ &= N! \sum_{n=N+1}^{\infty} \widehat{CG}_{n,\lambda}^{(N+1)} \frac{t^n}{n!}. \end{aligned} \tag{3.28}$$

By equation (3.10), (3.27) and (3.28), we finally get the explicit expression arising from nonlinear differential equation.

Theorem 3.2. *For $n \geq N+1$, we have*

$$\begin{aligned} \widehat{CG}_{n,\lambda}^{(N+1)} =& \frac{(-1)^N}{N!} \sum_{k=1}^{N} \sum_{M_3=N+1}^{n} \sum_{M_1=N+1}^{M_3} \sum_{l_2=0}^{n-M_3+k} \binom{n}{M_3} \binom{M_3}{M_1} \binom{n-M_3+k}{l_2} a_k(N) \\ & \times \lambda^{N-k+M_3-M_1} 2^N S_1(M_1, N+1) (k)_{M_3-M_1} b_{n-M_3-l_2+k} CG_{l_2+1,\lambda} \frac{1}{l_2+1}. \end{aligned}$$

4. Conclusion

T. Kim have studied some identities of Changhee numbers which are derived from generating function using nonlinear differential equation(see [8]). In this paper, we study some identities of the partially degenerate Changhee-Genocchi polynomials and the partially degenerate Changhee-Genocchi number arising from nonlinear differential equation. In **Theorem 2.3** and **Theorem 2.4**, we get the some identities of the partially degenerate Changhee-Genocchi polynomials. In **Theorem 3.1**, we get the solution of nonlinear differential equation arising from generating function of the partially degenerate Changhee-Genocchi numbers. In **Theorem 3.2**, we have explicit expression of the partially degenerate Changhee-Genocchi number from the result of **Theorem 3.1** using generating function and nonlinear differential equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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