Advanced Studies in Contemporary Mathematics
29 (2019), No. 4, pp. 521 - 535

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.4.521

SECOND HANKEL DETERMINANTS FOR SOME MA-MINDA SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we have investigated second Hankel determinants for some subclasses of Ma-Minda bi-univalent functions in the open unit disc Δ and these results are generalization of results in [7] and [9].

2010 MATHEMATICS SUBJECT CLASSIFICATION. 30C45, 30C50, 30C80.

KEYWORDS AND PHRASES. Analytic functions; Univalent functions; Bi-univalent functions; Second Hankel determinants; Subordination.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$ which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and $\mathbb{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. Let $\mathcal{P}$ be the family of analytic functions $p(z)$ in Δ such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0 (z \in \Delta)$. According to Koebe one-quarter theorem, every function $f \in \mathbb{S}$ has an inverse f^{-1} and f^{-1} need not be defined in the entire unit disc, satisfying

$$f^{-1}(f(z)) = z, (z \in \Delta), f(f^{-1}(w)) = w, (|w| < \frac{1}{4}).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ...$$

For any two functions f and g analytic in Δ, we say that the function f is subordinate to g in Δ and we write it as $f(z) \prec g(z)$, if there exists a Schwarz function w, in Δ such that $f(z) = g(w(z))$.
The well known relation between the function in the class $\mathcal{P}$ and the Schwarz function is given in [14] and is as follows.

$$p \in P \Longleftrightarrow p = \frac{1 + \omega}{1 - \omega}.$$

The work presented in this paper is partially supported by DST-FIST-Grant No.SR/FST/MSI-101/2014

Many researchers [25], [27], [28] defined various subclasses of analytic and univalent functions, such as starlike functions, convex functions, close to convex functions, spiral like functions and starlike or convex with respect to symmetric points etc. In particular the well known subclasses of starlike functions which were discussed by so many researchers are starlike functions, strongly starlike functions of order α, parabolic starlike and lemniscate starlike functions etc. and same type of subclasses were also discussed for convex functions in the unit disc Δ. All these functions were typically characterized by either of the quantities $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ lying in a certain domain starlike with respect to 1 in the right half plane. Using subordination Ma-Minda [19] unified all various subclasses of starlike and convex functions requiring that either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function ϕ, where the function ϕ is an analytic function with positive real part in Δ; such that $\phi(0)=1, \phi'(0)>0$ and with the property that $\phi(\Delta)$ is starlike with respect to 1 and symmetric with respect to the real axis. Therefore the function ϕ has series expansion of the form $\phi(z)=1+B_1z+B_2z^2+B_3z^3+.....$where $B_1>0, B_n's$ are real.
The class $S^*(\phi)$ of Ma-Minda starlike functions satisfies the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class $C(\phi)$ of Ma-Minda convex functions satisfy the subordination $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$. For results related to initial coefficients and Fekete-Szegö inequalities for analytic and univalent functions subordinate to ϕ one can refer [2], [4].
A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ, if both f and f^{-1} are univalent in Δ. Let Σ denote the class of all bi-univalent functions defined in the unit disc Δ. We notice that Σ is non empty. The behavior of the coefficients are remarkable when the bi-univalency condition is imposed on the function $f \in \mathcal{A}$. A function $f \in \Sigma$ is called Ma-Minda bi-starlike function or Ma-Minda bi-convex function if both f and f^{-1} are Ma-Minda starlike functions or Ma-Minda convex functions respectively. In 1967, Lewin [18] introduced the class Σ of bi-univalent functions and investigated second coefficient in Taylor-Maclaurin series expansion for every $f \in \Sigma$ and proved that $|a_2| < 1.51$. Subsequently, in 1967, Brannann and Clunie [6] introduced bi-starlike functions and bi-convex functions similar to the familiar subclasses of univalent functions as strongly starlike, strongly convex functions, starlike, convex etc. and obtained estimates on the initial coefficients and conjectured that $|a_2| \leq \sqrt{2}$ for bi-starlike functions, $|a_2| \leq 1$ for bi-convex functions. Only last estimate is sharp; equality occurs only for $f(z)=\frac{z}{1-z}$ or its rotation.

In recent years, Srivastava et al. rejuvenated the study of bi-univalent functions. He has found the way for many results, such as coefficient estimates, Fekete-Szegö inequalities for different subclasses of bi-univalent functions defined through differential subordination. Motivated by his research work in bi-univalent functions, so many authors have established several results for bi-univalent functions related to the coefficient inequalities, Fekete-Szegö inequalities([1], [8], [13], [30], [31], [32], [33], [34], [35], [36], [37]).

In 1976, Noonan and Thomas [21] defined q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$ which is stated by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} \cdots\cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} \cdots\cdots & a_{n+q} \\ a_{n+2} & a_{n+3} & a_{n+4} \cdots\cdots & a_{n+q+1} \\ -- & -- & ------ & --- \\ -- & -- & ------ & --- \\ a_{n+q-1} & a_{n+q} & a_{n+q+1} \cdots\cdots & a_{n+2q-2} \end{vmatrix} \tag{1.3}$$

Easily one can observe that $H_2(1) =\mid a_3 - a_2^2 \mid$ is a special case of the well known Fekete-Szegö functional $|a_3 - \mu a_2^2|$ where μ is real and for $\mu = 1$. Now for $q = 2, n = 2$, we get second Hankel determinant

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \tag{1.4}$$

This determinant has also been considered by several authors. For more study related to Hankel determinants refer ([11], [12], [16], [23]). Recently S. K. Lee, et al. [17] obtained the second Hankel determinant for Ma-Minda starlike and convex functions. In particular, upper bounds on $H_2(2)$ were obtained by the authors of articles [5], [26] for various subclasses of analytic and univalent functions. The concept of Hankel determinant were also applied for bi-univalent functions in the recent research works. The second Hankel determinant for bi-starlike and bi-convex functions of order β were estimated by E. Deniz[9]. Later, N. Magesh et al. [22] (also see [3], [7], [20], [24]) estimated the second Hankel determinant for some subclasses of bi-univalent functions of order β. In 2018, H. M. srivastava [29] estimated the second Hankel determinant for some subclass of bi-univalent functions defined by using a symmetric q-derivative operator.

Motivated by the above mentioned work, in this paper we have considered some subclasses of Ma-Minda bi-univalent functions and estimated second Hankel determinant. These results generalize the results of Deniz [9], Srivastava [7]and similar results are discussed for different functions of ϕ.

Definition 1.1 A function $f \in \Sigma$ is said to be in the class $S^*_\Sigma(\phi)$, if it satisfies the following conditions:

$$\frac{zf'(z)}{f(z)} \prec \phi(z), z \in \Delta, \tag{1.5}$$

$$\frac{wg'(w)}{g(w)} \prec \phi(w), w \in \Delta \tag{1.6}$$

where g is an extension of f^{-1} to Δ.

(1) If $\phi(z) = \frac{1+z}{1-z}$, then the class $S^*_\Sigma(\phi)$ reduces to the class S^*_Σ and satisfies the following conditions:

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in \Delta, \tag{1.7}$$

$$\mathrm{Re}\frac{wg'(w)}{g(w)} > 0, w \in \Delta \tag{1.8}$$

where g is an extension of f^{-1} to Δ. The class S^*_Σ is called class of bi-starlike functions.

(2) If $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}, 0 \leq \alpha < 1$, then the class $S^*_\Sigma(\phi)$ reduces to the class $S^*(\alpha)$ and satisfies the following conditions:

$$\text{Re}\frac{zf'(z)}{f(z)} > \alpha, z \in \Delta, \tag{1.9}$$

$$\text{Re}\frac{wg'(w)}{g(w)} > \alpha, w \in \Delta \tag{1.10}$$

where g is an extension of f^{-1} to Δ. The class $S^*(\alpha)$ is called class of bi-starlike functions of order α.

(3) If $\phi(z) = (\frac{1+z}{1-z})^\beta, 0 < \beta \leq 1$, then the class $S^*_\Sigma(\phi)$ reduces to the class $S^*(\beta)$ and satisfies the following conditions:

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}, z \in \Delta, \tag{1.11}$$

$$\left|\arg\frac{wg'(w)}{g(w)}\right| < \frac{\beta\pi}{2}, w \in \Delta \tag{1.12}$$

where g is an extension of f^{-1} to Δ. The class $S^*(\beta)$ is called class of strongly bi-starlike functions of order β.

(4) If $\phi(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$,then the class $S^*_\Sigma(\phi)$ reduces to the class of bi-starlike functions $S^*_\Sigma(\phi_{PAR})$ and satisfies the following conditions:

$$\text{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \Delta, \tag{1.13}$$

$$\text{Re}\frac{wg'(w)}{g(w)} > \left|\frac{wg'(w)}{g(w)} - 1\right|, w \in \Delta \tag{1.14}$$

where g is an extension of f^{-1} to Δ. The class $S^*_\Sigma(\phi_{PAR})$ is called class of parabolic bi-starlike functions.

Definition 1.2 A function $f \in \Sigma$ is said to be in the class $CV_\Sigma(\phi)$, if it satisfies the following conditions:

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \Delta, \tag{1.15}$$

$$1 + \frac{wg''(w)}{g'(w)} \prec \phi(w), w \in \Delta \tag{1.16}$$

where g is an extension of f^{-1} to Δ . The initial coefficients of bi-starlike and bi-convex functions were estimated by R. M. Ali. et al.[1].

(1) If $\phi(z) = \frac{1+z}{1-z}$, then the class $CV_\Sigma(\phi)$, reduces to the class CV_Σ, and satisfies the following conditions:

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, z \in \Delta, \tag{1.17}$$

$$\mathrm{Re}\Big(1 + \frac{wg''(w)}{g'(w)}\Big) > 0, w \in \Delta \tag{1.18}$$

where g is an extension of f^{-1} to Δ. The class CV_Σ is called class of bi-convex functions.

(2) If $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}, 0 \le \alpha < 1$, then the class $CV_\Sigma(\phi)$ reduces to the class $CV_\Sigma(\alpha)$ and satisfies the following conditions:

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha, z \in \Delta, \tag{1.19}$$

$$\mathrm{Re}\Big(1 + \frac{wg''(w)}{g'(w)}\Big) > \alpha, w \in \Delta \tag{1.20}$$

where g is an extension of f^{-1} to Δ. The class $CV_\Sigma(\alpha)$ is called class of bi-convex functions of order α.

(3) If $\phi(z) = (\frac{1+z}{1-z})^\beta, 0 < \beta \le 1$, then the class $CV_\Sigma(\phi)$ reduces to the class $CV_\Sigma(\beta)$ and satisfies the following conditions:

$$\Big|\arg\Big(1 + \frac{zf''(z)}{f'(z)}\Big)\Big| < \frac{\beta\pi}{2}, z \in \Delta, \tag{1.21}$$

$$\Big|\arg\Big(1 + \frac{wg''(w)}{g'(w)}\Big)\Big| < \frac{\beta\pi}{2}, w \in \Delta \tag{1.22}$$

where g is an extension of f^{-1} to Δ. The class $CV_\Sigma(\beta)$ is called class of strongly bi-convex functions of order β.

(4) If $\phi(z) = 1 + \frac{2}{\pi^2}\Big(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\Big)^2$, then the class $CV_\Sigma(\phi)$ reduces to the class of bi-starlike functions $CV_\Sigma(\phi_{PAR})$ and satisfies the following conditions:

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \Big|\frac{zf''(z)}{f'(z)}\Big|, z \in \Delta, \tag{1.23}$$

$$\mathrm{Re}\Big(1 + \frac{wg''(w)}{g'(w)}\Big) > \Big|\frac{wg''(w)}{g'(w)}\Big|, w \in \Delta \tag{1.24}$$

where g is an extension of f^{-1} to Δ. The class $CV_\Sigma(\phi_{PAR})$ is called class of parabolic bi-convex functions.

Definition 1.3 A function $f \in \Sigma$ is said to be in the class $H_\Sigma(\phi)$, if it satisfies the following conditions:

$$f'(z) \prec \phi(z), z \in \Delta, \tag{1.25}$$

$$g'(w) \prec \phi(w), w \in \Delta \tag{1.26}$$

where g is an extension of f^{-1} to Δ. The class $H_\Sigma(\phi)$ can be classified into different subclasses similar to that of classes $S^*_\Sigma(\phi)$, and $CV(\phi)$.

Definition 1.4 A function $f \in \Sigma$ is said to be in the class $R_{\Sigma}(\phi)$, if it satisfies the following conditions:

$$f'(z) + zf''(z) \prec \phi(z), z \in \Delta, \tag{1.27}$$

$$g'(w) + wg''(w) \prec \phi(w), w \in \Delta \tag{1.28}$$

where g is an extension of f^{-1} to Δ. The class $R_{\Sigma}(\phi)$ can be classified into different subclasses similar to that of classes $S^*_{\Sigma}(\phi)$, and $CV(\phi)$.

2. Preliminaries

Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

which are analytic in the open unit disc Δ and satisfying $\operatorname{Re} p(z) > 0$ for any $z \in \Delta$.

Lemma 2.1. [10] *If $p \in P$ then $|c_n| \leq 2$ for each $n \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. [15] *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc Δ to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & c_3 & ------- & c_n \\ c_{-1} & 2 & c_1 & c_2 & ------- & c_{n-1} \\ c_{-2} & c_{-1} & 2 & c_1 & ------- & c_{n-1} \\ --- & --- & --- & --- & ------- & --- \\ --- & --- & --- & --- & ------- & --- \\ c_{-n} & c_{-n+1} & c_{-n+2} & c_{-n+3} & --------- & 2 \end{vmatrix} ; \forall\, n \in \mathbb{N}$$

and $c_{-k} = \overline{c_k}$ are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_o(exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_o(z) = \frac{1+z}{1-z}$, in this case $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geq m$.

We may assume that without any restriction that $c_1 > 0$, on using lemma (2.2) for $n = 2$ and $n = 3$ respectively we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \\ \overline{c_2} & \overline{c_1} & 2 \end{vmatrix} = [8 + 2\operatorname{Re}\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \text{for some } x,\ |x| \leq 1. \tag{2.2}$$

If we consider the determinant

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c_1} & 2 & c_1 & c_2 \\ \overline{c_2} & \overline{c_1} & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & \overline{c_1} & 2 \end{vmatrix} \geq 0,$$

we get the following inequality

$$(2.3)\qquad |(4c_3-4c_1c_2+c_1^3)(4-c_1^2)+c_1(2c_2-c_1^2)^2|\leq 2(4-c_1^2)^2-2|(2c_2-c_1^2)|^2.$$

From (2.2) and (2.3), it is obtained that

$$(2.4)\qquad 4c_3=c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2c_1(4-c_1^2)(1-|x|^2)z,$$

for some z, $|z|\leq 1$.
Another required result is the optimal value of quadratic expression. Standard computations show that

$$\max_{0\leq t\leq 4}(Pt^2+Qt+R)=\begin{cases} R, & Q\leq 0, P\leq \frac{-Q}{4}\\ 16P+4Q+R, & Q\geq 0, P\geq \frac{-Q}{8} \text{ or } Q\leq 0, P\geq \frac{-Q}{4}\\ \frac{4PR-Q^2}{4P}, & Q>0, P\leq \frac{-Q}{8}\end{cases}$$

3. Main Results

Theorem 3.1. *Let $f\in S^*_\Sigma(\phi)$ and be of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$.*

(1) *If $4(\frac{B_1^2}{8}+\frac{|B_2|}{3})\leq \frac{B_1}{3}$, $\frac{B_1^3+|B_3|}{6}\leq \frac{B_1}{8}$ then $|a_2a_4-a_3^2|\leq \frac{B_1^2}{4}$.*
(2) *If $4(\frac{B_1^2}{8}+\frac{|B_2|}{3})\geq \frac{B_1}{3}$, $\frac{B_1^3+|B_3|}{6}-\frac{1}{2}(\frac{B_1^2}{8}+\frac{|B_2|}{3})-\frac{B_1}{12}\geq 0$ or $4(\frac{B_1^2}{8}+\frac{|B_2|}{3})\leq \frac{B_1}{3}$, $\frac{B_1^3+|B_3|}{6}\geq \frac{B_1}{8}$ then $|a_2a_4-a_3^2|\leq \frac{B_1(B_1^3+|B_3|)}{3}$.*
(3) *$4(\frac{B_1^2}{8}+\frac{|B_2|}{3})>\frac{B_1}{3}$, $\frac{B_1^3+|B_3|}{6}-\frac{1}{2}(\frac{B_1^2}{8}+\frac{|B_2|}{3})-\frac{B_1}{12}\leq 0$ then*
$$|a_2a_4-a_3^2|\leq \frac{B_1}{8}\Big[\frac{\frac{B_1}{3}(B_1^3+|B_3|)-\frac{4B_1}{3}(\frac{B_1^2}{8}+\frac{|B_2|}{3})-\frac{B_1^2}{9}-4(\frac{B_1^2}{8}+\frac{|B_2|}{3})^2}{\frac{B_1^3+|B_3|}{6}-(\frac{B_1^2}{8}+\frac{|B_2|}{3})-\frac{B_1}{24}}\Big].$$

Proof. Since $f\in S^*_\Sigma(\phi)$, there exists two Schwarz functions $u(z)$, $v(w)$ in Δ with $u(0)=0, v(0)=0$ and $|u(z)|\leq 1$, $|v(w)|\leq 1$ such that

$$(3.1)\qquad \frac{zf'(z)}{f(z)}=\{\phi[u(z)]\},$$

$$(3.2)\qquad \frac{wg'(w)}{g(w)}=\{\phi[v(w)]\}.$$

Define two functions $p(z)$, $q(w)$ such that

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+c_3z^3+\ldots\ldots$$

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+d_3w^3+\ldots\ldots$$

Then

$$\begin{aligned}\phi\left(\frac{p(z)-1}{p(z)+1}\right) =&1+\frac{B_1c_1z}{2}+\left(\frac{B_1}{2}(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}\right)z^2\\&+\left(\frac{B_1}{2}\left(\frac{c_1^3}{4}-c_1c_2+c_3\right)+\frac{B_2}{4}(2c_1c_2-c_1^3)+\frac{B_3}{8}c_1^3\right)z^3+\ldots\ldots\\ \phi\left(\frac{q(w)-1}{q(w)+1}\right) =&1+\frac{B_1d_1w}{2}+\left(\frac{B_1}{2}(d_2-\frac{d_1^2}{2})+\frac{B_2d_1^2}{4}\right)w^2\\&+\left(\frac{B_1}{2}\left(\frac{d_1^3}{4}-d_1d_2+d_3\right)+\frac{B_2}{4}(2d_1d_2-d_1^3)+\frac{B_3}{8}d_1^3\right)w^3+\ldots\ldots\end{aligned}$$

Then the equations (3.1) and (3.2) becomes

$$\frac{zf'(z)}{f(z)}=\phi\left(\frac{1+u(z)}{1-u(z)}\right), \tag{3.3}$$

$$\frac{wg'(w)}{g(w)}=\phi\left(\frac{1+v(w)}{1-v(w)}\right). \tag{3.4}$$

Now equate the coefficients in (3.3) and (3.4)

$$a_2=\frac{B_1c_1}{2}, \tag{3.5}$$

$$2a_3-a_2^2=\frac{B_1}{2}(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}, \tag{3.6}$$

$$3a_4-3a_2a_3+a_2^3=\frac{B_1}{2}\left(\frac{c_1^3}{4}-c_1c_2+c_3\right)+\frac{B_2}{4}\left(2c_1c_2-c_1^3\right)+\frac{B_3}{8}c_1^3 \tag{3.7}$$

and

$$-a_2=\frac{B_1d_1}{2}, \tag{3.8}$$

$$3a_2^2-2a_3=\frac{B_1}{2}(d_2-\frac{d_1^2}{2})+\frac{B_2d_1^2}{4}, \tag{3.9}$$

$$-3a_4+12a_2a_3-10a_2^3=\frac{B_1}{2}\left(\frac{d_1^3}{4}-d_1d_2+d_3\right)+\frac{B_2}{4}\left(2d_1d_2-d_1^3\right)+\frac{B_3}{8}d_1^3. \tag{3.10}$$

Now from (3.5) and (3.8)

$$c_1=-d_1 \tag{3.11}$$

and

$$a_2=\frac{B_1c_1}{2}. \tag{3.12}$$

Now from (3.6) and (3.9)

$$a_3=\frac{B_1^2c_1^2}{4}+\frac{B_1(c_2-d_2)}{8}. \tag{3.13}$$

Now from (3.7) and (3.10)

$$a_4 = \frac{B_1^3 c_1^3}{12} + \frac{5B_1^2 c_1(c_2-d_2)}{32} + \frac{B_2}{12}(c_1(c_2+d_2)-c_1^3)+$$

$$\frac{B_1}{12}\left[\frac{c_1^3}{2} - c_1(c_2+d_2) + (c_3-d_3)\right] + \frac{B_3 c_1^3}{24}. \tag{3.14}$$

Then

$$|a_2a_4 - a_3^2| = \frac{B_1}{8}\Big|\frac{-c_1^4}{6}[B_1^3 - B_1 - B_3 + 2B_2] + \frac{c_1 B_1(c_3-d_3)}{3} - \frac{B_1(c_2-d_2)^2}{8}$$

$$+ c^2\left[\frac{B_1^2(c_2-d_2)}{8} + \frac{(B_2-B_1)(c_2+d_2)}{3}\right]\Big|. \tag{3.15}$$

According to Lemma (2.2) we get that

$$2c_2 = c_1^2 + x(4-c_1^2), 2d_2 = d_1^2 + y(4-d_1^2),$$

$$\Longrightarrow (c_2-d_2) = \frac{(4-c_1^2)(x-y)}{2}; (c_2+d_2) = c_1^2 + \frac{(4-c_1^2)(x+y)}{2} \tag{3.16}$$

and

$$(c_3-d_3) = \frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4}$$

$$+ \frac{(4-c_1^2)}{2}[(1-|x|^2)z - (1-|y|^2)w]. \tag{3.17}$$

For some z, w with $|z| \leq 1, |w| \leq 1$. Using (3.16) and (3.17), we have

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{8}\Big[c_1^4 \frac{B_1^3 + |B_3|}{6} + \frac{c_1 B_1(4-c_1^2)}{3}\Big] + \frac{c_1^2 B_1(4-c_1^2)}{16}\Big(\frac{|B_2|}{3} + \frac{B_1^2}{8}\Big)(|x|+|y|)$$

$$+ \Big(\frac{B_1^2(c_1^2-2c)(4-c_1^2)}{96}\Big)(|x|^2+|y|^2) + \frac{B_1^2(4-c_1^2)^2}{256}(|x|+|y|)^2. \tag{3.18}$$

Since $p \in P$, so $|c_1| \leq 2$. Letting $c_1 = c$ we may assume without any restriction that $c \in [0,2]$. Thus for $\gamma_1 = |x| \leq 1$ and $\gamma_2 = |y| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq T_1 + T_2(\gamma_1+\gamma_2) + T_3(\gamma_1^2+\gamma_2^2) + T_4(\gamma_1+\gamma_2)^2 = F(\gamma_1,\gamma_2)$$

where

$$T_1 = \frac{B_1}{8}\Big[c^4\frac{B_1^3+|B_3|}{6} + \frac{cB_1(4-c^2)}{3}\Big],$$

$$T_2 = \frac{B_1 c^2(4-c^2)}{16}\Big(\frac{|B_2|}{3} + \frac{B_1^2}{8}\Big),$$

$$T_3 = \frac{B_1^2(4-c^2)(c^2-2c)}{96},$$

$$T_4 = \frac{B_1^2(4-c^2)^2}{256}.$$

Now we need to maximize $F(\gamma_1,\gamma_2)$ in the closed square $S = [0,1]\times[0,1]$ for any $c \in [0,2]$. We must investigate the maximum of $F(\gamma_1,\gamma_2)$ according to $c \in (0,2), c = 2$ and $c = 0$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$. First, let $c \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$, so we conclude that $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0$. Thus

the function F cannot have a local maximum in the interior of the square S. Now, we investigate the maximum of F on the boundary of the square S.
For $\gamma_1 = 0$ and $0 \le \gamma_2 \le 1$(Similarly $\gamma_2 = 0$ and $0 \le \gamma_1 \le 1$), we obtain

$$F(0, \gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2.$$

i. The case $T_3 + T_4 \ge 0$: In this case $0 \le \gamma_2 \le 1$ and for any fixed c with $0 < c < 2$, it is clear that $G'(\gamma_2) = 2(T_3 + T_4)\gamma_2 + T_2 > 0$; that is $G(\gamma_2)$ is an increasing function. Hence for any fixed $c \in (0, 2)$ the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$maxG(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

ii. The case $T_3 + T_4 < 0$: Since $2(T_3 + T_4) + T_2 \ge 0$ for $0 \le \gamma_2 \le 1$ and for any fixed c with $0 < c < 2$, it is clear that $2(T_3 + T_4) + T_2 < 2(T_3 + T_4)\gamma_2 + T_2 < T_2$ and so $G'(\gamma_2) > 0$. Hence for any fixed $c \in [0, 2)$ the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$. Also for $c = 2$ we obtain

$$F(\gamma_1, \gamma_2) = \frac{B_1}{3}(B_1^3 + |B_3|). \tag{3.19}$$

Taking into account the value of (3.19) and the case **i.** and case **ii.**, for $0 \le \gamma_2 \le 1$ and for any fixed c with $0 \le c \le 2$,

$$maxG(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\gamma_1 = 1$ and $0 \le \gamma_2 \le 1$ (similarly $\gamma_2 = 1$ and $0 \le \gamma_1 \le 1$), we obtain

$$F(1, \gamma_2) = H(\gamma_2) = (T_3 + T_3)\gamma_2^2 + (T_2 + 2T_4)\gamma_2 + T_1 + T_2 + T_3 + T_4.$$

Similarly to above case of $T_3 + T_4$, we get that

$$maxH(\gamma_2) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \le H(1)$ for $c \in [0, 2]$, $maxF(\gamma_1, \gamma_2) = F(1, 1)$ on the boundary of the square S. Thus the maximum of F occurs at $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square S.
Let $K : [0, 2] \longrightarrow \Re$

$$K(c) = maxF(\gamma_1, \gamma_2) = F(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Substituting the values of T_1, T_2, T_3, T_4 in above equation

$$K(c) = \frac{B_1}{8}\Big[c^4\Big(\frac{B_1^3 + |B_3|}{6} - \Big(\frac{B_1^2}{8} + \frac{|B_2|}{3}\Big) - \frac{B_1}{24}\Big) + c^2\Big(4\big(\frac{B_1^2}{8} + \frac{|B_2|}{3}\big) - \frac{B_1}{3}\Big) + 2B_1\Big].$$

Let

$$\begin{aligned} P =& \frac{B_1}{8}\Big[\frac{B_1^3 + B_3}{6} - \Big(\frac{B_1^2}{8} + \frac{|B_2|}{3}\Big) - \frac{B_1}{24}\Big], \\ Q =& \frac{B_1}{8}\Big[4\Big(\frac{B_1^2}{8} + \frac{|B_2|}{3}\Big) - \frac{B_1}{3}\Big], \\ R =& \frac{B_1^2}{4}. \end{aligned}$$

Then $K(c) = Pt^2 + Qt + R$ where $t = c^2$.
Then with help of optimal value of Quadratic expression, we get the required result. This completes the proof of the theorem. $\square$

Corollary 3.2. *If $f \in S^*_\Sigma(\alpha)$, then*

$$|a_2a_4 - a_3^2| \le \begin{cases} \dfrac{4(1-\alpha)^2(4\alpha^2-8\alpha+5)}{3}; & \text{if } \alpha \in [0, \frac{29-\sqrt{137}}{32}], \\ \dfrac{(1-\alpha)^2(13\alpha^2-14\alpha-7)}{(16\alpha^2-26\alpha+5)}; & \text{if } \alpha \in (\frac{29-\sqrt{137}}{32}, 1). \end{cases}$$

This result coincides with the result in [9].

Corollary 3.3. *If $f \in S^*_\Sigma$, then*

$$|a_2a_4 - a_3^2| \le \frac{20}{3}.$$

This result coincides with the result in [9].

Corollary 3.4. *If $f \in S^*_\Sigma(\beta)$, then*

$$|a_2a_4 - a_3^2| \le \begin{cases} \beta^2; & \text{if } 0 < \beta \le \frac{1}{7}, \\ \dfrac{7\beta^3(\beta-4)}{56\beta^2-42\beta+1}; & \text{if } \frac{1}{7} \le \beta \le \frac{3}{16} + \frac{\sqrt{889}}{112}, \\ \dfrac{4\beta^2(14\beta^3+1)}{9}; & \text{if } \frac{3}{16} + \frac{\sqrt{889}}{112} \le \beta \le 1. \end{cases}$$

Corollary 3.5. *If $f \in S^*_\Sigma(\phi_{PAR})$, then*

$$|a_2a_4 - a_3^2| \le \frac{\frac{1}{\pi^4}\left[\frac{3328}{3\pi^4} - \frac{1792}{9\pi^2} - \frac{11264}{405}\right]}{\left[\frac{256}{3\pi^4} - \frac{8}{\pi^2} - \frac{193}{135}\right]}.$$

Theorem 3.6. *Let $f \in CV_\Sigma(\phi)$ and be of the form$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.*

(1) *If* $2(\frac{B_1^2}{4} + |B_2|) \le \frac{B_1}{3}$, $\frac{B_1^3+4|B_3|}{8} \le \frac{B_1}{3}$ *then* $|a_2a_4 - a_3^2| \le \frac{B_1^2}{36}$.

(2) *If* $2(\frac{B_1^2}{4} + |B_2|) \ge \frac{B_1}{3}$, $\frac{B_1^3+4|B_3|}{8} - \frac{1}{2}(\frac{B_1^2}{4} + |B_2|) - \frac{B_1}{4} \ge 0$ *or* $2(\frac{B_1^2}{4} + |B_2|) \le \frac{B_1}{3}$, $\frac{B_1^3+|4B_3|}{8} \ge \frac{B_1}{3}$ *then* $|a_2a_4 - a_3^2| \le \frac{B_1(B_1^3+4|B_3|)}{96}$.

(3) $2(\frac{B_1^2}{4} + |B_2|) > \frac{B_1}{3}, \frac{B_1^3+4|B_3|}{8} - \frac{1}{2}(\frac{B_1^2}{4} + |B_2|) - \frac{B_1}{4} \le 0$ *then*

$$|a_2a_4 - a_3^2| \le \frac{B_1}{8}\left[\frac{\frac{2B_1}{3}(B_1^3 + 4|B_3|) - 4B_1(\frac{B_1^2}{4} + |B_2|) - B_1^2 - 4(\frac{B_1^2}{4} + |B_2|)^2}{\frac{B_1^3+|B_3|}{8} - (\frac{B_1^2}{4} + |B_2|) - \frac{B_1}{6}}\right].$$

Corollary 3.7. *If $f \in CV_\Sigma(\alpha)$, then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\alpha)^2(5\alpha^2+8\alpha-32)}{24(3\alpha^2-3\alpha-4)}.$$

This result coincides with the result in [9].

Corollary 3.8. *If $f \in CV_\Sigma$, then*

$$|a_2a_4 - a_3^2| \le \frac{1}{3}.$$

This result coincides with the result in [9].

Corollary 3.9. *If $f \in CV_\Sigma(\beta)$, then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \dfrac{\beta^2}{9}; & \text{if } 0 < \beta \leq \frac{1}{9}, \\ \dfrac{\beta(41\beta^2 + 54\beta + 1)}{72(9-5\beta)}; & \text{if } \frac{1}{9} \leq \beta \leq 1 \end{cases}$$

Corollary 3.10. *If $f \in CV_\Sigma(\phi_{PAR})$, then*

$$|a_2a_4 - a_3^2| \leq \frac{\frac{1}{\pi^4}\left[\frac{80}{9\pi^4} - \frac{56}{9\pi^2} - \frac{551}{405}\right]}{\left[\frac{8}{\pi^4} - \frac{2}{\pi^2} - \frac{26}{45}\right]}.$$

Theorem 3.11. *Let $f \in H_\Sigma(\phi)$ and be of the form $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$.*

(1) *If* $\left(\frac{B_1^2}{3} + 2|B_2|\right) \leq \frac{7B_1}{9}$, $\frac{|B_1^3+2B_3|}{4} \leq \frac{4B_1}{9}$ *then* $|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9}$.

(2) *If* $\left(\frac{B_1^2}{3} + 2|B_2|\right) \geq \frac{7B_1}{9}$, $\frac{|B_1^3+2B_3|}{4} - \frac{1}{2}\left(\frac{B_1^2}{6} + |B_2|\right) - \frac{B_1}{4} \geq 0$, *or* $(\frac{B_1^2}{3} + 2|B_2|) \leq \frac{7B_1}{9}$, $\frac{|B_1^3+2B_3|}{4} \geq \frac{4B_1}{9}$ *then* $|a_2a_4 - a_3^2| \leq \frac{B_1(B_1^3+2|B_3|)}{16}$.

(3) *If* $\left(\frac{B_1^2}{3} + 2|B_2|\right) > \frac{7B_1}{9}$, $\frac{B_1^3+2|B_3|}{4} - \frac{1}{2}\left(\frac{B_1^2}{6} + |B_2|\right) - \frac{B_1}{4} \leq 0$ *then*

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{64}\Big[\frac{\frac{16B_1}{9}(B_1^3 + 2|B_3|) - 4B_1\left(\frac{B_1^2}{6} + |B_2|\right) - 4\left(\frac{B_1^2}{6} + |B_2|\right)^2 - B_1^2}{\frac{B_1^3+2|B_3|}{4} - \left(\frac{B_1^2}{6} + |B_2|\right) - \frac{B_1}{18}}\Big].$$

Corollary 3.12. *If $f \in H_\Sigma(\alpha)$, then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \dfrac{(1-\alpha)^2(2\alpha^2 - 4\alpha + 3)}{2}, & \text{if } \alpha \in \left[0,\ \frac{11-\sqrt{37}}{12}\right], \\ \dfrac{(1-\alpha)^2(60\alpha^2 - 84\alpha - 25)}{16(9\alpha^2 - 15\alpha + 1)}, & \text{if } \alpha \in \left(\frac{11-\sqrt{37}}{12}, 1\right). \end{cases}$$

This result coincides with the result in [7].

Corollary 3.13. *If $f \in H_\Sigma$, then*

$$|a_2a_4 - a_3^2| \leq \frac{3}{2}.$$

This result coincides with the result in [7].

Corollary 3.14. *If $f \in H_\Sigma(\beta)$, then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \dfrac{4\beta^2}{9}; & \text{if } 0 < \beta \leq \frac{7}{24}, \\ \dfrac{\beta^2(64\beta^2 - 144\beta + 5)}{48(12\beta^2 - 12\beta + 1)}; & \text{if } \frac{7}{24} \leq \beta \leq \frac{1+\sqrt{2}}{4}, \\ \dfrac{\beta^2(8\beta^2 + 1)}{6}; & \text{if } \frac{1+\sqrt{2}}{4} \leq \beta \leq 1. \end{cases}$$

This result coincides with the result in [7].

Corollary 3.15. *If $f \in H_{\Sigma}(\phi_{PAR})$, then*

$$|a_2a_4 - a_3^2| \leq \frac{\frac{1}{3\pi^4}\left[\frac{2560}{\pi^4} - \frac{896}{3\pi^2} - \frac{11752}{135}\right]}{\left[\frac{128}{\pi^4} - \frac{32}{3\pi^2} - \frac{56}{15}\right]}.$$

Theorem 3.16. *Let $f \in R_{\Sigma}(\phi)$ and be of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.*

(1) *If $\frac{4}{9}(B_1^2 + 9|B_2|) \leq \frac{94B_1}{81}$, $\frac{B_1^3+4|B_3|}{8} \leq \frac{32B_1}{81}$ then $|a_2a_4 - a_3^2| \leq \frac{B_1^2}{81}$.*

(2) *If $\frac{4}{9}(B_1^2 + 9|B_2|) \geq \frac{94B_1}{81}$, $\frac{|B_1^3+4B_3|}{8} - \frac{1}{18}(B_1^2 + 9|B_2|) - \frac{B_1}{4} \geq 0$, or $\frac{4}{9}(B_1^2 + 9|B_2|) \leq \frac{94B_1}{81}$, $\frac{B_1^3+4|B_3|}{8} \geq \frac{32B_1}{81}$ then $|a_2a_4 - a_3^2| \leq \dfrac{B_1(B_1^3 + 4|B_3|)}{256}$.*

(3) *If $\frac{4}{9}(B_1^2 + 9|B_2|) > \frac{94B_1}{81}$, $\frac{B_1^3+4|B_3|}{8} - \frac{1}{18}(B_1^2 + 9|B_2|) - \frac{B_1}{4} \leq 0$ then*

$$|a_2a_4 - a_3^2| \leq \frac{\dfrac{B_1^2}{648}(B_1^3 + 4|B_3|) - \frac{B_1^2}{1152}(B_1^2 + 9|B_2|) - \frac{B_1(B_1^2+9|B_2|)^2}{10368} - \frac{B_1^3}{512}}{\frac{B_1^3+4|B_3|}{8} - \frac{B_1^2+9|B_2|}{9} - \frac{17B_1}{162}}.$$

Corollary 3.17. *If $f \in R_{\Sigma}(\alpha)$, then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{(1-\alpha)^2(\alpha^2-2\alpha+2)}{16}, & if\ \alpha \in \left[0,\ \frac{8}{9} - \frac{\sqrt{166}}{18}\right], \\ \dfrac{(1-\alpha)^2(240\alpha^2 - 264\alpha - 449)}{64(81\alpha^2 - 126\alpha - 53)}, & if\ \alpha \in \left(\frac{8}{9} - \frac{\sqrt{166}}{18}, 1\right). \end{cases}$$

Corollary 3.18. *If $f \in R_{\Sigma}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

Corollary 3.19. *If $f \in R_{\Sigma}(\beta)$, then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4\beta^2}{81}; & if\ 0 < \beta \leq \frac{188}{729}, \\ \dfrac{\beta^2(13 - 1188\beta - 172\beta^2)}{192(10 - 198\beta + 135\beta^2)}; & if\ \frac{188}{729} \leq \beta \leq \frac{11+\sqrt{211}}{30}, \\ \dfrac{\beta^2(5\beta^2+1)}{48}; & if\ \frac{11+\sqrt{211}}{30} \leq \beta \leq 1. \end{cases}$$

Corollary 3.20. *If $f \in R_{\Sigma}(\phi_{PAR})$, then*

$$|a_2a_4 - a_3^2| \leq \frac{\frac{1}{\pi^4}\left[\frac{1280}{\pi^4} - \frac{224}{\pi^2} - \frac{13957}{135}\right]}{\left[\frac{64}{\pi^4} - \frac{64}{9\pi^2} - \frac{1672}{405}\right]}.$$

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl. Math. Lett., **25**(2012), no.3, 344-351.

[2] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, *The Fekete-Szegö coefficient functional for transforms of analytic functions*, Bulletin of the Iranian Mathematical Society, **35**(2015), no.2, 119-142.

[3] Ş. Altinkaya, S. Yalçin, *Second Hankel Determinant for a General Subclass of Bi-Univalent Functions*, Twms J. Pure Appl. Math., **7**(2016), no.1, 98-104.

[4] K. O. Babalola, *On Coefficient Determinants with Fekete-Szegö Parameter*, Applied Mathematics E-Notes, **13**(2013), 92-99.

[5] R. Bharavi Sharma and M. Haripriya, *On a class of α-convex functions subordinate to a shell shaped region*, The Journal of Analysis, **25**(2017), no.1, 99-105.

[6] D. A. Brannan and J. G. Clunie, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Proceedings of the NATO Advanced Study Institute, (University of Durham), July (1-20, 1979) Academic Press, New York and London 1980.

[7] M. Çağlar, E. Deniz and H. M. Srivastava, *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turk. J. Math., **41**(2017), 694-706.

[8] E. Deniz, *Certain Subclasses of Bi-Univalent Functions Satisfying Subordinate Conditions*, J. Classical Anal., **2**(2013), no.1, 49-60.

[9] E. Deniz, M. Cağlar and H. Orhan, H., *Second Hankel determinant for bi-starlike and bi-convex functions of order beta*, Appl. Math. Comput., **271**(2015), 301-307.

[10] P. L. Duren, *Univalent Functions,* Grundehren der Math. Wiss., Springer Verlag,(1983) (New York Inc.).

[11] T. Hayami, S. Owa, *G*eneralized Hankel determinant for certain classes, Int. J. Math. Anal., **4**(2010), no.52, 2573-2585.

[12] M. M. EhHosh, *On the second Hankel determinant of univalent functions*, Int. J. Math. Anal., **9**(1986), no.2, 23-25.

[13] B. A. Frasin, M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**(2011), 1569-1573.

[14] I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker, New York, USA, 2003.

[15] U. Grenander, G. Szego, *Toeplitz forms and their applications*, II edition, Chelsa Publishing Co., (New York,1984).

[16] A. Janteng, S. A. Halim and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. (Ruse), **1**(2007), no.13, 619-625.

[17] S. K. Lee, V. Ravichandhran and S. Supramanim, *Bounds for the second Hankel determinant of certain univalent functions*, Journal of Inequalities and Applications, 2013, 2013:281

[18] Lewin,*On a Coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**(1967), 63-68.

[19] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis(Tiangin), Conference Proceedings, International press, Cambridge, USA, 157-169, 1992.

[20] G. Murugusudaramoorthy, K. Vijaya, *Second Hankel Determinant for Bi-univalent analytic functions associated with Hohlov operator*, International Journal of Analysis and Applications, **8**(2015), no.1, 22-29.

[21] J. W. Noonan, D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223**(1976), no.2, 337-346.

[22] H. Orhan, N. Magesh and J. Yamini, *Bounds for the second Hankel determinant of certain bi-univalent functions*, Turkish Journal of Mathematics, **40**(2016), 679-687.

[23] C. Pommerenkee, *On the coefficients and Hankel determinants of univalent functions*, J. London. Math. Soc., **41**(1966), 111-122.

[24] K. Rajya Laxmi, R. Bharavi Sharma, *Coefficient Inequalities of Second Hankel Determinants for Some Classes of Bi-Univalent Functions*, Mathematics, 2016, 4, 9; doi:10.3390/math4010009.

[25] T. Ram Reddy, *A study of certain subclasses of univalent analytic functions*, Thesis (1983), I.I.T. Kanpur.

[26] T. Ram Reddy, D. Vamshee krishna, *Coefficient inequalities for certain subclasses of analytic functions associated with hankel determinant*, Indian Journal of Pure Appl. Math., **46**(2015), no.1, 91-106.

[27] M. S. Robertson, *Certain classes of starlike functions*, Michigan Math. J., **76**(1954), no.1, 755-758.

[28] K. Sakaguchi, *On certain univalent mapping*, J. Math. Soc. Japan, **11**(1959), 72-75.

[29] H. M. Srivastava, S. Altinkaya and S. Yalcin, Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q-derivative operator, Filomat 32 (2018), 503-516.
[30] H. M. Srivastava, Dorina Răducanu and Pawel Zaprawa, *A Certain Subclass of Analytic Functions Defined by Means of Dfferential Subordination*, Filomat, **30**(2016), 3743-3757.
[31] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, Afrika Mat. 28 (2017), 693-706.
[32] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**(2010), 1188-1199.
[33] H. M.Srivastava, S. Sivasubramanian and R. Sivakumar, *Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions*, Tbilisi Math. J., **7**(2014), no.2, 1-10.
[34] H. M. Srivastava, S. Sumer Eker, S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iranian Math. Soc. 44 (2018), 149-157.
[35] H. Tang, H. M. Srivastava, S. Sivasubramanian and P. Gurusamy , *The Fekete-Szegö functional problems for some classes of m-fold symmetric bi-univalent functions*, J. Math. Inequal., **10**(2016), 1063-1092.
[36] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput., **218** (2012), 11461-11465.
[37] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, *Coefficient Estimates for a Certain Subclass of Analytic and Bi-Univalent Functions*, Applied Mathematics Letters, **25**(2012), 990-994.

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