Advanced Studies in Contemporary Mathematics
29 (2019), No. 4, pp. 505 – 520

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.4.505

A WRIT LARGE ANALYSIS OF COMPLEX ORDER COUPLED DIFFERENTIAL EQUATIONS IN THE COURSE OF COUPLED NON-LOCAL MULTI-POINT BOUNDARY CONDITIONS

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ABSTRACT. The primary objective of this paper is to validate the existence and uniqueness of coupled fractional differential equations of complex order along with coupled non-local multipoint boundary conditions. The convergence of the problem has validated with suitable examples.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 34A08, 34A12, 34B10.

KEYWORDS AND PHRASES. Fractional differential equations, Caputo derivatives, Multi-point, Non-local, Existence, Fixed point.

1. INTRODUCTION

The concept of fractional calculus is a generalization of differentiation, integration of real and complex order. Among various mathematicians, Liouville, Riemann, and Weyl gave important contributions to the theory of fractional calculus. The advantages of fractional calculus are perceptible in modeling, mechanical and electrical properties as well as in the description of properties of gases, liquids, and rocks, and in many other fields, we refer the reader to the texts [4, 7, 9, 11], and the references cited therein. There have been many contributions in this regard, going back to the recent work of Ahmad et.al. [1, 2], where they have shown the solvability of a coupled system of fractional differential equations by flux type integral boundary conditions. Also, they established the existence results for a coupled system of Caputo type sequential fractional differential equations with non-local integral boundary conditions. More important to the fractional boundary conditions were given by Liu et al.,[8], Henderson et al.,[5] and Wang et al.,[14] in the existence of solutions in a coupled system of the nonlinear fractional differential equation. Tariboon et al., [13] experimented on the non-integer order of Riemann-Liouville fractional derivatives and Hadamard fractional integrals on coupled fractional-order differential equations. For some recent work on the multiple orders of fractional derivatives and integrals on coupled fractional order differential equation was initiated by Suantai et al.,[12]. Later this year, discrete boundary conditions on coupled fractional-order differential equations were introduced by Alsaedi et al.,[3].

In view of the above, the reader can see that the authors have discussed the existence of solutions of fractional differential equations only on non-integer order. This provocative the inception of complex order differential equations and as defined by

$$
\begin{aligned}
&\mathfrak{D}^{\varsigma_1} v(z) = h_1(z, v(z), w(z)), \quad z \in \mathfrak{J} := [0,1] \;\; \varsigma_1 = \xi_1 + i\zeta_1, \\
&\mathfrak{D}^{\varsigma_2} w(z) = h_2(z, v(z), w(z)), \quad z \in \mathfrak{J} := [0,1] \;\; \varsigma_2 = \xi_2 + i\zeta_2,
\end{aligned} \tag{1}
$$

supplemented with the boundary conditions

$$(2)\qquad \begin{array}{rcl} v(0) &=& \Psi_1(w), \quad v^{'}(0) = \gamma_1 w^{'}(\delta_1), \quad v(1) = 0, \\ w(0) &=& \Psi_2(v), \quad w^{'}(0) = \gamma_2 v^{'}(\delta_2), \quad w(1) = 0, \end{array}$$

where $2 < \xi_1, \xi_2 \leq 3$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$ and $\mathfrak{D}^{\varsigma_1}$,$\mathfrak{D}^{\varsigma_2}$ are the Caputo fractional derivatives of order $\varsigma_1, \varsigma_2 \in \mathbb{C}$ and $h_1, h_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Psi_1, \Psi_2 : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, $0 < \delta_1, \delta_2 < 1$, γ_i, $(i = 1, 2)$ are positive real constants. The paper has been arranged systematically beginning with, preliminaries which have some fundamental concepts of fractional calculus with basic lemma related to the defined problem followed by the main results based on Leray-Schauder nonlinear alternative type and Banach fixed point theorem, finally with suitable examples as the validation of the results.

2. Preliminaries

Before commencing with the problem, we shall define some notations and lemmas of fractional calculus, which we may find useful for our problem. [6, 10]

Definition 2.1 The fractional integral of order $\varsigma \in \mathbb{C}$, $(\Re(\varsigma) > 0)$ with the lower limit zero for a function h is defined as

$$\mathfrak{I}^{\varsigma} h(z) \;=\; \frac{1}{\Gamma(\varsigma)} \int_0^z \frac{h(\theta)}{(z-\theta)^{1-\varsigma}} d\theta,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\varsigma) = \int_0^{\infty} z^{\varsigma-1} e^{-z} dz$.

Definition 2.2 The Stirling asymptotic formula of the Gamma function for $x \in \mathbb{C}$ is following

$$\Gamma(x) = (2\pi)^{\frac{1}{2}} x^{x-\frac{1}{2}} e^{-x} \left[1 + O\Big(\frac{1}{x}\Big) \right], \qquad (|arg(x)| < \pi; |x| \to \infty),$$

and its result for $|\Gamma(\xi + i\zeta)|$, $(\xi, \zeta \in \mathbb{R})$ is

$$(3)\qquad |\Gamma(\xi + i\zeta)| = (2\pi)^{\frac{1}{2}} |\zeta|^{\xi-\frac{1}{2}} e^{-\xi-\pi|\zeta|/2} \left[1 + O\Big(\frac{1}{\zeta}\Big) \right], \qquad (|\zeta| \to \infty).$$

Definition 2.3 The Caputo derivative of order $\varsigma \in \mathbb{C}$, $(\Re(\varsigma) > 0)$ for a function $h : [0, \infty) \to \mathbb{R}$ can be written as

$$\mathfrak{D}^{\varsigma} h(z) \;=\; \frac{1}{\Gamma(n-\varsigma)} \int_0^z (z-\theta)^{n-\varsigma-1} h^{(n)}(\theta) d\theta.$$

Lemma 2.1. *For $\varsigma \in \mathbb{C}$, $(\Re(\varsigma) > 0)$ the general solution of the fractional differential equation $\mathfrak{D}^{\varsigma} v(z) = 0$ is given by*

$$v(z) \;=\; d_1 + d_2 z + d_3 z^2 + \cdots + d_{n-1} z^{n-1}, \quad \text{where } d_i \in \mathbb{R},\ i = 1, 2, ..., n-1 (n = [\Re(\varsigma)] + 1).$$

In view of Lemma 2.1, it follows that

$$\mathfrak{I}^{\varsigma} \mathfrak{D}^{\varsigma} v(z) \;=\; v(z) + d_1 + d_2 z + d_3 z^2 + \cdots + d_{n-1} z^{n-1}, \quad \text{for some } d_i \in \mathbb{R},\ i = 1, 2, ..., n-1 (n = [\Re(\varsigma)] + 1).$$

Further, we proceed by furnishing a lemma which supports the continuation of the problem.

Lemma 2.2. *For* $\hat{h}_1, \hat{h}_2 \in C[0,1]$, *the solution of the linear system of fractional differential equations*

$$\begin{cases} \mathfrak{D}^{\varsigma_1} v(z) = \hat{h}_1(z), & z \in \mathfrak{J},\ \varsigma_1 = \xi_1 + i\zeta_1, \\ \mathfrak{D}^{\varsigma_2} w(z) = \hat{h}_2(z), & z \in \mathfrak{J},\ \varsigma_2 = \xi_2 + i\zeta_2, \end{cases} \tag{4}$$

for $2 < \xi_1, \xi_2 \leq 3$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$ *and supplemented with the boundary conditions (2) is equivalent to the system of integral equations,*

$$\begin{aligned} v(z) \ = \ & \int_0^z \frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)} \hat{h}_1(\theta) d\theta - \Phi_4(z)\Psi_2(v) + [\Phi_3(z)+1]\Psi_1(w) \\ & + \Phi_1(z)\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)} \hat{h}_2(\theta) d\theta\Big] \\ & + \Phi_2(z)\Big[\gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)} \hat{h}_1(\theta) d\theta\Big] \\ & + \Phi_3(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)} \hat{h}_1(\theta) d\theta\Big] \\ & - \Phi_4(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} \hat{h}_2(\theta) d\theta\Big] \end{aligned} \tag{5}$$

and

$$\begin{aligned} w(z) \ = \ & \int_0^z \frac{(z-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} \hat{h}_2(\theta) d\theta - \Phi_7(z)\Psi_1(w) + [\Phi_8(z)+1]\Psi_2(v) \\ & - \Phi_5(z)\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)} \hat{h}_2(\theta) d\theta\Big] \\ & - \Phi_6(z)\Big[\gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)} \hat{h}_1(\theta) d\theta\Big] \\ & - \Phi_7(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)} \hat{h}_1(\theta) d\theta\Big] \\ & + \Phi_8(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} \hat{h}_2(\theta) d\theta\Big] \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Phi_1(z) &= \frac{\kappa_1(z-z^2)}{\Delta}, \quad \Phi_2(z) = \frac{\kappa_2(z-z^2)}{\Delta}, \quad \Phi_3(z) = \frac{\kappa_3(z-z^2)}{\Delta} - z, \\ \Phi_4(z) &= \frac{\kappa_4(z-z^2)}{\Delta}, \quad \Phi_5(z) = \frac{\kappa_1'(z-z^2)}{\Delta}, \quad \Phi_6(z) = \frac{\kappa_2'(z-z^2)}{\Delta}, \\ \Phi_7(z) &= \frac{\kappa_3'(z-z^2)}{\Delta}, \quad \Phi_8(z) = \frac{\kappa_4'(z-z^2)}{\Delta} + z, \end{aligned} \tag{7}$$

$$\begin{aligned} \kappa_1 &= \Delta_2' - \Delta_2\gamma_2, \quad \kappa_2 = \Delta_2'\gamma_1 - \Delta_2, \\ \kappa_3 &= \Delta_2'\vartheta_1, \quad \kappa_4 = \Delta_2\vartheta_1, \end{aligned} \tag{8}$$

$$\kappa_1^{'} = \Delta_1^{'} - \Delta_1\gamma_2, \quad \kappa_2^{'} = \Delta_1^{'}\gamma_1 - \Delta_1,$$

(9) $$\kappa_3^{'} = \Delta_1^{'}\vartheta_1, \quad \kappa_4^{'} = \Delta_1\vartheta_1, \quad \vartheta_1 = 1 - \gamma_1\gamma_2,$$

$$\Delta_1 = \vartheta_1 + 2\gamma_1\gamma_2\delta_2, \quad \Delta_2 = 2\gamma_1\delta_1,$$

(10) $$\Delta_1^{'} = 2\gamma_2\delta_2, \quad \Delta_2^{'} = \vartheta_1 + 2\gamma_1\gamma_2\delta_1,$$

and

(11) $$\Delta = \Delta_1\Delta_2^{'} - \Delta_2\Delta_1^{'} \neq 0.$$

Proof. Solving the fractional differential equations (4) in a standard manner, we get

(12) $$v(z) = \int_0^z \frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta + b_1 + b_2 z + b_3 z^2,$$

(13) $$w(z) = \int_0^z \frac{(z-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta + b_4 + b_5 z + b_6 z^2,$$

where $b_j \in \mathbb{R}$, $j = 1, 2, ..., 6$ are arbitrary constants. Using the boundary conditions (2) in (12) and (13), together with notations (7)-(11), we find that $b_1 = \Psi_1(w)$, $b_4 = \Psi_2(v)$ and

(14) $$b_2 - \gamma_1 b_5 - 2\gamma_1\delta_1 b_6 = \gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\hat{h}_2(\theta)d\theta,$$

(15) $$b_5 - \gamma_2 b_2 - 2\gamma_2\delta_2 b_3 = \gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\hat{h}_1(\theta)d\theta,$$

(16) $$b_2 + b_3 = -\Psi_1(w) - \int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta,$$

(17) $$b_5 + b_6 = -\Psi_2(v) - \int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta.$$

Solving the system (14)-(17) for b_2, b_3, b_5 and b_6, we get

$$\begin{aligned} b_2 &= \frac{1}{\Delta}\Big[\kappa_1\gamma_1\Big(\int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\hat{h}_2(\theta)d\theta\Big) \\ &\quad +\kappa_2\gamma_2\Big(\int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\hat{h}_1(\theta)d\theta\Big) \\ &\quad -\kappa_4\Big(\Psi_2(v) + \int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta\Big)\Big] \\ &\quad +\Big[\frac{\kappa_3}{\Delta} - 1\Big]\Big[\Psi_1(w) + \int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta\Big], \\ b_3 &= \frac{1}{\Delta}\Big[-\kappa_1\gamma_1\Big(\int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\hat{h}_2(\theta)d\theta\Big) \\ &\quad -\kappa_2\gamma_2\Big(\int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\hat{h}_1(\theta)d\theta\Big) \\ &\quad +\kappa_4\Big(\Psi_2(v) + \int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta\Big) \\ &\quad -\kappa_3\Big(\Psi_1(w) + \int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta\Big)\Big], \end{aligned}$$

$$\begin{aligned}
b_5 &= \frac{1}{\Delta}\Big[-\kappa_1^{'}\gamma_1\Big(\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\hat{h}_2(\theta)d\theta\Big)\\
&\quad -\kappa_2^{'}\gamma_2\Big(\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\hat{h}_1(\theta)d\theta\Big)\\
&\quad -\kappa_3^{'}\Big(\Psi_1(w)+\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta\Big)\Big]\\
&\quad +\Big[\frac{\kappa_4^{'}}{\Delta}-1\Big]\Big[\Psi_2(v)+\int_0^{1}\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta\Big],\\
b_6 &= \frac{1}{\Delta}\Big[\kappa_1^{'}\gamma_1\Big(\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\hat{h}_2(\theta)d\theta\Big)\\
&\quad +\kappa_2^{'}\gamma_2\Big(\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\hat{h}_1(\theta)d\theta\Big)\\
&\quad +\kappa_3^{'}\Big(\Psi_1(w)+\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}\hat{h}_1(\theta)d\theta\Big)\\
&\quad -\kappa_4^{'}\Big(\Psi_2(v)+\int_0^{1}\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}\hat{h}_2(\theta)d\theta\Big)\Big],
\end{aligned}$$

where $\Phi_i(z)$ $(i=1,2,...,8)$, $\kappa_i,\ \kappa_i^{'},\ (i=1,2,3,4)$, $\Delta_i,\ \Delta_i^{'},\ (i=1,2)$, Δ are given by (7)-(11) respectively. Substituting the values of $b_i,\ (i=1,2,...,6)$ in (12) and (13), we get the solutions (5) and (6). This completes the proof. □

3. Existence and Uniqueness Results

In this section, we shall obtain the existence and uniqueness results of the boundary value problem (1) and (2).

We define spaces $\mathfrak{V}=\{v : v\in C(\mathfrak{J},\mathbb{R})\}$ and $\mathfrak{W}=\{w : w\in C(\mathfrak{J},\mathbb{R})\}$ equipped respectively with norms $\|v\|_{\mathfrak{V}}=\|v\|=\sup_{z\in[0,1]}|v(z)|$ and $\|w\|_{\mathfrak{W}}=\|w\|=\sup_{z\in[0,1]}|w(z)|$. Obviously, $(\mathfrak{V},\|\cdot\|_{\mathfrak{V}})$ and $(\mathfrak{W},\|\cdot\|_{\mathfrak{W}})$ are Banach space and consequently, the product space $(\mathfrak{V}\times\mathfrak{W},\|\cdot\|_{\mathfrak{V}\times\mathfrak{W}})$ is a Banach space with norm $\|(v,w)\|_{\mathfrak{V}\times\mathfrak{W}}=\|v\|_{\mathfrak{V}}+\|w\|_{\mathfrak{W}}$ for $(v,w)\in\mathfrak{V}\times\mathfrak{W}$. In view of Lemma 2.2, we define an operator $\mathfrak{G}:\mathfrak{V}\times\mathfrak{W}\to\mathfrak{V}\times\mathfrak{W}$ as

$$\mathfrak{G}(v,w)(z) = (\mathfrak{G}_1(v,w)(z),\mathfrak{G}_2(v,w)(z)), \tag{18}$$

where

$$\begin{aligned}
\mathfrak{G}_1(v,w)(z) &= \int_0^{z}\frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}h_1(\theta,v(\theta),w(\theta))d\theta\\
&\quad -\Phi_4(z)\Psi_2(v)+[\Phi_3(z)+1]\Psi_1(w)\\
&\quad +\Phi_1(z)\Big[\gamma_1\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}h_2(\theta,v(\theta),w(\theta))d\theta\Big]\\
&\quad +\Phi_2(z)\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}h_1(\theta,v(\theta),w(\theta))d\theta\Big]\\
&\quad +\Phi_3(z)\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}h_1(\theta,v(\theta),w(\theta))d\theta\Big]
\end{aligned}$$

$$-\Phi_4(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} h_2(\theta, v(\theta), w(\theta))d\theta\Big] \tag{19}$$

and

$$\begin{aligned}\mathfrak{G}_2(v,w)(z) &= \int_0^z \frac{(z-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} h_2(\theta, v(\theta), w(\theta))d\theta \\ &\quad -\Phi_7(z)\Psi_1(w) + [\Phi_8(z)+1]\Psi_2(v) \\ &\quad -\Phi_5(z)\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)} h_2(\theta, v(\theta), w(\theta))d\theta\Big] \\ &\quad -\Phi_6(z)\Big[\gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)} h_1(\theta, v(\theta), w(\theta))d\theta\Big] \\ &\quad -\Phi_7(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)} h_1(\theta, v(\theta), w(\theta))d\theta\Big] \\ &\quad +\Phi_8(z)\Big[\int_0^1 \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} h_2(\theta, v(\theta), w(\theta))d\theta\Big]. \end{aligned} \tag{20}$$

In the forthcoming analysis, we need the following assumptions:

$(\mathfrak{A}_1)$ The continuous functions h_1, h_2 are defined from $\mathfrak{J}\times\mathbb{R}^2$ to $\mathbb{R}$ and there exist constants ν_i and $\sigma_i \geq 0,\ i=1,2,\ \nu_0>0, \sigma_0>0$ such that $\forall\ p_i\in\mathbb{R},\ i=1,2,$ we have

$$|h_1(z,p_1,p_2)| \leq \nu_0+\nu_1|p_1|+\nu_2|p_2|,$$
$$|h_2(z,p_1,p_2)| \leq \sigma_0+\sigma_1|p_1|+\sigma_2|p_2|.$$

$(\mathfrak{A}_2)$ The continuous functions Ψ_1, Ψ_2 are defined from $C(\mathfrak{J},\mathbb{R})$ to $\mathbb{R}$ with $\Psi_1(0) = \Psi_2(0) = 0$ and there exist constants $\rho_i,\ i=1,2,$ we have

$$|\Psi_1(w)| \leq \rho_1\|w\|, \quad |\Psi_2(v)| \leq \rho_2\|v\|.$$

$(\mathfrak{A}_3)$ The continuous functions h_1, h_2 are defined from $\mathfrak{J}\times\mathbb{R}^2$ to $\mathbb{R}$ and there exist positive constants $\mathfrak{S}_i, \widehat{\mathfrak{S}}_i\ i=1,2$ such that for all $z\in\mathfrak{J}$ and $p_i, q_i\in\mathbb{R}(i=1,2)$, we have

$$|h_1(z,p_1,p_2)-h_1(z,q_1,q_2)| \leq \mathfrak{S}_1|p_1-q_1|+\mathfrak{S}_2|p_2-q_2|,$$
$$|h_2(z,p_1,p_2)-h_2(z,q_1,q_2)| \leq \widehat{\mathfrak{S}}_1|p_1-q_1|+\widehat{\mathfrak{S}}_2|p_2-q_2|.$$

$(\mathfrak{A}_4)$ The continuous functions Ψ_1, Ψ_2 are defined from $C(\mathfrak{J},\mathbb{R})$ to $\mathbb{R}$ with $\Psi_1(0) = \Psi_2(0) = 0$ and there exist constants $\upsilon_i,\ i=1,2,$ such that $\forall\ p_i\in C(\mathfrak{J},\mathbb{R}),\ i=1,2,$ we have

$$|\Psi_1(p_1)-\Psi_1(p_2)| \leq \upsilon_1|p_1-p_2|, \quad |\Psi_2(p_1)-\Psi_2(p_2)| \leq \upsilon_2|p_1-p_2|.$$

To avoid computational complexity,

$$\begin{aligned}\mathfrak{U}_1 &= (\mathfrak{P}_1+\mathfrak{Q}_1)\nu_1+(\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_1+\widehat{\Phi}_4\rho_2+[1+\widehat{\Phi}_8]\rho_2,\\ \mathfrak{U}_2 &= (\mathfrak{P}_1+\mathfrak{Q}_1)\nu_2+(\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_2+[1+\widehat{\Phi}_3]\rho_1+\widehat{\Phi}_7\rho_1,\end{aligned}$$

and

$$\widehat{\mathfrak{U}} = min\{1-\mathfrak{U}_1, 1-\mathfrak{U}_2\}, \tag{21}$$

where

$$\mathfrak{P}_1 = \frac{[1+\widehat{\Phi}_3]}{\xi_1\Gamma(\varsigma_1)}+\frac{\widehat{\Phi}_2\gamma_2\delta_2^{\xi_1-1}}{(\xi_1-1)\Gamma(\varsigma_1-1)},\ \mathfrak{P}_2 = \frac{\widehat{\Phi}_4}{\xi_2\Gamma(\varsigma_2)}+\frac{\widehat{\Phi}_1\gamma_1\delta_1^{\xi_2-1}}{(\xi_2-1)\Gamma(\varsigma_2-1)},$$

$$\mathfrak{Q}_1 = \frac{\widehat{\Phi}_7}{\xi_1\Gamma(\varsigma_1)} + \frac{\widehat{\Phi}_6\gamma_2\delta_2^{\xi_1-1}}{(\xi_1-1)\Gamma(\varsigma_1-1)}, \quad \mathfrak{Q}_2 = \frac{[1+\widehat{\Phi}_8]}{\xi_2\Gamma(\varsigma_2)} + \frac{\widehat{\Phi}_5\gamma_1\delta_1^{\xi_2-1}}{(\xi_2-1)\Gamma(\varsigma_2-1)}.$$

Lemma 3.1. *Let $\mathfrak{G} : \mathfrak{F} \to \mathfrak{F}$ be a completely continuous operator (i.e., a map that restricted to any bounded set in $\mathfrak{F}$ is compact). Let $\chi(\mathfrak{G}) = \{v \in \mathfrak{F} : v = \varepsilon\mathfrak{G}(v)$ for some $0 < \varepsilon < 1\}$. Then either the set $\chi(\mathfrak{F})$ is unbounded, or $\mathfrak{F}$ has at least one fixed point.*

Theorem 3.2. *Assume that $(\mathfrak{A}_1)$ and $(\mathfrak{A}_2)$ holds. Also let us understand that $\mathfrak{U}_1 < 1$, and $\mathfrak{U}_2 < 1$. Then there exists at least one solution for the boundary value problem (1)-(2) on $\mathfrak{J}$.*

Proof. First we show that the operator $\mathfrak{G} : \mathfrak{V}\times\mathfrak{W} \to \mathfrak{V}\times\mathfrak{W}$ is completely continuous. Note that $\mathfrak{G}$ is continuous, since the functions h_1 and h_2 are continuous.
Let $\Omega \subset \mathfrak{V}\times\mathfrak{W}$ be bounded. Then there exist positive constants $\mathfrak{E}_{h_1}$, $\mathfrak{E}_{h_2}$, $\mathfrak{E}_{\Psi_1}$ and $\mathfrak{E}_{\Psi_2}$ such that $|h_1(z,v(z),w(z))| \le \mathfrak{E}_{h_1}$, and $|h_2(z,v(z),w(z))| \le \mathfrak{E}_{h_2}$, $\forall\ (v,w) \in \Omega$, $|\Psi_1(w)| \le \mathfrak{E}_{\Psi_1}$ and $|\Psi_2(v)| \le \mathfrak{E}_{\Psi_2}$, $\forall\ (v,w) \in C(\mathfrak{J},\mathbb{R})$. So, for any $(v,w) \in \Omega$, we have

$$\begin{aligned}
|\mathfrak{G}_1(v,w)(z)| \le{}& \int_0^z \frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}|h_1(\theta,v(\theta),w(\theta))|d\theta \\
&+\Phi_4(z)|\Psi_2(v)| + [\Phi_3(z)+1]|\Psi_1(w)| \\
&+\Phi_1(z)\Big[\gamma_1\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|h_2(\theta,v(\theta),w(\theta))|d\theta\Big] \\
&+\Phi_2(z)\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|h_1(\theta,v(\theta),w(\theta))|d\theta\Big] \\
&+\Phi_3(z)\Big[\int_0^1\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}|h_1(\theta,v(\theta),w(\theta))|d\theta\Big] \\
&+\Phi_4(z)\Big[\int_0^1\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}|h_2(\theta,v(\theta),w(\theta))|d\theta\Big] \\
\le{}& \frac{\mathfrak{E}_{h_1}}{\Gamma(\varsigma_1)}\int_0^z (z-\theta)^{\xi_1-1}d\theta + \Phi_4(z)\mathfrak{E}_{\Psi_2} + [\Phi_3(z)+1]\mathfrak{E}_{\Psi_1} \\
&+\Phi_1(z)\frac{\mathfrak{E}_{h_2}}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big] \\
&+\Phi_2(z)\frac{\mathfrak{E}_{h_1}}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big] \\
&+\Phi_3(z)\frac{\mathfrak{E}_{h_1}}{\Gamma(\varsigma_1)}\Big[\int_0^1(1-\theta)^{\xi_1-1}d\theta\Big] \\
&+\Phi_4(z)\frac{\mathfrak{E}_{h_2}}{\Gamma(\varsigma_2)}\Big[\int_0^1(1-\theta)^{\xi_2-1}d\theta\Big] \\
\le{}& \mathfrak{E}_{h_1}\mathfrak{P}_1 + \mathfrak{E}_{h_2}\mathfrak{P}_2 + \widehat{\Phi}_4\mathfrak{E}_{\Psi_1} + [1+\widehat{\Phi}_3]\mathfrak{E}_{\Psi_2}.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
|\mathfrak{G}_2(v,w)(z)| \le{}& \frac{\mathfrak{E}_{h_2}}{\Gamma(\varsigma_2)}\int_0^z (z-\theta)^{\xi_2-1}d\theta \\
&+\Phi_7(z)\mathfrak{E}_{\Psi_1} + [\Phi_8(z)+1]\mathfrak{E}_{\Psi_2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Phi_5(z)\mathfrak{E}_{h_2}}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big]\\
&+\frac{\Phi_6(z)\mathfrak{E}_{h_1}}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big]\\
&+\frac{\Phi_7(z)\mathfrak{E}_{h_1}}{\Gamma(\varsigma_1)}\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big]\\
&+\frac{\Phi_8(z)\mathfrak{E}_{h_2}}{\Gamma(\varsigma_2)}\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big]\\
\leq\ & \mathfrak{E}_{h_2}\mathfrak{Q}_2+\mathfrak{E}_{h_1}\mathfrak{Q}_1+\widehat{\Phi}_7\mathfrak{E}_{\Psi_1}+[1+\widehat{\Phi}_8]\mathfrak{E}_{\Psi_2}.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathfrak{G}$ is uniformly bounded. Next, we show that $\mathfrak{G}$ is equicontinuous. Let $z_1, z_2 \in \mathfrak{J}$ with $z_1 < z_2$. Then we have

$$
\begin{aligned}
&|\mathfrak{G}_1(v,w)(z_2)-\mathfrak{G}_1(v,w)(z_1)|\\
\leq\ &\left|\int_0^{z_1}\frac{[(z_2-\theta)^{\varsigma_1-1}-(z_1-\theta)^{\varsigma_1-1}]}{\Gamma(\varsigma_1)}\times h_1(\theta,v(\theta),w(\theta))d\theta\right|\\
&+\left|\int_{z_1}^{z_2}\frac{(z_2-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}h_1(\theta,v(\theta),w(\theta))d\theta\right|\\
&+|\Phi_4(z_2-z_1)|\mathfrak{E}_{\Psi_2}+|\Phi_3(z_2-z_1)|\mathfrak{E}_{\Psi_1}\\
&+|\Phi_1(z_2-z_1)|\Big[\gamma_1\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|h_2(\theta,v(\theta),w(\theta))|d\theta\Big]\\
&+|\Phi_2(z_2-z_1)|\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|h_1(\theta,v(\theta),w(\theta))|d\theta\Big]\\
&+|\Phi_3(z_2-z_1)|\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}|h_1(\theta,v(\theta),w(\theta))|d\theta\Big]\\
&+|\Phi_4(z_2-z_1)|\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}|h_2(\theta,v(\theta))|d\theta\Big]\\
\leq\ &\Big[\frac{(z_2-z_1)+(z_2^2-z_1^2)}{\Delta}\Big]\times\Bigg\{\kappa_4\Big[\frac{\mathfrak{E}_{h_2}}{\xi_2\Gamma(\varsigma_2)}+\mathfrak{E}_{\Psi_2}\Big]\\
&+\kappa_3\Big[\frac{\mathfrak{E}_{h_1}}{\xi_1\Gamma(\varsigma_1)}+\mathfrak{E}_{\Psi_1}\Big]+\frac{\kappa_1\mathfrak{E}_{h_2}\gamma_1\delta_1^{\xi_2-1}}{(\xi_2-1)\Gamma(\varsigma_2-1)}\\
&+\frac{\kappa_2\mathfrak{E}_{h_1}\gamma_2\delta_2^{\xi_1-1}}{(\xi_1-1)\Gamma(\varsigma_1-1)}\Bigg\}+(z_2-z_1)\Big[\frac{\mathfrak{E}_{h_1}}{\xi_1\Gamma(\varsigma_1)}+\mathfrak{E}_{\Psi_1}\Big]\\
&+\frac{\mathfrak{E}_{h_1}}{\xi_1\Gamma(\varsigma_1)}[2(z_2-z_1)^{\xi_1}+(z_2^{\xi_1}-z_1^{\xi_1})].
\end{aligned}
$$

Analogously, we obtain

$$
\begin{aligned}
&|\mathfrak{G}_2(v,w)(z_2)-\mathfrak{G}_2(v,w)(z_1)|\\
\leq\ &\left|\int_0^{z_1}\frac{[(z_2-\theta)^{\varsigma_2-1}-(z_1-\theta)^{\varsigma_2-1}]}{\Gamma(\varsigma_2)}\times h_2(\theta,v(\theta),w(\theta))d\theta\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_{z_1}^{z_2} \frac{(z_2-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} h_2(\theta, v(\theta), w(\theta))d\theta\right| \\
&+|\Phi_7(z_2-z_1)|\mathfrak{E}_{\Psi_1} + |\Phi_8(z_2-z_1)|\mathfrak{E}_{\Psi_2} \\
&+|\Phi_5(z_2-z_1)|\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)} |h_2(\theta, v(\theta), w(\theta))|d\theta\Big] \\
&+|\Phi_6(z_2-z_1)|\Big[\gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)} |h_1(\theta, v(\theta), w(\theta))|d\theta\Big] \\
&+|\Phi_7(z_2-z_1)|\Big[\int_0^{1} \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)} |h_1(\theta, v(\theta), w(\theta))|d\theta\Big] \\
&+|\Phi_8(z_2-z_1)|\Big[\int_0^{1} \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)} |h_2(\theta, v(\theta), w(\theta))|d\theta\Big] \\
\leq\ & \Big[\frac{(z_2-z_1)+(z_2^2-z_1^2)}{\Delta}\Big] \times \Bigg\{\kappa_4^{'}\Big[\frac{\mathfrak{E}_{h_2}}{\xi_2\Gamma(\varsigma_2)} + \mathfrak{E}_{\Psi_2}\Big] \\
&+\kappa_3^{'}\Big[\frac{\mathfrak{E}_{h_1}}{\xi_1\Gamma(\varsigma_1)} + \mathfrak{E}_{\Psi_1}\Big] + \frac{\kappa_1^{'}\mathfrak{E}_{h_2}\gamma_1\delta_1^{\xi_2-1}}{(\xi_2-1)\Gamma(\varsigma_2-1)} \\
&+\frac{\kappa_2^{'}\mathfrak{E}_{h_1}\gamma_2\delta_2^{\xi_1-1}}{(\xi_1-1)\Gamma(\varsigma_1-1)}\Bigg\} + (z_2-z_1)\Big[\frac{\mathfrak{E}_{h_2}}{\xi_2\Gamma(\varsigma_2)} + \mathfrak{E}_{\Psi_2}\Big] \\
&+\frac{\mathfrak{E}_{h_2}}{\xi_2\Gamma(\varsigma_2)}[2(z_2-z_1)^{\xi_2} + (z_2^{\xi_2} - z_1^{\xi_2})].
\end{aligned}
$$

Therefore, the operator $\mathfrak{G}(v,w)$ is equicontinuous, and thus the operator $\mathfrak{G}(v,w)$ is completely continuous, by Arzela-Ascoli theorem. Finally, it will be verified that the set $\Lambda = \{(v,w) \in \mathfrak{V} \times \mathfrak{W} | (v,w) = \tau\mathfrak{G}(v,w), 0 < \tau < 1\}$ is bounded. Let $(v,w) \in \Lambda$, then $(v,w) = \tau\mathfrak{G}(v,w)$. For any $z \in \mathfrak{J}$, we have $v(z) = \tau\mathfrak{G}_1(v,w)(z)$, $w(z) = \tau\mathfrak{G}_2(v,w)(z)$. Then

$$
\begin{aligned}
|v(z)| &= |\tau\mathfrak{G}_1(v,w)(z)| \\
&\leq \frac{(\nu_0+\nu_1\|v\|+\nu_2\|w\|)}{\Gamma(\varsigma_1)} \int_0^z (z-\theta)^{\xi_1-1}d\theta \\
&\quad +\Phi_4(z)\rho_2\|v\| + [\Phi_3(z)+1]\rho_1\|w\| \\
&\quad +\frac{\Phi_1(z)(\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big] \\
&\quad +\frac{\Phi_2(z)(\nu_0+\nu_1\|v\|+\nu_2\|w\|)}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big] \\
&\quad +\frac{\Phi_3(z)(\nu_0+\nu_1\|v\|+\nu_2\|w\|)}{\Gamma(\varsigma_1)}\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big] \\
&\quad +\frac{\Phi_4(z)(\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)}{\Gamma(\varsigma_2)}\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big] \\
&\leq (\nu_0+\nu_1\|v\|+\nu_2\|w\|)\mathfrak{P}_1 + (\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)\mathfrak{P}_2 \\
&\quad +\widehat{\Phi}_4\rho_2\|v\| + [1+\widehat{\Phi}_3]\rho_1\|w\|
\end{aligned}
$$

and

$$\begin{aligned}
|w(z)| &= |\tau\mathfrak{G}_2(v,w)(z)| \\
&\leq \frac{(\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)}{\Gamma(\varsigma_2)}\int_0^z (z-\theta)^{\xi_2-1}d\theta \\
&\quad +\Phi_7(z)\rho_1\|w\| + [\Phi_8(z)+1]\rho_2\|v\| \\
&\quad +\frac{\Phi_5(z)(\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big] \\
&\quad +\frac{\Phi_6(z)(\nu_0+\nu_1\|v\|+\nu_2\|w\|)}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big] \\
&\quad +\frac{\Phi_7(z)(\nu_0+\nu_1\|v\|+\nu_2\|w\|)}{\Gamma(\varsigma_1)}\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big] \\
&\quad +\frac{\Phi_8(z)(\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)}{\Gamma(\varsigma_2)}\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big] \\
&\leq (\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)\mathfrak{Q}_2 + (\nu_0+\nu_1\|v\|+\nu_2\|w\|)\mathfrak{Q}_1 \\
&\quad +\widehat{\Phi}_7\rho_1\|w\| + [1+\widehat{\Phi}_8]\rho_2\|v\|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|v\| &\leq (\nu_0+\nu_1\|v\|+\nu_2\|w\|)\mathfrak{P}_1 + (\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)\mathfrak{P}_2 \\
&\quad +\widehat{\Phi}_4\rho_2\|v\| + [1+\widehat{\Phi}_3]\rho_1\|w\|
\end{aligned}$$

and

$$\begin{aligned}
\|w\| &\leq (\sigma_0+\sigma_1\|v\|+\sigma_2\|w\|)\mathfrak{Q}_2 + (\nu_0+\nu_1\|v\|+\nu_2\|w\|)\mathfrak{Q}_1 \\
&\quad +\widehat{\Phi}_7\rho_1\|w\| + [1+\widehat{\Phi}_8]\rho_2\|v\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|v\|+\|w\| &\leq (\mathfrak{P}_1+\mathfrak{Q}_1)\nu_0 + (\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_0 \\
&\quad +\Big((\mathfrak{P}_1+\mathfrak{Q}_1)\nu_1 + (\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_1 + \widehat{\Phi}_4\rho_2 + [1+\widehat{\Phi}_8]\rho_2\Big)\|v\| \\
&\quad +\Big((\mathfrak{P}_1+\mathfrak{Q}_1)\nu_2 + (\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_2 + [1+\widehat{\Phi}_3]\rho_1 + \widehat{\Phi}_7\rho_1\Big)\|w\|.
\end{aligned}$$

Consequently,

$$\|(v,w)\| \leq \frac{(\mathfrak{P}_1+\mathfrak{Q}_1)\nu_0 + (\mathfrak{P}_2+\mathfrak{Q}_2)\sigma_0}{\widehat{\mathfrak{U}}},$$

for any $z \in \mathfrak{J}$, where $\widehat{\mathfrak{U}}$ is defined by (21), which proves that Λ is bounded. Thus, by Lemma 3.1, the operator $\mathfrak{G}$ has at least one fixed point. Hence the boundary value problem (1)-(2) has at least one solution on $\mathfrak{J}$. □

Example 3.3 Consider the coupled fractional differential equations of complex order given by

$$\begin{aligned}
&\mathfrak{D}^{\frac{8}{3}+i}v(z) = \frac{\tan^{-1} z}{4\sqrt{z+9}} + \frac{\sin v(z)}{150} + \frac{w(z)|v(z)|}{275(1+|v(z)|)}, \quad z\in\mathfrak{J}, \\
&\mathfrak{D}^{\frac{9}{4}+i}w(z) = \frac{\sin z}{3\sqrt{z^2+25}} + \frac{\cos v(z)}{180(1+w(z)))} + \frac{\tan^{-1} w(z)}{9\sqrt{z+625}}, \quad z\in\mathfrak{J},
\end{aligned} \tag{22}$$

subject to the boundary conditions

$$(23)\qquad \begin{aligned} v(0) &= \frac{1}{100}w(z), \quad v'(0)=\gamma_1 w'(\delta_1), \quad v(1)=0,\\ w(0) &= \frac{1}{111}v(z), \quad w'(0)=\gamma_2 v'(\delta_2), \quad w(1)=0. \end{aligned}$$

Clearly, h_1, h_2 are continuous functions. Now, we get

$$\begin{aligned} |h_1(z,v(z),w(z))| &\le \frac{1}{12}+\frac{1}{150}\|v\|+\frac{1}{275}\|w\|,\\ |h_2(z,v(z),w(z))| &\le \frac{1}{15}+\frac{1}{180}\|v\|+\frac{1}{225}\|w\|. \end{aligned}$$

Here, $\xi_1=\frac{8}{3}$, $\xi_2=\frac{9}{4}$, $\zeta_1,\zeta_2=1$, $\rho_1=\frac{1}{100}$, $\rho_2=\frac{1}{111}$, $\gamma_1=\frac{1}{3}$, $\gamma_2=\frac{1}{5}$, $\delta_1=\frac{1}{2}$, $\delta_2=\frac{3}{4}$, $\nu_0=\frac{1}{12}$, $\nu_1=\frac{1}{150}$, $\nu_2=\frac{1}{275}$, $\sigma_0=\frac{1}{15}$, $\sigma_1=\frac{1}{180}$, $\sigma_2=\frac{1}{225}$.
In view of relation (3), since $|\Gamma(\varsigma_1+i)|>1$ and $|\Gamma(\varsigma_2+i)|>1$. With the given data, we find that $\widehat{\mathfrak{U}} = \min\{1-\mathfrak{U}_1, 1-\mathfrak{U}_2\}<1$. Thus, the assumptions of Theorem 3.2 holds and the problem (22)-(23) has at least one solution on $\mathfrak{J}$.

Next, we shall roll the ball towards the uniqueness of solutions using the Banach fixed point theorem for the problem (1) and (2).

Theorem 3.4. *Assume that* $(\mathfrak{A}_3)$ *and* $(\mathfrak{A}_4)$ *holds. Also let us understand that*

$$\begin{aligned} \mathfrak{K}_1 &= (\mathfrak{S}_1+\mathfrak{S}_2)\mathfrak{P}_1+(\widehat{\mathfrak{S}}_1+\widehat{\mathfrak{S}}_2)\mathfrak{P}_2+\widehat{\Phi}_4\upsilon_2+[1+\widehat{\Phi}_3]\upsilon_1,\\ \mathfrak{K}_2 &= (\widehat{\mathfrak{S}}_1+\widehat{\mathfrak{S}}_2)\mathfrak{Q}_2+(\mathfrak{S}_1+\mathfrak{S}_2)\mathfrak{Q}_1+\widehat{\Phi}_7\upsilon_1+[1+\widehat{\Phi}_8]\upsilon_2, \end{aligned}$$

then the boundary value problem (1)-(2) has a unique solution on $\mathfrak{J}$.

Proof. Define $\sup_{z\in[0,1]}|h_1(z,0,0)|=\mathfrak{M}_1<\infty$ and $\sup_{z\in[0,1]}|h_2(z,0,0)|=\mathfrak{M}_2<\infty$ and choose a positive real number ϱ, such that

$$\varrho \ \ge\ \max\left\{\frac{\mathfrak{P}_1\mathfrak{M}_1+\mathfrak{P}_2\mathfrak{M}_2}{1-\mathfrak{K}_1}, \frac{\mathfrak{Q}_1\mathfrak{M}_1+\mathfrak{Q}_2\mathfrak{M}_2}{1-\mathfrak{K}_2}\right\}.$$

First, we show that $\mathfrak{G}\mathfrak{B}_\varrho\subset\mathfrak{B}_\varrho$, where $\mathfrak{B}_\varrho=\{(v,w)\in\mathfrak{V}\times\mathfrak{W}:\|(v,w)\|\le\varrho\}$. In view of the assumption $(\mathfrak{A}_3)$ and $(\mathfrak{A}_4)$ for $(v,w)\in\mathfrak{B}_\varrho$, $z\in\mathfrak{J}$, we have

$$\begin{aligned} |h_1(z,v(z),w(z))| &\le |h_1(z,v(z),w(z))-h_1(z,0,0)|+|h_1(z,0,0)|\\ &\le \mathfrak{S}_1|v(z)|+\mathfrak{S}_2|w(z)|+\mathfrak{M}_1\\ &\le \mathfrak{S}_1\|v\|+\mathfrak{S}_2\|w\|+\mathfrak{M}_1\le(\mathfrak{S}_1+\mathfrak{S}_2)\varrho+\mathfrak{M}_1, \end{aligned}$$

and

$$|\Psi_1(w)| \ \le\ \upsilon_1\|w\|\le\upsilon_1\varrho, \quad |\Psi_2(v)|\le\upsilon_2\|v\|\le\upsilon_2\varrho,$$

and

$$\begin{aligned} |h_2(z,v(z),w(z))| &\le |h_2(z,v(z),w(z))-h_2(z,0,0)|+|h_2(z,0,0)|\\ &\le \widehat{\mathfrak{S}}_1\|v\|+\widehat{\mathfrak{S}}_2\|w\|+\mathfrak{M}_2\le(\mathfrak{S}_1+\mathfrak{S}_2)\varrho+\mathfrak{M}_2. \end{aligned}$$

This guides to

$$\begin{aligned} |\mathfrak{G}_1(v,w)(z)| \ \le\ \sup_{z\in[0,1]}\Bigg\{ &\int_0^z \frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}(|h_1(\theta,v(\theta),w(\theta))\\ &-h_1(\theta,0,0)|+|h_1(\theta,0,0)|)d\theta \end{aligned}$$

$$
\begin{aligned}
&+\Phi_4(z)|\Psi_2(v)| + [\Phi_3(z)+1]|\Psi_1(w)| \\
&+\Phi_1(z)\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}(|h_2(\theta,v(\theta),w(\theta)) \\
&-h_2(\theta,0,0)| + |h_2(\theta,0,0)|)d\theta\Big] \\
&+\Phi_2(z)\Big[\gamma_2 \int_0^{\delta_2} \frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}(|h_1(\theta,v(\theta),w(\theta)) \\
&-h_1(\theta,0,0)| + |h_1(\theta,0,0)|)d\theta\Big] \\
&+\Phi_3(z)\Big[\int_0^{1} \frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}(|h_1(\theta,v(\theta),w(\theta)) \\
&-h_1(\theta,0,0)| + |h_1(\theta,0,0)|)d\theta\Big] \\
&+\Phi_4(z)\Big[\int_0^{1} \frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}(|h_2(\theta,v(\theta),w(\theta)) \\
&-h_2(\theta,0,0)| + |h_2(\theta,0,0)|)d\theta\Big]\Bigg\} \\
\leq\ & \frac{(\mathfrak{S}_1\|v\| + \mathfrak{S}_2\|w\| + \mathfrak{M}_1)}{\Gamma(\varsigma_1)}\int_0^z (z-\theta)^{\xi_1-1}d\theta \\
&+\Phi_4(z)\upsilon_2\|v\| + [\Phi_3(z)+1]\upsilon_1\|w\| \\
&+\frac{\Phi_1(z)(\widehat{\mathfrak{S}}_1\|v\| + \widehat{\mathfrak{S}}_2\|w\| + \mathfrak{M}_2)}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big] \\
&+\frac{\Phi_2(z)(\mathfrak{S}_1\|v\| + \mathfrak{S}_2\|w\| + \mathfrak{M}_1)}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\xi_1-2}}{d}\theta\Big] \\
&+\frac{\Phi_3(z)(\mathfrak{S}_1\|v\| + \mathfrak{S}_2\|w\| + \mathfrak{M}_1)}{\Gamma(\varsigma_1)}\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big] \\
&+\frac{\Phi_4(z)(\widehat{\mathfrak{S}}_1\|v\| + \widehat{\mathfrak{S}}_2\|w\| + \mathfrak{M}_2)}{\Gamma(\varsigma_2)}\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big] \\
\leq\ & (\mathfrak{S}_1\|v\| + \mathfrak{S}_2\|w\| + \mathfrak{M}_1)\mathfrak{P}_1 + (\widehat{\mathfrak{S}}_1\|v\| + \widehat{\mathfrak{S}}_2\|w\| + \mathfrak{M}_2)\mathfrak{P}_2 \\
&+\widehat{\Phi}_4\upsilon_2\|v\| + [1+\widehat{\Phi}_3]\upsilon_1\|w\| \\
\leq\ & \mathfrak{K}_1 r + \mathfrak{P}_1\mathfrak{M}_1 + \mathfrak{P}_2\mathfrak{M}_2 \leq \varrho.
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
|\mathfrak{G}_2(v,w)(z)| \leq\ & \sup_{z\in[0,1]}\Bigg\{\int_0^z \frac{(z-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}(|h_2(\theta,v(\theta),w(\theta)) \\
&-h_2(\theta,0,0)| + |h_2(\theta,0,0)|)d\theta \\
&+\Phi_7(z)|\Psi_1(w)| + [\Phi_8(z)+1]|\Psi_2(v)| \\
&+\Phi_5(z)\Big[\gamma_1 \int_0^{\delta_1} \frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}(|h_2(\theta,v(\theta),w(\theta)) \\
&-h_2(\theta,0,0)| + |h_2(\theta,0,0)|)d\theta\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi_6(z)\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}(|h_1(\theta,v(\theta),w(\theta))\\
&-h_1(\theta,0,0)|+|h_1(\theta,0,0)|)d\theta\Big]\\
&+\Phi_7(z)\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}(|h_1(\theta,v(\theta),w(\theta))\\
&-h_1(\theta,0,0)|+|h_1(\theta,0,0)|)d\theta\Big]\\
&+\Phi_8(z)\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}(|h_2(\theta,v(\theta),w(\theta))\\
&-h_2(\theta,0,0)|+|h_2(\theta,0,0)|)d\theta\Big]\Bigg\}\\
\leq\;& \frac{(\widehat{\mathfrak{S}}_1\|v\|+\widehat{\mathfrak{S}}_2\|w\|+\mathfrak{M}_2)}{\Gamma(\varsigma_2)}\int_0^z(z-\theta)^{\xi_2-1}d\theta\\
&+\Phi_7(z)\upsilon_1\|w\|+[\Phi_8(z)+1]\upsilon_2\|v\|\\
&+\frac{\Phi_5(z)(\widehat{\mathfrak{S}}_1\|v\|+\widehat{\mathfrak{S}}_2\|w\|+\mathfrak{M}_2)}{\Gamma(\varsigma_2-1)}\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big]\\
&+\frac{\Phi_6(z)(\mathfrak{S}_1\|v\|+\mathfrak{S}_2\|w\|+\mathfrak{M}_1)}{\Gamma(\varsigma_1-1)}\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big]\\
&+\frac{\Phi_7(z)(\mathfrak{S}_1\|v\|+\mathfrak{S}_2\|w\|+\mathfrak{M}_1)}{\Gamma(\varsigma_1)}\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big]\\
&+\frac{\Phi_8(z)(\widehat{\mathfrak{S}}_1\|v\|+\widehat{\mathfrak{S}}_2\|w\|+\mathfrak{M}_2)}{\Gamma(\varsigma_2)}\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big]\\
\leq\;& (\widehat{\mathfrak{S}}_1\|v\|+\widehat{\mathfrak{S}}_2\|w\|+\mathfrak{M}_2)\mathfrak{Q}_2+(\mathfrak{S}_1\|v\|+\mathfrak{S}_2\|w\|+\mathfrak{M}_1)\mathfrak{Q}_1\\
&+\widehat{\Phi}_7\upsilon_1\|w\|+[1+\widehat{\Phi}_8]\upsilon_2\|v\|\\
\leq\;& \mathfrak{K}_2r+\mathfrak{Q}_2\mathfrak{M}_2+\mathfrak{Q}_1\mathfrak{M}_1\leq\varrho.
\end{aligned}
$$

In view of $(\mathfrak{A}_3)$ and $(\mathfrak{A}_4)$, it follows that

$$
\begin{aligned}
|h_1(\theta,v_1,w_1)-h_1(\theta,v_2,w_2)| &\leq \mathfrak{S}_1\|v_1-v_2\|+\mathfrak{S}_2\|w_1-w_2\|\\
|\Psi_1(w_1)-\Psi_1(w_2)| &\leq \upsilon_1\|w_1-w_2\|,\\
|h_2(\theta,v_1,w_1)-h_2(\theta,v_2,w_2,w_2)| &\leq \widehat{\mathfrak{S}}_1\|v_1-v_2\|+\widehat{\mathfrak{S}}_2\|w_1-w_2\|\\
|\Psi_2(v_1)-\Psi_2(v_2)| &\leq \upsilon_2\|v_1-v_2\|.
\end{aligned}
$$

Now for $(v_2,w_2),(v_1,w_1)\in\mathfrak{V}\times\mathfrak{W}$, and for any $z\in\mathfrak{J}$, we get
$|\mathfrak{G}_1(v_2,w_2)(z)-\mathfrak{G}_1(v_1,w_1)(z)|$

$$
\begin{aligned}
\leq\;& \int_0^z\frac{(z-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}|h_1(\theta,v_2(\theta),w_2(\theta))-h_1(\theta,v_1(\theta),w_1(\theta))|d\theta\\
&+\Phi_3(z)|\Psi_1(v_2)-\Psi_1(v_1)|+[\Phi_4(z)+1]|\Psi_2(w_2)-\Psi_2(w_1)|\\
&+\Phi_1(z)\Big[\gamma_1\int_0^{\delta_1}\frac{(\delta_1-\theta)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|h_2(\theta,v_2(\theta),w_2(\theta))-h_2(\theta,v_1(\theta),w_1(\theta))|d\theta\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi_2(z)\Big[\gamma_2\int_0^{\delta_2}\frac{(\delta_2-\theta)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|h_1(\theta,v_2(\theta),w_2(\theta))-h_1(\theta,v_1(\theta),w_1(\theta))|d\theta\Big]\\
&+\Phi_3(z)\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_1-1}}{\Gamma(\varsigma_1)}|h_1(\theta,v_2(\theta),w_2(\theta))-h_1(\theta,v_1(\theta),w_1(\theta))|d\theta\Big]\\
&+\Phi_4(z)\Big[\int_0^{1}\frac{(1-\theta)^{\varsigma_2-1}}{\Gamma(\varsigma_2)}|h_2(\theta,v_2(\theta),w_2(\theta))-h_2(\theta,v_1(\theta),w_1(\theta))|d\theta\Big]\\
\le\ &\frac{(\mathfrak{S}_1\|v_2-v_1\|+\mathfrak{S}_2\|w_2-w_1\|)}{\Gamma(\varsigma_1)}\int_0^z(z-\theta)^{\xi_1-1}d\theta\\
&+\Phi_3(z)\upsilon_1\|v_2-v_1\|+[\Phi_4(z)+1]\upsilon_2\|w_2-w_1\|\\
&+\frac{\Phi_1(z)}{\Gamma(\varsigma_2-1)}(\widehat{\mathfrak{S}}_1\|v_2-v_1\|+\widehat{\mathfrak{S}}_2\|w_2-w_1\|)\Big[\gamma_1\int_0^{\delta_1}(\delta_1-\theta)^{\xi_2-2}d\theta\Big]\\
&+\frac{\Phi_2(z)}{\Gamma(\varsigma_1-1)}(\mathfrak{S}_1\|v_2-v_1\|+\mathfrak{S}_2\|w_2-w_1\|)\Big[\gamma_2\int_0^{\delta_2}(\delta_2-\theta)^{\xi_1-2}d\theta\Big]\\
&+\frac{\Phi_3(z)}{\Gamma(\varsigma_1)}(\mathfrak{S}_1\|v_2-v_1\|+\mathfrak{S}_2\|w_2-w_1\|)\Big[\int_0^{1}(1-\theta)^{\xi_1-1}d\theta\Big]\\
&+\frac{\Phi_4(z)}{\Gamma(\varsigma_2)}(\widehat{\mathfrak{S}}_1\|v_2-v_1\|+\widehat{\mathfrak{S}}_2\|w_2-w_1\|)\Big[\int_0^{1}(1-\theta)^{\xi_2-1}d\theta\Big]\\
\le\ &(\mathfrak{S}_1\|v_2-v_1\|+\mathfrak{S}_2\|w_2-w_1\|)\mathfrak{P}_1+(\widehat{\mathfrak{S}}_1\|v_2-v_1\|+\widehat{\mathfrak{S}}_2\|w_2-w_1\|)\mathfrak{P}_2\\
&+\Phi_3(z)\upsilon_1\|v_2-v_1\|+[\Phi_4(z)+1]\upsilon_2\|w_2-w_1\|\\
\le\ &\mathfrak{K}_1(\|v_2-v_1\|+\|w_2-w_1\|),
\end{aligned}
$$

and consequently we obtain

$$\|\mathfrak{G}_1(v_2,w_2)(z)-\mathfrak{G}_1(v_1,w_1)(z)\| \le \mathfrak{K}_1(\|v_2-v_1\|+\|w_2-w_1\|). \tag{24}$$

Similarly,

$$\|\mathfrak{G}_2(v_2,w_2)(z)-\mathfrak{G}_2(v_1,w_1)(z)\| \le \mathfrak{K}_2(\|v_2-v_1\|+\|w_2-w_1\|). \tag{25}$$

It follows from (24) and (25) that

$$\|\mathfrak{G}(v_2,w_2)(z)-\mathfrak{G}(v_1,w_1)(z)\| \le (\mathfrak{K}_1+\mathfrak{K}_2)(\|v_2-v_1\|+\|w_2-w_1\|).$$

Since $(\mathfrak{K}_1+\mathfrak{K}_2)<1$, therefore, $\mathfrak{G}$ is a contraction operator. So, by Banach fixed point theorem, the operator $\mathfrak{G}$ has a unique fixed point, which is the unique solution of the problem (1)-(2). □

Example 3.5 Consider the coupled fractional differential equations of complex order given by

$$
\begin{aligned}
&\mathfrak{D}^{\frac{5}{2}+i}v(z)=z+\frac{p(z)}{4\sqrt{z^2+4}}+\frac{\sin q(z)}{3\sqrt{16+z}},\quad z\in\mathfrak{J},\\
&\mathfrak{D}^{\frac{7}{3}+i}w(z)=\sin z+\frac{\cos p(z)}{\sqrt{z^2+81}}+\frac{\cos q(z)}{13},\quad z\in\mathfrak{J},
\end{aligned}
\tag{26}
$$

subject to the boundary conditions (23).
Using the given data, it is found that $\mathfrak{S}_1=\frac{1}{8}$, $\mathfrak{S}_2=\frac{1}{12}$, $\widehat{\mathfrak{S}}_1=\frac{1}{9}$, $\widehat{\mathfrak{S}}_2=\frac{1}{13}$, $\upsilon_1=\frac{1}{100}$, $\upsilon_2=\frac{1}{111}$.
It is clear that h_1,h_2 are continuous functions. Now, we get

$|h_1(z,v_1,v_2) - h_1(z,w_1,w_2)| \leq \frac{1}{8}|v_1 - w_1| + \frac{1}{12}|v_2 - w_2|,$
$|h_2(z,v_1,v_2) - h_2(z,w_1,w_2)| \leq \frac{1}{9}|v_1 - w_1| + \frac{1}{13}|v_2 - w_2|.$

With the given data, we get

$$\begin{aligned}
\mathfrak{K}_1 &= 0.0400791 + \frac{22}{117}\left[\frac{2\left(\frac{1}{2} - \frac{28}{41\sqrt{3}}\right)}{3\Gamma(\frac{3}{2}+i)} + \frac{56}{205\Gamma(\frac{5}{2}+i)}\right] \\
&\quad + \frac{5}{24}\left[\frac{4}{123\sqrt{3}\Gamma(\frac{3}{2}+i)} + \frac{64}{41\Gamma(\frac{5}{2}+i)} + \frac{141}{82\Gamma(\frac{8}{3}+i)}\right], \\
\mathfrak{K}_2 &= 0.0426895 + \frac{5}{24}\left[\frac{9}{205 \times 7^{\frac{1}{3}}\Gamma(\frac{4}{3}+i)} + \frac{72}{1435\Gamma(\frac{7}{3}+i)}\right] \\
&\quad + \frac{22}{117}\left[\frac{3\left(\frac{1}{5} - \frac{2}{41\times 7^{\frac{1}{3}}}\right)}{4\Gamma(\frac{4}{3}+i)} + \frac{2448}{1435\Gamma(\frac{7}{3}+i)}\right].
\end{aligned}$$

Thus, the assumption of Theorem 3.4 holds and hence the problem (26) with the boundary conditions (23) has a unique solution on $\mathfrak{J}$.

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