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EXTENDED FORMS OF LEGENDRE-GOULD-HOPPER-APPELL POLYNOMIALS

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Abstract. In this article, the extended class of Legendre-Gould-Hopper-Appell polynomials are introduced using integral transform and operational method which are fairly useful tools to deal with new families of special polynomials. Some important properties including operational representation, generating function, series definitions, determinant definition and some other identities of this class are established. Certain members of extended Legendre-Gould-Hopper-Appell polynomials are also considered.

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1. Introduction and preliminaries

Fractional calculus with integral transforms has emerged as an important interdisciplinary subject during the last four decades mainly due to its applications in various fields of science and engineering. The most interesting and useful applications of the combined use of integral transforms and fractional derivatives are establishing new generalizations of hybrid type polynomials [3, 9, 8].

One of the starting point of the theory of fractional operators is the operator raised to a fractional power which was given by Srivastava and Manocha [18] and the possibility of using integral transforms to deal with fractional derivative in a wider context is discussed by Dattoli *et al.* [9, 8], that is the identity (called Euler's integral)

$$\mathfrak{a}^{-\upsilon} = \frac{1}{\Gamma(\upsilon)} \int_0^\infty e^{-\mathfrak{a}\xi} \, \xi^{\upsilon-1} \, d\xi. \tag{1}$$

The fractional operators can be treated as an efficient way by combining the properties of exponential operators and suitable integral representations. The class of Appell polynomial sequences [1] is one of the interesting and important classes of special polynomials. The Appell polynomial sequences have been an active research area since they have many applications in analytic number theory, approximation theory, theoretical physics and several other mathematical branches. The recent applications of Appell polynomials in probability theory and statistics are considered in [2, 17]. The generalized Appell polynomials as tools for approximating 3D mappings were introduced for the first time in [15] in combination with Clifford analysis methods. The representation theoretic results like those of [4, 14] provide new examples of applications of Appell polynomials and gave evidence to the central role of Appell polynomials as orthogonal polynomials. Representation theory is also the tool for their applications in quantum physics as explained in [19]. The set of Appell sequences is an abelian group under the binomial convolution. The Appell polynomials are defined by means of the generating function

$$\mathcal{A}(t)\ exp(xt) = \sum_{m=0}^{\infty} \mathcal{A}_m(x)\ \frac{t^m}{m!}. \tag{2}$$

Here

$$\mathcal{A}(t) = \sum_{m=0}^{\infty} \mathcal{A}_m\ \frac{t^m}{m!},\ \mathcal{A}_0 \neq 0 \tag{3}$$

is an analytic function at $t = 0$ and $\mathcal{A}_m := \mathcal{A}_m(0)$ denotes the Appell numbers. It is directly seen that for any $\mathcal{A}(t)$, the derivative of $\mathcal{A}_m(x)$ satisfies

$$\mathcal{A}_m^{'}(x) = m\mathcal{A}_{m-1}(x). \tag{4}$$

The class of Appell sequences contains a large number of classical polynomials sequences such as Bernoulli, Euler, Genocchi, *etc.* Recently, certain new classes of hybrid special polynomials related to Appell sequences are introduced and studied [13, 10, 12, 11, 20]. These hybrid polynomials are important due to the fact that they posses important properties such as differential equation, generating function, series definition, determinant definition, *etc.*

Recently, Yasmin *et al.* have introduced the Legendre-Gould-Hopper-Appell polynomials (LeGHAP) ${}_{RH^{(s)}}A_m(x,y,z)$ which are defined by the following generating function [20]:

$$\mathcal{A}(t)exp(zt^s)\ C_0(xt)\ C_0(-yt) = \sum_{m=0}^{\infty} {}_{RH^{(s)}}\mathcal{A}_m(x,y,z)\frac{t^m}{m!}, \tag{5}$$

where $C_0(x)$ denotes the Tricomi function of order zero. The m^{th} order Tricomi functions $C_m(x)$ are defined by means of the generating function

$$exp\left(t - \frac{x}{t}\right) = \sum_{m=0}^{\infty} C_m(x)t^m, \tag{6}$$

for $t \neq 0$ and for all finite x and are defined by the following series [18]:

$$C_m(x) = \sum_{s=0}^{\infty} \frac{(-1)^s\ x^s}{s!\ (m+s)!},\ m = 0, 1, 2, ... \tag{7}$$

The series definition of the LeGHAP ${}_{RH^{(s)}}\mathcal{A}_m(x,y,z)$ is given as:

$${}_{RH^{(s)}}\mathcal{A}_m(x,y,z) = m!\ \sum_{k=0}^{m}\sum_{l=0}^{[\frac{m}{s}]} \frac{(-1)^k x^k z^l \mathcal{A}_{m-sl-k}(D_y^{-1})}{l!(k!)^2(m-sl-k)!}, \tag{8}$$

or, equivalently

$$(9) \qquad {}_{R}H^{(s)}\mathcal{A}_m(x,y,z) = m!\sum_{k=0}^{m}\sum_{l=0}^{[\frac{m}{s}]}\frac{(-1)^k x^k y^{m-sl-k}\mathcal{A}_l(z)}{l!(k!)^2((m-sl-k)!)^2}.$$

The LeGHAP ${}_{R}H^{(s)}\mathcal{A}_m(x,y,z)$ satisfy the following differential equation

$$(10) \quad \left(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}}+D_y^{-1}\frac{\partial}{\partial D_y^{-1}}+sz\frac{\partial^s}{\partial y^{s-1}\partial D_y^{-1}}+\frac{A'(D_y y D_y)}{A(D_y y D_y)}\frac{\partial}{\partial D_y^{-1}}-n\right){}_{R}H^{(s)}A_n(x,y,z)=0,$$

where $D_x := \frac{\partial}{\partial x}$, $\frac{\partial}{\partial D_x^{-1}} = \frac{\partial}{\partial x}x\frac{\partial}{\partial x}$, $D_x^{-n}\{1\} = \frac{x^n}{n!}$.

Also, the LeGHAP ${}_{R}H^{(s)}\mathcal{A}_m(x,y,z)$ are defined by means of the following operational representations:

$$(11) \qquad {}_{R}H^{(s)}\mathcal{A}_m(x,y,z) = exp\left(z\frac{\partial^s}{\partial D_y^{-s}}\right)\{{}_{R}\mathcal{A}_m(x,y)\},$$

$$(12) \qquad {}_{R}H^{(s)}\mathcal{A}_m(x,y,z) = exp\left((-1)^s z\frac{\partial^s}{\partial D_x^{-s}}\right)\{{}_{R}\mathcal{A}_m(x,y)\},$$

where ${}_{R}\mathcal{A}_m(x,y)$ are the 2-variable Legendre-Appell polynomials (2VLeAP) [20], defined by means of the generating function

$$(13) \qquad \mathcal{A}(t)\ C_0(xt)\ C_0(-yt) = \sum_{m=0}^{\infty}{}_{R}\mathcal{A}_m(x,y)\frac{t^m}{m!}.$$

The Legendre-Gould-Hopper polynomials $\frac{{}_{R}H_m^{(s)}(x,y,z)}{m!}$ are defined by means of the following generating function [21]:

$$(14) \qquad exp(zt^s)\ C_0(xt)\ C_0(-yt) = \sum_{m=0}^{\infty}\frac{{}_{R}H_m^{(s)}(x,y,z)}{m!}\frac{t^m}{m!}$$

and operational representation of the form

$$(15) \qquad \frac{{}_{R}H_m^{(s)}(x,y,z)}{m!} = exp\left(z\frac{\partial^s}{\partial D_y^{-s}}\right)\left\{\frac{R_m(x,y)}{m!}\right\}.$$

The 2-variable Legendre polynomials $R_m(x,y)$ are defined by the following series [7]:

$$(16) \qquad R_m(x,y) = (m!)^2\sum_{r=0}^{m}\frac{(-1)^{m-r}y^r x^{m-r}}{(r!)^2((m-r)!)^2},$$

and specified by the generating function

$$(17) \qquad C_0(xt)\ C_0(-yt) = \sum_{m=0}^{\infty}\frac{R_m(x,y)}{m!}\frac{t^m}{m!}.$$

Further, we recall the following identities [16]

$$(18) \qquad (1-u)^{-\alpha} = \sum_{m=0}^{\infty}\frac{(\alpha)_m\ u^m}{m!},$$

$$(m-k)! = \frac{(-1)^k \, m!}{(-m)_k}, \ 0 \leq k \leq m, \tag{19}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} B(k,m) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} B(k,m-k). \tag{20}$$

Further, for some choices of the indices and variables, the LeGHAP ${}_{R}H^{(s)}A_m(x,y,z)$ reduce to certain special polynomials which are mentioned as special cases in Table 1.

TABLE 1. Special cases of the LeGHAP ${}_{R}H^{(s)}A_n(x,y,z)$

S. No.	**Values of the indices and variables**	**Relation between LeGHAP ${}_{R}H^{(s)}A_n(x,y,z)$ and its special case**	**Name of the special polynomials**
I.	$z=0$	${}_{R}H^{(s)}A_m(x,y,0) = {}_{R}A_m(x,y)$	2-Variable Legendre based Appell polynomials (2VLeAP)
II.	$s=n;\ y=0,\ z\to y$	${}_{R}H^{(n)}A_m(x,0,y) = {}_{[n]}LA_m(x,y)$	2-Variable generalized Laguerre type based Appell polynomials (2VGLTAP)
III.	$s=2;\ x=0,$ $y\to x,\ z\to y$	${}_{R}H^{(2)}A_m(0,x,y) = {}_{G}A_m(x,y)$	Hermite type based Appell polynomials (HTAP)
IV.	$s=1;\ x\to(\frac{1-x}{2}),$ $y\to(\frac{1+x}{2}),\ z=0$	${}_{R}H^{(1)}A_m(\frac{1-x}{2},\frac{1+x}{2},0) = {}_{P}A_m(x)$	Legendre based Appell polynomials (LeAP)

In the present article, the extended forms of Legendre-Gould-Hopper-Appell polynomials are introduced and discussed with the help of fractional operators. In Section 2, the extended Legendre-Gould-Hopper-Appell polynomials are introduced by means of the operational representation and generating function using Euler's integral and operational rules. In Section 3, the determinant definition of extended Legendre-Gould-Hopper-Appell polynomials is established. In Section 4, certain members of these polynomials are also considered as special cases.

2. EXTENDED LEGENDRE-GOULD-HOPPER-APPELL POLYNOMIALS

In this section, we first establish an operational representation between the extended Legendre-Gould-Hopper-Appell polynomials and Legendre-Appell polynomials by proving the following results:

Theorem 2.1. *The following operational representations between the ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z)$ and 2VLeAP ${}_{R}\mathcal{A}_m(x,y)$ hold true:*

$$\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} {}_{R}\mathcal{A}_m(x,y) = {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha), \tag{21}$$

$$\left[\alpha - z(-1)^s\frac{\partial^s}{\partial D_x^{-s}}\right]^{-\upsilon} {}_{R}\mathcal{A}_m(x,y) = {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha). \tag{22}$$

Proof. On multiplying both sides of equation (1) by $g(x,y)$ and then replacing $\mathfrak{a}$ by $\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]$ in the resultant equation, we get

$$\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} g(x,y) = \frac{1}{\Gamma(\upsilon)}\int_0^{\infty} e^{-\alpha\xi}\ \xi^{\upsilon-1} exp\left(z\xi\frac{\partial^s}{\partial D_y^{-s}}\right) g(x,y)\ d\xi. \tag{23}$$

Further, taking $g(x,y) = {}_R\mathcal{A}_m(x,y)$ in the above equation, we get

(24)
$$\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} {}_R\mathcal{A}_m(x,y) = \frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\ \xi^{\upsilon-1} exp\left(z\xi\frac{\partial^s}{\partial D_y^{-s}}\right){}_R\mathcal{A}_m(x,y)\ d\xi,$$

which on using operational relation (11), becomes

$$\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} {}_R\mathcal{A}_m(x,y) = \frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\ \xi^{\upsilon-1}\ {}_{RH^{(s)}}\mathcal{A}_m(x,y,z\xi)\ d\xi. \tag{25}$$

Now, we note that the integral transform in the r.h.s. of equation (25) generate a new special polynomials. Denoting these polynomials by ${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$, we get

$${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha) = \frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\ \xi^{\upsilon-1}\ {}_{RH^{(s)}}\mathcal{A}_m(x,y,z\xi)\ d\xi. \tag{26}$$

Finally, in view of equations (25) and (26), we get assertion (21). Using a similar argument as in the proof of assertion (21) by making use of operational identity (12), we get assertion (22). □

Theorem 2.2. *For the ELeGHAP* ${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$*, the following series formula holds true:*

$${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha) = \frac{m!}{\alpha^\upsilon}\sum_{k=0}^{m}\sum_{l=0}^{[\frac{m}{s}]}\frac{(\upsilon)_k(-1)^k x^k z^l \mathcal{A}_{m-sl-k}(D_y^{-1})}{\alpha^k l!(k!)^2(m-sl-k)!}. \tag{27}$$

Proof. In view of series definition (8), equation (26) becomes

(28)
$${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha) = \frac{m!}{\Gamma(\upsilon)}\sum_{k=0}^{m}\sum_{l=0}^{[\frac{m}{s}]}\frac{(-1)^k x^k z^l \mathcal{A}_{m-sl-k}(D_y^{-1})}{l!(k!)^2(m-sl-k)!}\int_0^\infty e^{-\alpha\xi}\ \xi^{\upsilon+k-1}\ d\xi.$$

Now, using equation (1) in the r.h.s. of equation (28), yields to assertion (27). □

Theorem 2.3. *The following generating function for the ELeGHAP* ${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$*, holds true:*

$$\frac{A(\tau)\ C_0(x\tau)\ C_0(-y\tau)}{(\alpha - z\tau^s)^\upsilon} = \sum_{m=0}^{\infty} {}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!}. \tag{29}$$

Proof. On multiplying both sides of equation (26) by $\frac{\tau^m}{m!}$ and then summing up over m, we get

(30)
$$\sum_{m=0}^{\infty} {}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!} = \frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\ \xi^{\upsilon-1}\left(\sum_{m=0}^{\infty} {}_{RH^{(s)}}\mathcal{A}_m(x,y,z\xi)\frac{\tau^m}{m!}\right)d\xi,$$

which in view of equations (1) and (5), becomes

(31)
$$\sum_{m=0}^{\infty} {}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!} = A(\tau)\ C_0(x\tau)\ C_0(-y\tau)\frac{1}{\alpha^\upsilon}\sum_{k=0}^{\infty}\frac{(\upsilon)_k(z\xi^s)^k}{k!}.$$

Finally, using identity (18) in the r.h.s. of equation (31), yields to assertion (29). □

Theorem 2.4. *The following explicit summation formula for the ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$, holds true:*

(32)
$$ {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=\frac{1}{\alpha^{\upsilon}}\sum_{r=0}^{\infty}\sum_{n,k=0}^{m}\frac{(-1)^{n}(-m)_{n+k}\ (\upsilon)_{r}\ x^{k}\ y^{n}\ (z\xi^{s})^{r}\mathcal{A}_{m-n-k}}{r!(k!)^{2}(n!)^{2}}. $$

Proof. In view of equations (3), (7) and (31), we get

(33)
$$ \sum_{m=0}^{\infty}{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\frac{1}{\alpha^{\upsilon}}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{r}(\upsilon)_{k}x^{r}y^{n}(z\xi^{s})^{k}\mathcal{A}_{m}\ \tau^{m+n+r}}{m!\ k!(n!)^{2}(r!)^{2}}. $$

Next, replacing m by $m-r$ in r.h.s. of equation (33) and then using relation (20) in the resultant equation, we find

(34)
$$ \sum_{m=0}^{\infty}{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\frac{1}{\alpha^{\upsilon}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m}\frac{(-1)^{r}(\upsilon)_{k}x^{r}y^{n}(z\xi^{s})^{k}\mathcal{A}_{m-r}\ \tau^{m+n}}{(m-r)!\ k!(n!)^{2}(r!)^{2}}. $$

Again, replacing m by $m-n$ in the r.h.s. of equation (34), then using relation (20) in the resultant equation, we get

(35)
$$ \sum_{m=0}^{\infty}{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\frac{1}{\alpha^{\upsilon}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{m}\sum_{r=0}^{m}\frac{(-1)^{r}(\upsilon)_{k}x^{r}y^{n}(z\xi^{s})^{k}\mathcal{A}_{m-n-r}\ \tau^{m}}{(m-n-r)!\ k!(n!)^{2}(r!)^{2}}. $$

Finally, using identity (19) in the r.h.s. of the above equation and then equating the coefficients of $\frac{\tau^{m}}{m!}$ in the resultant equation, we arrive at assertion (32). □

Theorem 2.5. *The extended Legendre-Gould-Hopper-Appell polynomials ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ are defined by the series:*

(36)
$$ {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=\frac{1}{\alpha^{\upsilon}}\sum_{n=0}^{\infty}\sum_{k=0}^{m}\frac{(-1)^{k}(-m)_{k}(\upsilon)_{n}(z\xi^{s})^{n}\mathcal{A}_{k}R_{m-k}(x,y)}{k!\ n!\ (m-k)!}. $$

Proof. The proof is direct use of equations (3) and (17) in the r.h.s. of equation (31) and then equating the coefficients of $\frac{\tau^{m}}{m!}$ in the resultant equation. □

Remark 2.6. Using a similar argument as in the proof of Theorem 2.1 and in view of equations (1) and (15) we can drive the following operational result for the extended Legendre-Gould-Hopper polynomials $\frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!}$:

(37)
$$ \left[\alpha-z\frac{\partial^{s}}{\partial D_{y}^{-s}}\right]^{-\upsilon}\frac{R_{m}(x,y)}{m!}=\frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!}. $$

Further, using a similar argument as in the proof of Theorem 2.3 and using equations (1), (14) and (18) we can drive the following generating function for the extended Legendre-Gould-Hopper polynomials $\frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!}$:

(38)
$$ \frac{C_{0}(x\tau)\ C_{0}(-y\tau)}{(\alpha-z\tau^{s})^{\upsilon}}=\sum_{m=0}^{\infty}\frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!}\frac{\tau^{m}}{m!}. $$

Theorem 2.7. *For the ELeGHAP* ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$*, the following explicit summation formula holds true:*

$$(39)\qquad {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=\sum_{n=0}^{m}\frac{(-1)^{n}(-m)_{n}\ {}_{R}H^{(s)}_{n,\upsilon}(x,y,z;\alpha)\mathcal{A}_{m-n}}{(n!)^{2}}.$$

Proof. In view of equations (3), (29) and (38), we have

$$(40)\qquad \sum_{m=0}^{\infty}{}_{R}H^{(s)}A_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{{}_{R}H^{(s)}_{n,\upsilon}(x,y,z;\alpha)\mathcal{A}_{m}\ \tau^{m+n}}{(n!)^{2}\ m!}.$$

Now, replacing m by $m-n$ in the r.h.s. of the above equation and then using identities (19) and (20) in the resultant equation, we get

$$(41)\qquad \sum_{m=0}^{\infty}{}_{R}H^{(s)}A_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{n}(-m)_{n}\ {}_{R}H^{(s)}_{n,\upsilon}(x,y,z;\alpha)\mathcal{A}_{m-n}}{(n!)^{2}}\frac{\tau^{m}}{m!}.$$

Finally, equating the coefficients of $\frac{\tau^{m}}{m!}$ in the above equation, we arrive at assertion (39). □

Theorem 2.8. *For the ELeGHAP* ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$*, the following explicit summation formula holds true:*

$$(42)\qquad {}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=\sum_{n=0}^{m}\sum_{k=0}^{m-n}\frac{(-1)^{n+k}(-m)_{n+k}\ u^{n}\mathcal{A}_{k}(u){}_{R}H^{(s)}_{m-n-k,\upsilon}(x,y,z;\alpha)}{k!n!(m-n-k)!}.$$

Proof. In view of equations (2) and (38), we have

$$(43)\qquad \frac{A(\tau)e^{u\tau}\ C_{0}(x\tau)\ C_{0}(-y\tau)}{(\alpha-z\tau^{s})^{\upsilon}}=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{A}_{k}(u)\frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)\tau^{m+k}}{k!(m!)^{2}}.$$

Now, shifting the exponential to the r.h.s. and then in the resultant equation replacing m by $m-k$ in the r.h.s. and using equation (29) in the l.h.s., we get

$$(44)\qquad \sum_{m=0}^{\infty}{}_{R}H^{(s)}A_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^{m}}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m}\frac{u^{n}\mathcal{A}_{k}(u){}_{R}H^{(s)}_{m-k,\upsilon}(x,y,z;\alpha)\tau^{m+n}}{n!k!((m-k)!)^{2}}.$$

Finally, replacing m by $m-n$ in the r.h.s. of equation (44), then using identity (19) and equating the coefficients of $\frac{\tau^{m}}{m!}$ in the resultant equation, we arrive at assertion (42). □

Remark 2.9. The ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following differential recurrence relations*:*

$$\frac{\partial}{\partial D_x^{-1}}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=-m\,{}_{R}H^{(s)}\mathcal{A}_{m-1,\upsilon}(x,y,z;\alpha), \tag{45}$$

$$\frac{\partial}{\partial D_y^{-1}}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=m\,{}_{R}H^{(s)}\mathcal{A}_{m-1,\upsilon}(x,y,z;\alpha), \tag{46}$$

$$\begin{aligned}\frac{\partial}{\partial z}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)&\\ &=\upsilon m(m-1)(m-2)...(m-s+1)\,{}_{R}H^{(s)}\mathcal{A}_{m-s,\upsilon+1}(x,y,z;\alpha),\end{aligned} \tag{47}$$

$$\frac{\partial}{\partial \alpha}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=-\upsilon\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon+1}(x,y,z;\alpha). \tag{48}$$

Further, we have

$$\frac{\partial}{\partial z}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=(-1)^{s+1}\frac{\partial^{s+1}}{\partial\alpha\;\partial D_x^{-s}}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha), \tag{49}$$

$$\frac{\partial}{\partial z}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=-\frac{\partial^{s+1}}{\partial\alpha\;\partial D_y^{-s}}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha). \tag{50}$$

Theorem 2.10. *The ELeGHAP* ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ *satisfy the following differential equation:*

$$\begin{aligned}\Big(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}}+D_y^{-1}\frac{\partial}{\partial D_y^{-1}}-sz\frac{\partial^{s+1}}{\partial\alpha\partial y^{s-1}\partial D_y^{-1}}&\\ +\frac{A'(D_yyD_y)}{A(D_yyD_y)}\frac{\partial}{\partial D_y^{-1}}-m\Big)\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)=0.&\end{aligned} \tag{51}$$

Proof. On replacing z by zt in equation (10), multiplying by $\frac{1}{\Gamma(\upsilon)}e^{-\alpha\xi}\xi^{\upsilon-1}$ and integrating the resultant equation w.r.t. ξ with limits from 0 to ∞, we get

$$\begin{aligned}\Big(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}}+D_y^{-1}\frac{\partial}{\partial D_y^{-1}}+\frac{A'(D_yyD_y)}{A(D_yyD_y)}\frac{\partial}{\partial D_y^{-1}}-m\Big)\frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\xi^{\upsilon-1}\,{}_{R}H^{(s)}\mathcal{A}_m(x,y,z\xi)d\xi&\\ +sz\frac{\partial^s}{\partial y^{s-1}\partial D_y^{-1}}\frac{1}{\Gamma(\upsilon)}\int_0^\infty e^{-\alpha\xi}\;\xi^{\upsilon}\,{}_{R}H^{(s)}\mathcal{A}_m(x,y,z\xi)\;d\xi=0.&\end{aligned} \tag{52}$$

Further, in view of equation (26), equation (52) becomes

$$\begin{aligned}\Big(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}}+D_y^{-1}\frac{\partial}{\partial D_y^{-1}}+\frac{A'(D_yyD_y)}{A(D_yyD_y)}\frac{\partial}{\partial D_y^{-1}}-m\Big)\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)&\\ +sz\upsilon\frac{\partial^s}{\partial y^{s-1}\partial D_y^{-1}}\,{}_{R}H^{(s)}\mathcal{A}_{m,\upsilon+1}(x,y,z;\alpha)=0,&\end{aligned} \tag{53}$$

which on using equation (48) yields to assertion (51). □

In the next section, we established the determinant definition for the extended Legendre-Gould-Hopper-Appell polynomials ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$.

3. Determinant Approach

The determinant approach is equivalent to the corresponding approach based on operational methods. However, the simplicity of this approach allows non-specialists to use its applications and it is also suitable for computation. In 2010, Costabile and Longo introduced the determinant definition of the Appell polynomials [5, p. 1533]. In order to derive the determinant definition for the extended Legendre-Gould-Hopper-Appell polynomials ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$, we first establish the determinant definition for the 2-variable Legendre-Appell polynomials ${}_{R}\mathcal{A}_{m}(x,y)$:

Theorem 3.1. *The 2-variable Legendre-Appell polynomials* ${}_{R}\mathcal{A}_{m}(x,y)$ *of degree* m *are defined by*

$$ {}_{R}\mathcal{A}_{0}(x,y)=\frac{1}{\beta_0},\ \beta_0=\frac{1}{\mathcal{A}_0}, \tag{54}$$

$$ {}_{R}\mathcal{A}_{m}(x,y)=\frac{(-1)^m}{(\beta_0)^{m+1}}\begin{vmatrix} 1 & R_1(x,y) & \frac{R_2(x,y)}{2!} & \cdots & \frac{R_{m-1}(x,y)}{(m-1)!} & \frac{R_m(x,y)}{m!} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{m-1} & \beta_m \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{m-1}{1}\beta_{m-2} & \binom{m}{1}\beta_{m-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{m-1}{2}\beta_{m-3} & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{m}{m-1}\beta_1 \end{vmatrix}, \tag{55}$$

$$\beta_m=-\frac{1}{\mathcal{A}_0}\left(\sum_{k=1}^{m}\binom{m}{k}A_k\beta_{m-k}\right),\ m=1,2,3,...,$$

where $\beta_0,\beta_1,...,\beta_m\in\mathbb{R},\beta_0\neq 0$ and $\frac{R_m(x,y)}{m!}(m=0,1,2,...,)$ are the 2-variable Legendre polynomials defined by equation (16).

Proof. The proof is the direct use of the identities [20, 21]

$$ {}_{R}H^{(s)}A_m(x,y,0)={}_{R}A_m(x,y)\ \text{and}\ {}_{R}H^{(s)}_{m}(x,y,0)=R_m(x,y), \tag{56}$$

in both sides of the determinant definition of Legendre-Gould-Hopper-Appell polynomials which was introduced by the authors in [20]. □

Next, we introduce the determinant form of the extended Legendre-Gould-Hopper-Appell polynomials ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ by proving the following result:

Theorem 3.2. *The extended Legendre-Gould-Hopper-Appell polynomials* ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ *of degree* m *are defined by*

$$ {}_{R}H^{(s)}\mathcal{A}_{0,\upsilon}(x,y,z;\alpha)=\frac{1}{\beta_0}{}_{R}H^{(s)}_{0,\upsilon}(x,y,z;\alpha),\ \beta_0=\frac{1}{\mathcal{A}_0}, \tag{57}$$

(58) ${}_{R}H^{(s)}\mathcal{A}_{m,v}(x,y,z;\alpha)$

$$= \frac{(-1)^m}{(\beta_0)^{m+1}} \begin{vmatrix} 1 & {}_RH^{(s)}_{1,v}(x,y,z;\alpha) & \frac{{}_RH^{(s)}_{2,v}(x,y,z;\alpha)}{2!} & \dots & \frac{{}_RH^{(s)}_{m-1,v}(x,y,z;\alpha)}{(m-1)!} & \frac{{}_RH^{(s)}_{m,v}(x,y,z;\alpha)}{m!} \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{m-1}{1}\beta_{m-2} & \binom{m}{1}\beta_{m-1} \\ 0 & 0 & \beta_0 & \dots & \binom{m-1}{2}\beta_{m-3} & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{m}{m-1}\beta_1 \end{vmatrix},$$

$$\beta_m = -\frac{1}{\mathcal{A}_0}\left(\sum_{k=1}^{m}\binom{m}{k}\mathcal{A}_k\beta_{m-k}\right), \quad m = 1, 2, 3, ...,$$

where $\beta_0, \beta_1, ..., \beta_m \in \mathbb{R}, \beta_0 \neq 0$ and $\frac{{}_RH^{(s)}_{m,v}(x,y,z;\alpha)}{m!}(m = 0, 1, 2, ...,)$ are the extended Legendre-Gould-Hopper polynomials defined by equation (38).

Proof. Taking $m = 0$ in equation (39) and in view of equation (54), assertion (57) is proved.

Next, consider the determinant definition of the 2VLeAP given by equations (54) and (55). Expanding the determinant in the r.h.s. of equation (55) w.r.t. the first row, gives

$$(59) \qquad {}_R\mathcal{A}_m(x,y) = \frac{(-1)^m}{(\beta_0)^{m+1}} \begin{vmatrix} \beta_1 & \beta_2 & \dots & \beta_{m-1} & \beta_m \\ \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{m-1}{1}\beta_{m-2} & \binom{m}{1}\beta_{m-1} \\ 0 & \beta_0 & \dots & \binom{m-1}{2}\beta_{m-3} & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \beta_0 & \binom{m}{m-1}\beta_1 \end{vmatrix}$$

$$-\frac{(-1)^m R_1(x,y)}{(\beta_0)^{m+1}} \begin{vmatrix} \beta_0 & \beta_2 & \dots & \beta_{m-1} & \beta_m \\ 0 & \binom{2}{1}\beta_1 & \dots & \binom{m-1}{1}\beta_{m-2} & \binom{m}{1}\beta_{m-1} \\ 0 & \beta_0 & \dots & \binom{m-1}{2}\beta_{m-3} & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \beta_0 & \binom{m}{m-1}\beta_1 \end{vmatrix}$$

$$+\frac{(-1)^m R_2(x,y)}{(\beta_0)^{m+1}2!} \begin{vmatrix} \beta_0 & \beta_1 & \dots & \beta_{m-1} & \beta_m \\ 0 & \beta_0 & \dots & \binom{m-1}{1}\beta_{m-2} & \binom{m}{1}\beta_{m-1} \\ 0 & 0 & \dots & \binom{m-1}{2}\beta_{m-3} & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \beta_0 & \binom{m}{m-1}\beta_1 \end{vmatrix}$$

$$+ ... + \frac{(-1)^{2m+1} R_{m-1}(x,y)}{(\beta_0)^{m+1}(m-1)!}$$

$$\times \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & ... & \beta_m \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{m}{1}\beta_{m-1} \\ 0 & 0 & \beta_0 & ... & \binom{m}{2}\beta_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \binom{m}{m-1}\beta_1 \end{vmatrix} + \frac{R_m(x,y)}{(\beta_0)^{m+1} m!} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & ... & \beta_{m-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{m-1}{1}\beta_{m-2} \\ 0 & 0 & \beta_0 & ... & \binom{m-1}{2}\beta_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \beta_0 \end{vmatrix}.$$

Since each minor in equation (59) is independent of x, therefore on applying the operator $\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon}$ on both sides of equation (60), using equations (21) and (37) and combining the terms in the r.h.s. of the resultant equation, we arrive at assertion (58). □

4. Examples

In this section, we give some important examples as members of extended Legendre-Gould-Hopper-Appell family ${}_RH^{(s)}A_{m,\upsilon}(x,y,z;\alpha)$ such as extended Legendre-Gould-Hopper-Bernoulli polynomials (ELeGHBP) ${}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$, extended Legendre-Gould-Hopper-Euler polynomials (ELeGHEP) ${}_RH^{(s)}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ and extended Legendre-Gould-Hopper-Genocchi polynomials (ELeGHGP) ${}_RH^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$.

4.1. Extended Legendre-Gould-Hopper-Bernoulli polynomials.

Since, for $\mathcal{A}(t) = \frac{t}{e^t-1}$, the Appell polynomials (AP) $\mathcal{A}_m(x)$ reduce to the Bernoulli polynomials (BP) $\mathfrak{B}_m(x)$. Therefore, for the same choice of $\mathcal{A}(t)$ the ELeGHAP ${}_RH^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ reduce to ELeGHBP ${}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ which in view of the equation (29) are defined by means of following generating function:

(60) $$\frac{\tau\ C_0(x\tau)\ C_0(-y\tau)}{(e^\tau-1)(\alpha - z\tau^s)^\upsilon} = \sum_{m=0}^{\infty} {}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!}.$$

In view of equations (21) and (22), the operational representations for the ELeGHBP ${}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ are given as:

(61) $$\left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} {}_R\mathfrak{B}_m(x,y) = {}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha),$$

(62) $$\left[\alpha - z(-1)^s\frac{\partial^s}{\partial D_x^{-s}}\right]^{-\upsilon} {}_R\mathfrak{B}_m(x,y) = {}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha).$$

Further, in view of equation (32), the ELeGHBP ${}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ are defined by the series

(63) $${}_RH^{(s)}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = \frac{1}{\alpha^\upsilon}\sum_{r=0}^{\infty}\sum_{n,k=0}^{m}\frac{(-1)^n(-m)_{n+k}\ (\upsilon)_r\ x^k\ y^n\ (z\xi^s)^r\mathfrak{B}_{m-n-k}}{r!(k!)^2(n!)^2}.$$

The ELeGHBP ${}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following recurrence relations:

$$\frac{\partial}{\partial D_x^{-1}}\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = -m\,{}_{RH^{(s)}}\mathfrak{B}_{m-1,\upsilon}(x,y,z;\alpha), \tag{64}$$

$$\frac{\partial}{\partial D_y^{-1}}\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = m\,{}_{RH^{(s)}}\mathfrak{B}_{m-1,\upsilon}(x,y,z;\alpha), \tag{65}$$

$$\frac{\partial}{\partial z}\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = \upsilon m(m-1)(m-2)\dots(m-s+1)\,{}_{RH^{(s)}}\mathfrak{B}_{m-s,\upsilon+1}(x,y,z;\alpha), \tag{66}$$

$$\frac{\partial}{\partial \alpha}\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = -\upsilon\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon+1}(x,y,z;\alpha). \tag{67}$$

Also, the ELeGHBP ${}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\Bigg(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}} + D_y^{-1}\frac{\partial}{\partial D_y^{-1}} - sz\frac{\partial^{s+1}}{\partial\alpha\partial y^{s-1}\partial D_y^{-1}} + \frac{e^{(D_y y D_y)}(1-D_y y D_y)-1}{(e^{(D_y y D_y)}-1)} - m\Bigg)\,{}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) = 0. \tag{68}$$

Further, we note that for $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}, (i = 1,2,3,...,m)$, the determinant definition for the Appell polynomials reduces to determinant definition of the Bernoulli polynomials [6], therefore taking $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}, (i = 1,2,3,...,m)$ in equations (57) and (58), we get the following determinant definition of the ELeGHBP ${}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$:

Definition 4.1.1. The extended Legendre-Gould-Hopper-Bernoulli polynomials ${}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha)$ of degree m are defined by

$${}_{RH^{(s)}}\mathfrak{B}_{0,\upsilon}(x,y,z;\alpha) = {}_RH^{(s)}_{0,\upsilon}(x,y,z;\alpha), \tag{69}$$

$${}_{RH^{(s)}}\mathfrak{B}_{m,\upsilon}(x,y,z;\alpha) \tag{70}$$

$$= \frac{(-1)^m}{(\beta_0)^{m+1}}\begin{vmatrix} 1 & {}_RH^{(s)}_{1,\upsilon}(x,y,z;\alpha) & \frac{{}_RH^{(s)}_{2,\upsilon}(x,y,z;\alpha)}{2!} & \dots & \frac{{}_RH^{(s)}_{m-1,\upsilon}(x,y,z;\alpha)}{(m-1)!} & \frac{{}_RH^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!} \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{m} & \frac{1}{m+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \dots & \binom{m-1}{1}\frac{1}{m-1} & \binom{m}{1}\frac{1}{m} \\ 0 & 0 & 1 & \dots & \binom{m-1}{2}\frac{1}{m-2} & \binom{m}{2}\frac{1}{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{m}{m-1}\frac{1}{2} \end{vmatrix},$$

$$m = 1,2,3,\dots.$$

4.2. **Extended Legendre-Gould-Hopper-Euler polynomials.** Since, for $\mathcal{A}(t) = \frac{2}{e^t+1}$, the AP $\mathcal{A}_m(x)$ reduce to the Euler polynomials (EP) $\mathcal{E}_m(x)$. Therefore, for the same choice of $\mathcal{A}(t)$ the ELeGHAP ${}_{RH^{(s)}}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$

reduce to ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ which in view of the equation (29) are defined by means of following generating function:

$$\frac{2\ C_0(x\tau)\ C_0(-y\tau)}{(e^\tau+1)(\alpha-z\tau^s)^\upsilon}=\sum_{m=0}^{\infty}{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!}. \tag{71}$$

Further, in view of equations (21) and (22), the operational representations for the ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ are given as:

$$\left[\alpha-z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon}{}_R\mathcal{E}_m(x,y)={}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha), \tag{72}$$

$$\left[\alpha-z(-1)^s\frac{\partial^s}{\partial D_x^{-s}}\right]^{-\upsilon}{}_R\mathcal{E}_m(x,y)={}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha). \tag{73}$$

Next, in view of equation (32), the ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ are defined by the series

$$ {}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)=\frac{1}{\alpha^\upsilon}\sum_{r=0}^{\infty}\sum_{n,k=0}^{m}\frac{(-1)^n(-m)_{n+k}\ (\upsilon)_r\ x^k\ y^n\ (z\xi^s)^r\mathcal{E}_{m-n-k}}{r!(k!)^2(n!)^2}. \tag{74}$$

The ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following recurrence relations:

$$\frac{\partial}{\partial D_x^{-1}}{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)=-m\,{}_{_RH^{(s)}}\mathcal{E}_{m-1,\upsilon}(x,y,z;\alpha), \tag{75}$$

$$\frac{\partial}{\partial D_y^{-1}}{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)=m\,{}_{_RH^{(s)}}\mathcal{E}_{m-1,\upsilon}(x,y,z;\alpha), \tag{76}$$

$$\begin{aligned}\frac{\partial}{\partial z}{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)\\ =\upsilon m(m-1)(m-2)...(m-s+1)\,{}_{_RH^{(s)}}\mathcal{E}_{m-s,\upsilon+1}(x,y,z;\alpha),\end{aligned} \tag{77}$$

$$\frac{\partial}{\partial \alpha}{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)=-\upsilon\,{}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon+1}(x,y,z;\alpha). \tag{78}$$

Also, the ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\begin{aligned}\Bigg(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}}+D_y^{-1}\frac{\partial}{\partial D_y^{-1}}-sz\frac{\partial^{s+1}}{\partial\alpha\partial y^{s-1}\partial D_y^{-1}}\\ +\frac{e^{(D_yyD_y)}(1-e^{(D_yyD_y)})-1}{(e^{(D_yyD_y)}-1)}-m\Bigg){}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)=0.\end{aligned} \tag{79}$$

Further, in fact for $\beta_0=1$ and $\beta_i=\frac{1}{2},(i=1,2,3,...,m)$, the determinant definition for the Appell polynomials reduces to determinant definition of the Euler polynomials [5], therefore taking $\beta_0=1$ and $\beta_i=\frac{1}{2},(i=1,2,3,...,m)$ in equations (57) and (58), we get the following determinant definition of the ELeGHEP ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$:

Definition 4.2.1. The extended Legendre-Gould-Hopper-Euler polynomials ${}_{_RH^{(s)}}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$ of degree m are defined by

$$(80)\qquad {}_{R}H^{(s)}\mathcal{E}_{0,\upsilon}(x,y,z;\alpha) = {}_{R}H^{(s)}_{0,\upsilon}(x,y,z;\alpha),$$

$$(81)\qquad {}_{R}H^{(s)}\mathcal{E}_{m,\upsilon}(x,y,z;\alpha)$$

$$= \frac{(-1)^m}{(\beta_0)^{m+1}} \begin{vmatrix} 1 & {}_{R}H^{(s)}_{1,\upsilon}(x,y,z;\alpha) & \frac{{}_{R}H^{(s)}_{2,\upsilon}(x,y,z;\alpha)}{2!} & \cdots & \frac{{}_{R}H^{(s)}_{m-1,\upsilon}(x,y,z;\alpha)}{(m-1)!} & \frac{{}_{R}H^{(s)}_{m,\upsilon}(x,y,z;\alpha)}{m!} \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{m-1}{1}\frac{1}{2} & \binom{m}{1}\frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{m-1}{2}\frac{1}{2} & \binom{m}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{m}{m-1}\frac{1}{2} \end{vmatrix},$$

$$m = 1, 2, 3, \ldots.$$

4.3. **Extended Legendre-Gould-Hopper-Genocchi polynomials.** Since, for $\mathcal{A}(t) = \frac{2t}{e^t+1}$, the AP $\mathcal{A}_m(x)$ reduce to the Genocchi polynomials (GP) $\mathcal{G}_m(x)$. Therefore, for the same choice of $\mathcal{A}(t)$ the ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ reduce to ELeGHGP ${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$ which in view of the equation (29) are defined by means of following generating function:

$$(82)\qquad \frac{2\tau\ C_0(x\tau)\ C_0(-y\tau)}{(e^\tau+1)(\alpha - z\tau^s)^\upsilon} = \sum_{m=0}^{\infty} {}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)\frac{\tau^m}{m!}.$$

Further, in view of equations (21) and (22), the operational representations for the ELeGHGP ${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$ are given as:

$$(83)\qquad \left[\alpha - z\frac{\partial^s}{\partial D_y^{-s}}\right]^{-\upsilon} {}_{R}\mathcal{G}_m(x,y) = {}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha),$$

$$(84)\qquad \left[\alpha - z(-1)^s\frac{\partial^s}{\partial D_x^{-s}}\right]^{-\upsilon} {}_{R}\mathcal{G}_m(x,y) = {}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha).$$

Next, in view of equation (32), the ELeGHGP ${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$ are defined by the series

(85)

$${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha) = \frac{1}{\alpha^\upsilon}\sum_{r=0}^{\infty}\sum_{n,k=0}^{m} \frac{(-1)^n(-m)_{n+k}\ (\upsilon)_r\ x^k\ y^n\ (z\xi^s)^r\mathcal{G}_{m-n-k}}{r!(k!)^2(n!)^2}.$$

The ELeGHGP ${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following recurrence relations:

$$(86)\qquad \frac{\partial}{\partial D_x^{-1}}\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha) = -m\,{}_{R}H^{(s)}\mathcal{G}_{m-1,\upsilon}(x,y,z;\alpha),$$

$$(87)\qquad \frac{\partial}{\partial D_y^{-1}}\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha) = m\,{}_{R}H^{(s)}\mathcal{G}_{m-1,\upsilon}(x,y,z;\alpha),$$

$$\begin{aligned}(88)\quad & \frac{\partial}{\partial z}\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)\\ &\qquad = \upsilon m(m-1)(m-2)...(m-s+1)\,{}_{R}H^{(s)}\mathcal{G}_{m-s,\upsilon+1}(x,y,z;\alpha),\end{aligned}$$

$$(89)\quad \frac{\partial}{\partial \alpha}\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha) = -\upsilon\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon+1}(x,y,z;\alpha).$$

Also, the ELeGHGP ${}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\begin{aligned}(90)\quad & \Bigg(-D_x^{-1}\frac{\partial}{\partial D_y^{-1}} + D_y^{-1}\frac{\partial}{\partial D_y^{-1}} - sz\frac{\partial^{s+1}}{\partial\alpha\partial y^{s-1}\partial D_y^{-1}}\\ &\qquad + \frac{e^{(D_y y D_y)}(1-D_y y D_y)+1}{(e^{(D_y y D_y)}+1)} - m\Bigg)\,{}_{R}H^{(s)}\mathcal{G}_{m,\upsilon}(x,y,z;\alpha) = 0.\end{aligned}$$

Similarly, for the other members of the Appell family (see [20, Table 1]), we can establish new special polynomials belonging to the extended Legendre-Gould-Hopper-Appell family. The operational representations, generating function and other properties of these special polynomials can be obtained from the results derived in the second and third sections.

Appendix

We have mentioned special cases of the LeGHAP ${}_{R}H^{(s)}A_n(x,y,z)$ in Table 1. Now, for the same choice of the variables and indices the ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$ reduce to the corresponding special cases. We mention these new special polynomials in Table 2.

Table 2. Special cases of the ELeGHAP ${}_{R}H^{(s)}\mathcal{A}_{m,\upsilon}(x,y,z;\alpha)$

S. No.	**Values of the indices and variables**	**Generating function**	**Name of the special polynomials**
I.	$z=0$	$\frac{1}{\alpha^{\upsilon}}A(\tau)\ C_0(x\tau)\ C_0(-y\tau) = \sum_{m=0}^{\infty} {}_{R}A_{m,\upsilon}(x,y;\alpha)\frac{\tau^m}{m!}$	Extended 2-variable Legendre-Appell polynomials (E2VLeAP)
II.	$s=n;\ y=0,\ z\to y$	$\frac{A(\tau)\ C_0(x\tau)}{(\alpha-y\tau^n)^{\upsilon}} = \sum_{m=0}^{\infty} {}_{[n]L}A_{m,\upsilon}(x,y;\alpha)\frac{\tau^m}{m!}$	Extended 2-variable generalized Laguerre type-Appell polynomials (E2VGLTAP)
III.	$s=2;\ x=0,$ $y\to x,\ z\to y$	$\frac{A(\tau)\ C_0(-x\tau)}{(\alpha-y\tau^2)^{\upsilon}} = \sum_{m=0}^{\infty} {}_{G}A_{m,\upsilon}(x,y;\alpha)\frac{\tau^m}{m!}$	Extended Hermite type-Appell polynomials (EHTAP)
IV.	$s=1;\ x\to(\frac{1-x}{2}),$ $y\to(\frac{1+x}{2}),\ z=0$	$\frac{1}{\alpha^{\upsilon}}A(\tau)\ C_0(\frac{1-x}{2}\tau)\ C_0(\frac{-1-x}{2}\tau) = \sum_{m=0}^{\infty} {}_{P}A_{m,\upsilon}(x;\alpha)\frac{\tau^m}{m!}$	Extended Legendre-Appell polynomials (ELeAP)

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