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ON THE EXISTENCE AND UNIQUENESS OF SOLUTION OF STOCHASTIC DIFFERENTIAL SYSTEM DRIVEN BY G-BROWNIAN MOTION

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Abstract. In this paper we prove the existence and the uniqueness of the solution of system of stochastic differential equations driven by G-Brownian motion by using the Caratheodory approximation scheme.

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1. Introduction

The Caratheodory approximation scheme has been used by several mathematicians to prove the existence theorem for the solutions of ordinary differential equations under mild regularity conditions. N. Caratheodory [2] was the first to introduce this approximation for ordinary differential equations. Then, K.Liu [10] has used the Caratheodory solution for a class of infinite-dimensional stochastic evolution equations with time delays. Moreover, H.Young [18] has discussed Caratheodory's and Euler Maruyam's approximate solutions to stochastic differential delay equation.

Furthermore, in [3] F. Faiz Ullah proves that the Caratheodory approximation solution for vector valued stochastic differential equation driven by a $G-$Brownian motion ($G-$SDEs) converges to the unique solution of the following $G-$SDEs under the Lipschitz and the linear growth conditions:

(1.1)
$$X_t = X_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, d\langle B\rangle_s + \int_0^t h(s, X_s)\, dB_s,\ t \in [0, T]$$

In 2006, S. Peng (for more details see [12], [13], [14], [15] and [16]) introduced the theory of non linear expectation, $G-$Brownian motion and defined the related stochastic calculus, especially stochastic integrals of Itô's type with respect to $G-$Brownian motion and derived the related Itô's formula [16]. In addition, the notion of $G-$normal distribution plays the same important role in the theory of non linear expectation as that of normal distribution with the classical probability.

The existence and the uniqueness of the solution X_t, for $G-$SDEs (1.1) under different conditions was proved in ([1], [4], [5], [7], [8], [9], [11], [14] and [16]).

In this paper, we present both the existence and the uniqueness of the solution for the following system of stochastic differential equations driven by a $G-$Brownian motion (S$G-$SDEs):

$$
\begin{cases} X_t = X_0 + \int_0^t f_1(s, X_s, Y_s)\, ds + \\ + \int_0^t f_2(s, X_s, Y_s)\, d\langle B\rangle_s + \int_0^t f_3(s, X_s, Y_s)\, dB_s \\ Y_t = Y_0 + \int_0^t g_1(s, X_s, Y_s)\, ds + \\ + \int_0^t g_2(s, X_s, Y_s)\, d\langle B\rangle_s + \int_0^t g_3(s, X_s, Y_s)\, dB(s) \end{cases} \tag{1.2}
$$

Where (X_0, Y_0) is a given initial condition, $(\langle B_t\rangle)_{t\geq 0}$ is the quadratic variation process of the $G-$Brownian motion $(B_t)_{t\geq 0}$ and all the coefficients $f_i(t,x,y)$, $g_i(t,x,y)$, for $i = 1,2,3$, satisfy the Lipschitz and the linear growth conditions with respect to (x,y). These results are obtained by using the technics adopted by F. Faiz Ullah [3] in the case where the Lipschitz and the linear growth constants are time dependant.

This paper is divided into three sections. The second section gives the necessary notations and results that we will use in this work. The third section proves the existence and the uniqueness of the solution of (1.2) by using the Caratheodory approximation scheme.

2. Preliminaries

In this section we recall some basic notions and theorems which we use in this work and deal with sublinear expectation and $G-$stochastic calculus. More details concerning this section may be found in [12], [13], [14] and [16].

Let Ω be a given non-empty set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1. *A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all X, Y in $\mathcal{H}$, c in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:*
(i) (Monotonicity): if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$,
(ii) (Constant preserving): $\mathbb{E}[c] = c$,
(iii) (Sub-additivity): $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$,
(iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called sublinear expectation space.

We assume that if $X_1,\ X_2,\ ...,\ X_n \in\ \mathcal{H}$, then $\varphi(\ X_1, X_2, ..., X_n) \in \mathcal{H}$, for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$, the set of functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the condition: $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x-y|$ for all $x,\ y \in \mathbb{R}^n$, where C is a positive constant and $m \in \mathbb{N}^*$ depending only on φ.

Definition 2.2. *Let X, Y be two $n-$dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if $\mathbb{E}_2[\varphi(Y)] = \mathbb{E}_1[\varphi(X)]$, for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$.*

Definition 2.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ if $\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}]\quad \forall \varphi \in C_{l,lip}(\mathbb{R}^{m+n})$.*

$\tilde{X}$ is called an independent copy of X, if $\tilde{X} \overset{d}{=} X$ and $\tilde{X}$ is independent from X.

Remark 1. *It is important to note that Y is independent from X does not imply that X is independent from Y, (see* [6]*).*

Let Γ be a closed bounded and convex subset of $\mathbb{S}_+(d)$ the set of positive and symmetric $d-$dimensional matrix. Let

$$\Sigma = \{AA^* : A \in \Gamma \text{ and } A^* \text{ is the transpose of } A\}$$

and let $G : \mathbb{S}_+(d) \longrightarrow \mathbb{R}$ defined by:

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} Tr(\gamma A)$$

Definition 2.4. *In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a $d-$dimensional vector of random variables $X \in \mathcal{H}^d$ is $G-$normal distributed, if for each $\varphi \in C_{l,lip}(\mathbb{R}^d)$, the function $u(t,x) = \mathbb{E}\left(\varphi\left(x + \sqrt{t}X\right)\right)$ is the unique viscosity solution of the following parabolic equation called the $G-$heat equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = G\left(D^2 u\right) \\ u(0,x) = \varphi(x) \end{cases}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

where $D^2 u = \left(\partial^2_{x_i x_j} u\right)_{i,j}^{d}$ is the Hessian matrix of u.

Remark 2. *In fact, if $d = 1$ we have $G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right)$, where $\overline{\sigma}^2 = \mathbb{E}[X^2]$, $\underline{\sigma}^2 = -\mathbb{E}[-X^2]$, $\alpha^+ = \max(\alpha, 0)$ and $\alpha^- = \max(-\alpha, 0)$ (For more details see* [16]*). We write $X \sim \mathcal{N}\left(0; [\underline{\sigma}^2, \overline{\sigma}^2]\right)$.*

Definition 2.5. *A process $(B_t)_{t\geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, is called a $G-$Brownian motion if the following properties are satisfied:*
(i) $B_0 = 0$,
(ii) For each t, $s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(0; [\underline{\sigma}^2 s, \overline{\sigma}^2 s]-$distributed and is independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_n \leq t$.

We denote by $\Omega = C_0(\mathbb{R})$ the space of all $\mathbb{R}-$valued continuous functions ω, defined on $\mathbb{R}_+$ such that $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \max_{t\in[0,i]} \left[\left|\left(\omega_t^1 - \omega_t^2\right) \wedge 1\right|\right]$$

For each fixed $T > 0$, let

$$\Omega_T = \{\omega_{.\wedge T},\ \omega \in \Omega\}$$

$$Lip(\Omega_T) = \{\varphi(B_{t_1}, ..., B_{t_m}),\ m \geq 1,\ t_1, ..., t_m \in [0,T],\ \varphi \in C_{l;lip}(\mathbb{R}^m)\}$$

and let

$$Lip(\Omega) = \bigcup_{n=1}^{\infty} Lip(\Omega_n)$$

In [16] Peng constructs a sublinear expectation $\mathbb{E}$ on $(\Omega, Lip(\Omega))$ under which the canonical process $(B_t)_{t\geq 0}$ $(i.e. B_t(\omega) = \omega_t)$ is a $G-$Brownian motion. In what follows, we consider this $G-$Brownian motion.

We denote by $L_G^p(\Omega_T)$, $p \geq 1$, the completion of $Lip(\Omega_T)$ under the norm $||X||_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote $L_G^p(\Omega)$ the completion space of

$Lip(\Omega)$. It was shown in [12] and [16] that there exists a family of probability measures $\mathcal{P}$ on Ω such that

$$\mathbb{E}[X] = \sup_{P\in\mathcal{P}} E^P[X], \quad \text{for } X \in L^1_G(\Omega)$$

where E^P stands for the linear expectation under the probability P. We say that a property holds quasi surely ($q.s.$) if it holds for each $P \in \mathcal{P}$.

For a finite partition of $[0,T]$, $\pi_T = \{t_0, t_1, ..., t_N\}$, we set

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i|,\ i = 0, 1, ..., N-1\}$$

Consider the collection $M^{p,0}_G(0,T)$ of simple processes defined by:

$$\eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) I_{[t_i,t_{i+1}[}(t)$$

where

$$\xi_i \in L^p_G(\Omega_{t_i}),\ i = 0, 1, ..., N-1 \text{ and } p \geq 1$$

The completion of $M^{p,0}_G(0,T)$ under the norm

$$\|\eta\| = \left\{\frac{1}{T}\int_0^T \mathbb{E}[|\eta_t|^p]\, dt\right\}^{\frac{1}{p}}$$

is denoted by $M^p_G(0,T)$. Note that

$$M^qG(0,T) \subset M^p_G(0,T),\ \text{for } 1 \leq p \leq q$$

Definition 2.6. *For each $\eta \in M^{2;0}_G(0,T)$, the $G-$Itô's integral is defined by*

$$I(\eta) = \int_0^T \eta_v dB_v = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i})$$

The mapping $\eta \longmapsto I(\eta)$ can be extends continuously to $M^2_G(0,T)$.

Definition 2.7. *The increasing continuous process $\left(\langle B\rangle_t\right)_{t\geq 0}$ with $\langle B\rangle_0 = 0$ defined by*

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_v dB_v$$

is called the quadratic variation process of $(B_t)_{t\geq 0}$. Note that $\langle B\rangle_t$ can be regarded as the limit in $L^2_G(\Omega_t)$ of $\sum\limits_{i=0}^{N}\left(B_{t^N_{i+1}} - B_{t^N_i}\right)^2$, where $\pi^N_T = \{t^N_0, t^N_1, ..., t^N_k\}$, is a sequence of partitions of $[0,T]$ such that $\mu(\pi^N_T)$ tends to 0 when N goes to infinity.

Burkholder–Davis–Gundy (BDG) inequalities play an important role in the study of $G-$stochastic differential equations. There has been an increased interest in the following lemmas, see [5] and [17]:

Lemma 2.8. *Let $p \geq 1$, $\eta \in M^p_G(0,T)$ and $0 \leq s \leq t \leq T$. Then*

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta_r d\langle B\rangle_r\right|^p\right] \leq C_1 (t-s)^{p-1}\int_s^t \mathbb{E}[|\eta_u|^p]\, du \tag{2.1}$$

where $C_1 > 0$ is a constant independent of η.

Lemma 2.9. *Let* $p \geq 2$, $\eta \in M_G^p(0,T)$ *and* $0 \leq s \leq t \leq T$. *Then*

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta_r dB_r\right|^p\right] \leq C_2 \left|t-s\right|^{\frac{p}{2}-1}\int_s^t \mathbb{E}\left[\left|\eta_u\right|^p\right] du \tag{2.2}$$

where $C_2 > 0$ *is a constant independent of* η.

3. Existence and uniqueness theorem

In this section, we give the existence and the uniqueness of the solution to the $SG-$SDE (1.2), where the initial conditions $X_0,\ Y_0 \in M_G^2\left(0,T;\mathbb{R}^d\right)$, and $f_i,\ g_i : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ for $i = 1,2,3$.

The Caratheodory approximation scheme for (1.2) is given as follows: For any integer $k \geq 1$, we define

$$X_t^k = X_0,\ Y_t^k = Y_0 \quad \text{if } t \in \left]-1,0\right]$$

and

(3.1)

$$\begin{cases} X_t^k = X_0 + \int_0^t f_1\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) ds + \int_0^t f_2\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) d\left\langle B\right\rangle_s \\ +\int_0^t f_3\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) dB_s \\ Y_t^k = Y_0 + \int_0^t g_1\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) ds + \int_0^t g_2\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) d\left\langle B\right\rangle_s \\ +\int_0^t g_3\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) dB_s \end{cases}$$

if $t \in \left]0,T\right]$.

We make the following assumptions: For $h = f_i,\ g_i$, respectively, $i = 1,2,3$

(A1)

$$\left|h\left(t,x,y\right)\right|^2 \leq \varphi\left(t\right)\left(1+\left|x\right|^2+\left|y\right|^2\right) \tag{3.2}$$

for each $x,\ y \in \mathbb{R}^d$ and $t \in [0,T]$, where φ is a positive and continuous function on $[0,T]$.

(A2)

$$\left|h\left(t,x_2,\ y_2\right)-h\left(t,x_1,y_1\right)\right|^2 \leq \psi\left(t\right)\left(\left|x_2-x_1\right|^2+\left|y_2-y_1\right|^2\right) \tag{3.3}$$

for each $x_1, y_1, x_2,\ y_2 \in \mathbb{R}^d$ and $t \in [0,T]$, where ψ is a positive and continuous function on $[0,T]$.

In the following we equip the space of processes in $M_G^2\left(0,T;\mathbb{R}^d\right) \times M_G^2\left(0,T;\mathbb{R}^d\right)$ with the norm

$$||(X,Y)|| = \mathbb{E}^{\frac{1}{2}}\left[\sup_{0\leq t\leq T}\left(\left|X_t\right|^2+\left|Y_t\right|^2\right)\right]$$

Note that this space is a Banach space.

Now we give our main results:

Theorem 3.1. *Under the assumptions* (3.2) *and* (3.3)*, the system* (1.2) *has a unique solution q.s.*

$$\left(X_t, Y_t\right) \in M_G^2\left(0,T;\mathbb{R}^d\right) \times M_G^2\left(0,T;\mathbb{R}^d\right)$$

In order to prove Theorem 3.1,we need some lemmas:

Lemma 3.2. *For all integer $k \geq 1$ and $0 \leq s < t \leq T$*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|X_t^k\right|^2 + \left|Y_t^k\right|^2\right] \leq K \exp\left(C\int_0^T \varphi(t)\,dt\right) \tag{3.4}$$

where $K = \left[1+4(\mathbb{E}\left[|X_0|^2\right] + \mathbb{E}\left[|Y_0|^2\right])\right]$ and C is a constant depending only on C_1, C_2 and T.

Lemma 3.3. *For all integer $k \geq 1$ and $0 \leq s < t \leq T$,*

$$\mathbb{E}\left[\left|X_t^k - X_s^k\right|^2 + \left|Y_t^k - Y_s^k\right|^2\right] \leq K_1\left(\Phi(t) - \Phi(s)\right) \tag{3.5}$$

where $\Phi(t) = \int_0^t \varphi(s)\,ds$ and $K_1 = \frac{3}{4}C\left(1 + K\exp\left(C\int_0^T \varphi(t)\,dt\right)\right)$.

Proof. (of the lemma (3.2)). By using the formula (3.1) and the fact that $\left(\sum_{i=1}^n a_i\right)^2 \leq n\sum_{i=1}^n a_i^2$ for each positive constants a_i, $i = 1, 2, ..., n$, we have for all $t \in [0, T]$,

$$\begin{aligned}\left|X_t^k\right|^2 &\leq 4|X_0|^2 + 4\left|\int_0^t f_1\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)ds\right|^2 \\ &\quad +4\left|\int_0^t f_2\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)d\langle B\rangle_s\right|^2 + 4\left|\int_0^t f_3\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)dB_s\right|^2\end{aligned}$$

and

$$\begin{aligned}\left|Y_t^k\right|^2 &\leq 4|Y_0|^2 + 4\left|\int_0^t g_1\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)ds\right|^2 + \\ &\quad +4\left|\int_0^t g_2\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)d\langle B\rangle_s\right|^2 + 4\left|\int_0^t g_3\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right)dB_s\right|^2\end{aligned}$$

which implies that, by using the Lemmas (2.8) and (2.9), the $G-$Hölder inequality and the assumption (3.2),

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t^k\right|^2\right] \leq 4\mathbb{E}\left[|X_0|^2\right] +$$

$$\begin{aligned}&C_3\int_0^t \varphi(s)\left(1+\mathbb{E}\left[\left|X^k_{s-\frac{1}{k}}\right|^2\right] + \mathbb{E}\left[\left|Y^k_{s-\frac{1}{k}}\right|^2\right]\right)ds \\ \leq\ & 4\mathbb{E}\left[|X_0|^2\right] + C_3\int_0^t \varphi(v)\left(1+\sup_{0\leq v\leq s}\mathbb{E}\left[\left|X_v^k\right|^2\right] + \sup_{0\leq v\leq s}\mathbb{E}\left[\left|Y_v^k\right|^2\right]\right)dv\end{aligned}$$

where $C_3 = 4(T + C_1T + C_2)$. Similarly, we have

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|Y_t^k\right|^2\right] \leq 4\mathbb{E}\left[|Y_0|^2\right] + C_3\int_0^t \varphi(v)\left(1+\sup_{0\leq v\leq s}\mathbb{E}\left[\left|X_v^k\right|^2\right] + \sup_{0\leq v\leq s}\mathbb{E}\left[\left|Y_v^k\right|^2\right]\right)dv$$

Thus

$$\begin{aligned} & 1+\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t^k\right|^2\right]+\sup_{0\le t\le T}\mathbb{E}\left[\left|Y_t^k\right|^2\right] \\ \le\ & K+C\int_0^t\varphi(v)\left(1+\sup_{0\le v\le s}\mathbb{E}\left[\left|X_v^k\right|^2\right]+\sup_{0\le v\le s}\mathbb{E}\left[\left|Y_v^k\right|^2\right]\right)dv \end{aligned}$$

where $C=2C_3$.We conclude, by Gronwall's lemma, that

$$1+\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t^k\right|^2\right]+\sup_{0\le t\le T}\mathbb{E}\left[\left|Y_t^k\right|^2\right]\le K\exp\left\{C\int_0^t\varphi(s)\,ds\right\}$$

and consequently

$$\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t^k\right|^2+\left|Y_t^k\right|^2\right]\le K\exp\left\{C\int_0^T\varphi(t)\,dt\right\}$$

□

Proof. (of the lemma (3.3)). We have

$$\begin{aligned} X_t^k-X_s^k\ =\ & \int_s^t f_1\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)dw \\ & +\int_s^t f_2\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)d\langle B\rangle_w+\int_s^t f_3\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)dB_w \end{aligned}$$

and so, we have for each $0\le s\le u\le t\le T$:

$$\begin{aligned} & \mathbb{E}\left[\sup_{s\le u\le t}\left|X_u^k-X_s^k\right|^2\right] \\ \le\ & 3\mathbb{E}\left[\left|\int_s^u f_1\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)dw\right|^2\right] \\ & +3\mathbb{E}\left[\sup_{s\le u\le t}\left|\int_s^u f_2\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)d\langle B\rangle_w\right|^2\right] \\ & +3\mathbb{E}\left[\sup_{s\le u\le t}\left|\int_s^u f_3\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)dB_w\right|^2\right] \end{aligned}$$

Thanks to the lemmas (2.8), (2.9) and the assumption (3.2), we obtain that

$$\mathbb{E}\left[\sup_{s\le u\le t}\left|X_u^k-X_s^k\right|^2\right]\le 3T\int_s^t\mathbb{E}\left[\left|f_1\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)\right|^2\right]dw$$

$$+3C_1T\int_s^t\mathbb{E}\left[\left|f_2\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)\right|^2\right]dw+3C_2\int_s^t\mathbb{E}\left[\left|f_3\left(w,X^k_{w-\frac{1}{k}},Y^k_{w-\frac{1}{k}}\right)\right|^2\right]dw$$

$$\le C\int_s^t\varphi(w)\left(1+\mathbb{E}\left[\left|X^k_{w-\frac{1}{k}}\right|^2\right]+\mathbb{E}\left[\left|Y^k_{w-\frac{1}{k}}\right|^2\right]\right)dw$$

$$\le C\left(\Phi(t)-\Phi(s)\right)+C\int_s^t\varphi(w)\left(\mathbb{E}\left[\left|X^k_{w-\frac{1}{k}}\right|^2\right]+\mathbb{E}\left[\left|Y^k_{w-\frac{1}{k}}\right|^2\right]\right)dw$$

By using lemma (3.2), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leq u\leq t}\left|X_u^k-X_s^k\right|^2\right] &\leq C\left(1+K\exp\left\{C\int_0^T\varphi(t)\,dt\right\}\right)(\Phi(t)-\Phi(s))\\ &\leq \frac{K_1}{2}(\Phi(t)-\Phi(s))\end{aligned}$$

where $K_1=\frac{3}{4}C\left(1+K\exp\left\{C\int_0^T\varphi(t)\,dt\right\}\right)$.

By the same way, we have

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|Y_u^k-Y_s^k\right|^2\right]\leq\frac{K_1}{2}(\Phi(t)-\Phi(s))$$

Thus

$$\mathbb{E}\left[\left|X_t^k-X_s^k\right|^2+\left|Y_t^k-Y_s^k\right|^2\right]\leq K_1(\Phi(t)-\Phi(s))$$

The proof is complete. □

Proof. (of the theorem 3.1). We will prove the theorem in three steps:

Step 1. Suppose that $(X_{1,t},Y_{1,t})$ and $(X_{2,t},Y_{2,t})$ are two solutions of the system (1.2) with initial conditions $(X_{1,0},Y_{1,0})$ and $(X_{2,0},Y_{2,0})$ respectively. Then we have

$$\begin{aligned}&|X_{1,t}-X_{2,t}|^2\\ \leq\ & 4|X_{1,0}-X_{2,0}|^2+4\left|\int_0^t f_1(s,X_{1,s},Y_{1,s})-f_1(s,X_{2,s},Y_{2,s})\,ds\right|^2\\ &+4\left|\int_0^t f_2(s,X_{1,s},Y_{1,s})-f_2(s,X_{2,s},Y_{2,s})\,d\langle B\rangle_s\right|^2\\ &+4\left|\int_0^t f_3(s,X_{1,s},Y_{1,s})-f_3(s,X_{2,s},Y_{2,s})\,dB_s\right|^2\end{aligned}$$

By using the Lemmas (2.8), (2.9) and the assumption (3.3), we have for $0\leq r\leq t\leq T$,

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^r(f_1(s,X_{1,s},Y_{1,s})-f_1(s,X_{2,s},Y_{2,s}))\,ds\right|^2\right]\\ \leq\ & T\int_0^t\mathbb{E}\left[|f_1(s,X_{1,s},Y_{1,s})-f_1(s,X_{2,s},Y_{2,s})|^2\right]ds\\ \leq\ & T\int_0^t\psi(s)\,\mathbb{E}\left[|X_{1,s}-X_{2,s}|^2+|Y_{1,s}-Y_{2,s}|^2\right]ds\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq r\leq t}\left|\int_0^r f_2(s,X_{1,s},Y_{1,s})-f_2(s,X_{2,s},Y_{2,s})\,d\langle B\rangle_s\right|^2\right]\\ \leq\ & C_1T\int_0^t\mathbb{E}\left[|f_2(s,X_{1,s},Y_{1,s})-f_2(s,X_{2,s},Y_{2,s})|^2\right]ds\\ \leq\ & C_1T\int_0^t\psi(s)\,\mathbb{E}\left[|X_{1,s}-X_{2,s}|^2+|Y_{1,s}-Y_{2,s}|^2\right]ds,\end{aligned}$$

and

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|\int_0^r f_3\left(s,X_{1,s},Y_{1,s}\right)-f_3\left(s,X_{2,s},Y_{2,s}\right)dB_s\right|^2\right]$$

$$\begin{aligned}&\le C_2\int_0^t \mathbb{E}\left[\left|f_3\left(s,X_{1,s},Y_{1,s}\right)-f_3\left(s,X_{2,s},Y_{2,s}\right)\right|^2\right]ds\\&\le C_2\int_0^t \psi\left(s\right)\mathbb{E}\left[\left|X_{1,s}-X_{2,s}\right|^2+\left|Y_{1,s}-Y_{2,s}\right|^2\right]ds\end{aligned}$$

Therefore

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|X_{1,r}-X_{2,r}\right|^2\right]$$

$$\le 4\left|X_{1,0}-X_{2,0}\right|^2+C_3\int_0^t \psi\left(s\right)\mathbb{E}\left[\left|X_{1,s}-X_{2,s}\right|^2+\left|Y_{1,s}-Y_{2,s}\right|^2\right]ds$$

where $C_3=4\left(T+C_1T+C_2\right)$.

By the same way, we have

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|Y_{1,r}-Y_{2,r}\right|^2\right]$$

$$\le 4\left|Y_{1,0}-Y_{2,0}\right|^2+C_3\int_0^t \psi\left(s\right)\mathbb{E}\left[\left|X_{1,s}-X_{2,s}\right|^2+\left|Y_{1,s}-Y_{2,s}\right|^2\right]ds$$

Finally, we obtain that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le r\le t}\left(\left|X_{1,r}-X_{2,r}\right|^2+\left|Y_{1,r}-Y_{2,r}\right|^2\right)\right]\\&\le 4\left(\left|X_{1,0}-X_{2,0}\right|^2+\left|Y_{1,0}-Y_{2,0}\right|^2\right)\\&\quad +C\int_0^t \psi\left(s\right)\mathbb{E}\left[\left|X_{1,s}-X_{2,s}\right|^2+\left|Y_{1,s}-Y_{2,s}\right|^2\right]ds\end{aligned}$$

where $C=2C_3$.

By using Gronwall's lemma, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le r\le t}\left(\left|X_{1,r}-X_{2,r}\right|^2+\left|Y_{1,r}-Y_{2,r}\right|^2\right)\right]\\&\le 4\left(\left|X_{1,0}-X_{2,0}\right|^2+\left|Y_{1,0}-Y_{2,0}\right|^2\right)\exp\left(C\int_0^t \psi\left(s\right)ds\right)\end{aligned}$$

Now taking $\left(X_{1,0},Y_{1,0}\right)=\left(X_{2,0},Y_{2,0}\right)$, we see that for $t=T$

$$\mathbb{E}\left[\sup_{0\le r\le T}\left(\left|X_{1,r}-X_{2,r}\right|^2+\left|Y_{1,r}-Y_{2,r}\right|^2\right)\right]=0$$

which

$$\left(X_{1,t},Y_{1,t}\right)=\left(X_{2,t},Y_{2,t}\right)\quad q.s.\text{ for each } t\in[0,T]$$

Step 2: We will prove that $\left(X_t^k,Y_t^k\right)_{k\ge1}$ is an $M_G^2\left(0,T;\mathbb{R}^d\right)\times M_G^2\left(0,T;\mathbb{R}^d\right)$ Cauchy sequence for each $t\in[0,T]$. By the same arguments used in step 1, we have for each $m > n$,

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\le t\le T}\left(|X_t^m - X_t^n|^2 + |Y_t^m - Y_t^n|^2\right)\right] \\ \le\ & \frac{3}{4}C\int_0^T \psi(s)\,\mathbb{E}\left[\left|X^m_{s-\frac{1}{m}} - X^n_{s-\frac{1}{n}}\right|^2 + \left|Y^m_{s-\frac{1}{m}} - Y^n_{s-\frac{1}{n}}\right|^2\right] ds \end{aligned}$$

Since

$$\begin{aligned} & \mathbb{E}\left[\left|X^m_{s-\frac{1}{m}} - X^n_{s-\frac{1}{n}}\right|^2 + \left|Y^m_{s-\frac{1}{m}} - Y^n_{s-\frac{1}{n}}\right|^2\right] \\ \le\ & 2\mathbb{E}\left[\left|X^m_{s-\frac{1}{m}} - X^n_{s-\frac{1}{m}}\right|^2 + \left|X^n_{s-\frac{1}{m}} - X^n_{s-\frac{1}{n}}\right|^2\right. \\ & \left. + \left|Y^m_{s-\frac{1}{m}} - Y^n_{s-\frac{1}{m}}\right|^2 + \left|Y^n_{s-\frac{1}{m}} - Y^n_{s-\frac{1}{n}}\right|^2\right] \\ \le\ & 2\mathbb{E}\left[\sup_{0\le u\le s}\left(|X_u^m - X_u^n|^2 + |Y_u^m - Y_u^n|^2\right)\right] \\ & + 2\mathbb{E}\left[\left(\left|X^n_{s-\frac{1}{m}} - X^n_{s-\frac{1}{n}}\right|^2 + \left|Y^n_{s-\frac{1}{m}} - Y^n_{s-\frac{1}{n}}\right|^2\right)\right] \end{aligned}$$

then, by using the lemma (3.3), we have

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\le t\le T}\left(|X_t^m - X_t^n|^2 + |Y_t^m - Y_t^n|^2\right)\right] \\ \le\ & \frac{3}{2}C\int_0^T \psi(r)\,\mathbb{E}\left[\sup_{0\le u\le r}\left(|X_u^m - X_u^n|^2 + |Y_u^m - Y_u^n|^2\right)\right] dr + \\ & + \frac{3}{2}CK_1\left[\Phi\left(s - \frac{1}{m}\right) - \Phi\left(s - \frac{1}{n}\right)\right]\int_0^T \psi(r)\,dr \end{aligned}$$

Thus, by Gronwall's lemma,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(|X_t^m - X_t^n|^2 + |Y_t^m - Y_t^n|^2\right)\right] \le \widetilde{K}\left(\frac{1}{n} - \frac{1}{m}\right)\exp\left(\frac{3C}{2}\int_0^T \psi(s)\,ds\right)$$

Where $\widetilde{K} = \frac{3}{2}K_1CT\sup_{0\le t\le T}[\varphi(t)]\sup_{0\le t\le T}[\psi(t)]$

which means that $\left(X_t^k, Y_t^k\right)_{k\ge1}$ is a Chauchy sequence.

Step 3: We will prove that the limit (X_t, Y_t) in $M_G^2\left(0,T;\mathbb{R}^d\right)\times M_G^2\left(0,T;\mathbb{R}^d\right)$ of $\left(X_t^k, Y_t^k\right)$ is the solution of the system (1.2). For the existence, let $(X_0, Y_0) \in M_G^2\left(0,T;\mathbb{R}^d\right)\times M_G^2\left(0,T;\mathbb{R}^d\right)$ be an initial condition such $\mathbb{E}\left[|X_0|^2 + |Y_0|^2\right] < \infty$.

$$\begin{aligned} \left|X_u - X_u^k\right|^2 \le\ & 3\left|\int_0^u f_1\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) - f_1(s, X_s, Y_s)\,ds\right|^2 \\ & + 3\left|\int_0^u f_2\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) - f_2(s, X_s, Y_s)\,d\langle B\rangle_s\right|^2 \\ & + 3\left|\int_0^u f_3\left(s, X^k_{s-\frac{1}{k}}, Y^k_{s-\frac{1}{k}}\right) - f_3(s, X_s, Y_s)\,dB_s\right|^2 \end{aligned}$$

By using the lemmas (2.8) , (2.9) and the assumption (3.3), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|X_u^k-X_u\right|^2\right)\right]\\
\leq\ & C\int_0^T\psi(s)\,\mathbb{E}\left[\left|X_{s-\frac{1}{k}}^k-X_s\right|^2+\left|Y_{s-\frac{1}{k}}^k-Y_s\right|^2\right]ds\\
\leq\ & 2C\int_0^T\psi(s)\,\mathbb{E}\left[\left|X_{s-\frac{1}{k}}^k-X_s^k\right|^2+\left|Y_{s-\frac{1}{k}}^k-Y_s^k\right|^2\right]ds\\
&+2C\int_0^T\psi(s)\,\mathbb{E}\left[\left|X_s^k-X_s\right|^2+\left|Y_s^k-Y_s\right|^2\right]ds
\end{aligned}$$

Thus, by using the lemma (3.3),

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|X_u^k-X_u\right|^2\right)\right]\\
\leq\ & \frac{M}{k}+2C\int_0^T\psi(s)\,\mathbb{E}\left[\sup_{0\leq u\leq s}\left(\left|X_u^k-X_u\right|^2+\left|Y_u^k-Y_u\right|^2\right)\right]ds,
\end{aligned}$$

where $M=\widetilde{K}C\int_0^T\psi(s)\,ds\exp\left\{C\int_0^T\varphi(s)\,ds\right\}$. Similarly, we have the same formula with $\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|Y_u^k-Y_u\right|^2\right)\right]$ instead of $\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|X_u^k-X_u\right|^2\right)\right]$, which implies that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|X_u^k-X_u\right|^2+\left|Y_u^k-Y_u\right|^2\right)\right]\\
\leq\ & \frac{M}{k}+2C\int_0^T\psi(s)\,\mathbb{E}\left[\sup_{0\leq u\leq s}\left(\left|X_u^k-X_u\right|^2+\left|Y_u^k-Y_u\right|^2\right)\right]ds
\end{aligned}$$

and consequently, by suing Gronwall's lemma,

$$\mathbb{E}\left[\sup_{0\leq u\leq T}\left(\left|X_u^k-X_u\right|^2+\left|Y_u^k-Y_u\right|^2\right)\right]\leq\frac{\widetilde{K}}{k}\exp\left(\frac{3}{2}C\int_0^T\psi(s)\,ds\right)$$

The proof is now complete. □

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