Advanced Studies in Contemporary Mathematics
29 (2019), No. 4, pp. 455 - 459
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.4.455

ONE-DIMENSIONAL FINALLY PRECONTINUOUS PSEUDOREPRESENTATIONS OF ALMOST CONNECTED LOCALLY COMPACT GROUPS WHOSE IDENTITY COMPONENT HAS NO HERMITIAN SYMMETRIC QUOTIENTS

A. I. Shtern

Abstract. The structure of one-dimensional finally precontinuous pseudorepresentations of almost connected locally compact groups whose identity component has no Hermitian symmetric semisimple quotient Lie groups is studied.

For generalities concerning quasirepresentations, pseudorepresentations, pseudocharacters, and quasicharacters, see [1–3].

§ 1. Introduction

The structure of one-dimensional pseudorepresentations of connected locally compact groups was described in [4].

In this paper, we consider the structure of one-dimensional pseudorepresentations with sufficiently small defect of almost connected locally compact groups whose connected component has no Hermitian symmetric semisimple quotient Lie groups. As in [4], using the same consideration, one can see that, if π is a pseudorepresentation of this kind, then π must be either an ordinary one-dimensional representation of G (bounded or unbounded) or a bounded one-dimensional pseudorepresentation of the group. Below we describe the

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25, 22A10.
Key words and phrases. almost connected locally compact group, one-dimensional pseudorepresentation, Hermitian symmetric semisimple Lie group.

structure of bounded one-dimensional finally precontinuous pseudorepresentations with small defect of almost connected locally compact groups whose connected component has no Hermitian symmetric quotient semisimple Lie groups.

§ 2. Preliminaries

We recall the following notion.

Lemma 1. *Let G be an almost connected locally compact group, and let $\mathcal{N}$ be the family of compact normal subgroups $N \neq \{e\}$ such that G/N is a (not necessarily connected) Lie group. Then $\mathcal{N}$ is a nontrivial filter basis convergent to $\{e\}$.*

Proof. This follows immediately from the Gleason–Montgomery–Zippin–Yamabe theorem.

Recall that a one-dimensional locally bounded (in particular, bounded) pseudorepresentation of G is said to be *finally precontinuous* if the related set $\mathrm{FDG}(\pi) = \bigcap_{N\in\mathcal{N}} \overline{\pi(N)}$ is the singleton formed by the number 1 [1]. As is known, this condition implies that the restriction of π to the commutator subgroup G' of G is continuous with respect to the intrinsic Lie topology of the Lie group G' [1, 5, 7].

Recall that a pseudorepresentation is said to be pure if its restriction to every commutative subgroup is an ordinary representation of the subgroup.

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [6]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0 is compact) admits a totally disconnected compact subgroup D such that $G = G_0D$.

§ 3. Main theorem

Let us describe the structure of bounded one-dimensional finally precontinuous pseudorepresentations with sufficiently small defect of almost connected locally compact groups whose identity component admits no Hermitian symmetric quotient semisimple Lie groups.

Theorem. *Let G be an almost connected locally compact group, let G_0 be the connected component of the identity element e of G, and let π be a one-dimensional finally precontinuous pseudorepresentation of G with sufficiently small defect ε ($|\pi(g_1g_2) - \pi(g_1)\pi(g_2)| \leq \varepsilon$ for all $g_1, g_2 \in G$, where $\varepsilon <$*

1/12 [7]). *Let N be a compact normal subgroup of G such that the quotient G/N is a Lie group and the restriction of π to N is the identity representation of N. Let $(G/N)_0$ be the connected component of the identity element $e_{G/N}$ of G/N and let $(G/N)_0 = SR$ be a Levi decomposition of the Lie group $(G/N)_0$, where R is the radical of $(G/N)_0$ and S stands for a Levi subgroup of $(G/N)_0$. Denote the pseudorepresentation of $(G/N)_0$ naturally defined by the pseudorepresentation π of G by π_0. Then π_0 is an ordinary one-dimensional representation of $(G/N)_0$ defined by the restriction of π to G_0 and π_0 is given by a (not necessarily continuous) central unitary character φ of R (i.e., $\varphi(k) = \varphi(gkg^{-1})$ for all $k \in R$ and $g \in (G/N)_0$; see* [1, 3]). *The representation π_0 is invariant with respect to the inner automorphisms of G, and π is an ordinary representation of G completely determined by π_0 and by the restriction ρ of π to the finite Dong Hoon Lee's group D such that $G/N = (G/N)_0 D$ (ρ is an ordinary representation of D). Moreover,*

$$\pi(g) = \pi(g_0 d) = \pi_0(g_0)\rho(d), \qquad g = g_0 d, \quad g_0 \in G_0, \quad d \in D.$$

Thus, the above assertion is a kind of "triviality theorem" (cf. [8]).

Proof. We can consider the group G/N instead of G, and thus we may assume that G is a Lie group with finite quotient group G/G_0. The group D can be chosen to be finite by Lemma 2.12 of [5]. Since every finite group is amenable, and thus the pseudorepresentation ρ can be regarded as a representation, and the restriction of π to G_0 is an ordinary representation by [8], it follows that the pseudorepresentation π is completely determined by the representations ρ and π_0, and both the representations are one-dimensional characters of the corresponding groups. Let us compare the characters of G_0 given by the rules $g_0 \mapsto \pi_0(g_0)$, $g_0 \in G_0$, and $g_0 \mapsto \pi_0(gg_0g^{-1})$, $g_0 \in G_0$, $g \in G$. Since we may assume that π_0 is unitary, we see (because the restriction of π to the Abelian subgroup generated by g for every $g \in G$ is a unitary character of the subgroup) that

$$\begin{aligned}
&|\pi_0(g_0) - \pi_0(gg_0g^{-1})| \le |\pi_0(g_0) - \pi(gg_0)\pi(g^{-1})| + \varepsilon \\
&\le |\pi_0(g_0) - \pi(g)\pi_0(g_0)\pi(g^{-1})| + 2\varepsilon = |\pi_0(g_0) - \pi_0(g_0)| + 2\varepsilon \\
&\le 2\varepsilon < 1/6,
\end{aligned}$$

and, since two close characters are equal [9], we have $\pi_0(g_0) = \pi_0(gg_0g^{-1})$ for every $g_0 \in G_0$ and $g \in G$. This proves the desired invariance of π_0 with respect to inner automorphisms of G.

Let us define a mapping $\sigma : G \to \mathbb{C}$ by the rule

$$\sigma(g) = \sigma(g_0 d) = \pi_0(g_0)\rho(d), \qquad g = g_0 d, \quad g_0 \in G_0, \quad d \in D.$$

Using the fact that G_0 is a normal subgroup of G, we see that, for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\sigma((g_0 d)^n) = \sigma(g_0(dg_0d^{-1})\cdots(d^{n-1}g_0d^{-n+1})d^n) \\
&= \pi_0(g_0(dg_0d^{-1})\cdots(d^{n-1}g_0d^{-n+1}))\rho(d^n) \\
&= \pi_0(g_0)\pi_0(dg_0d^{-1})\cdots\pi_0(d^{n-1}g_0d^{-n+1})\rho(d)^n \\
&= \pi_0(g_0)^n\rho(d)^n = (\sigma(g_0 d))^n,
\end{aligned}$$

and thus the mapping σ, which is obviously a quasirepresentation, turns out to be a pseudorepresentation. A similar formula for $\sigma(g_{01}d_1g_{02}d_2)$ shows that σ is an ordinary representation. Since two close one-dimensional pseudorepresentations are equal, as was noted above, we see that $\sigma = \pi$, which completes the proof of the theorem.

§ 4. Concluding remarks

Recall that, by [6], we have

Theorem 2. *If the defect of a one-dimensional pseudorepresentation* π *is less than* $\sqrt{3}$ *(i.e.,* $|\pi(gh) - \pi(g)\pi(h)| \leq q < \sqrt{3}$ *for some* q *and all* $g, h \in G$*), then* π *is pure.*

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was partially supported by the Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN), theme NIR 0065-2019-0007.

References

1. A. I. Shtern, *Finite-dimensional quasi-representations of connected Lie groups and Mishchenko's conjecture*, J. Math. Sci. **159** (2009), no. 5, 653–751.

2. A. I. Shtern, *Remarks on finite-dimensional locally bounded finally precontinuous quasirepresentations of locally compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 4, 469–480.
3. A. I. Shtern, *A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups*, Izv. Math. **72** (2008), no. 1, 169–205.
4. A. I. Shtern, *One-dimensional finally precontinuous pseudorepresentations of connected locally compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 3, 449–453.
5. A. I. Shtern, *Continuity conditions for locally bounded finite-dimensional representations of almost connected locally compact groups*, Proc. Jangjeon Math. Soc. **21** (2018), no. 4, 779–782.
6. D. H. Lee, *Supplements for the identity component in locally compact groups*, Math. Z. **104** (1968), no. 1, 28–49.
7. A. I. Shtern, *Specific properties of one-dimensional pseudorepresentations of groups*, J. Math. Sci. (N.Y.) **233** (2018), no. 5, 770–776.
8. A. I. Shtern, *Locally bounded finally precontinuous finite-dimensional quasirepresentations of connected locally compact groups*, Mat. Sb. **208** (2017), no. 10, 149–170; English transl., Sb. Math. **208** (2017), no. 10, 1557–1576.
9. A. I. Shtern, *On operators in Fréchet spaces that are similar to isometries*, Vestnik MGU Ser. Matem. Mech. (1991), no. 4, 67–70; English transl., Moscow Univ. Math. Bull. **46** (1991), no. 4, 47–48.
10. A. I. Shtern, *Quasi-symmetry. I*, Russ. J. Math. Phys. **2 issue3** (1994), 353–382.
11. A. I. Shtern, *Triviality and continuity of pseudocharacters and pseudorepresentations*, Russ. J. Math. Phys. **5** (1998), no. 1, 135–138.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru