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NUMERICAL SOLUTION OF TIME-DELAYED BURGERS' EQUATIONS USING HAAR WAVELETS

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Abstract. In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve the time-delayed Burgers' equations. The quasilinearization technique is used to conveniently handle the nonlinear terms in the time-delayed Burgers' equations. The basic idea of Haar wavelet collocation method is to convert the partial differential equation into a system of algebraic equations that involves a finite number of variables. The solutions obtained by Haar wavelet collocation method are compared with the exact solutions and are found to be in good agreement. The error analysis of the Haar wavelet method shows that the accuracy of the method improves as the level of resolution of the Haar wavelet is increased.

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1. Introduction

In a model, delay occurs whenever there is a time gap between the control action and its impact on the system. The partial differential equations (PDEs) in which the rate of change of a time-dependent process is determined by both present and past states are known as time delay partial differential equations [1]. The time delay PDEs play an important role in describing many physical phenomena arising in physical, chemical, economical and biological sciences, such as study of physiological diseases, chemical technology, population dynamics, optimal control, circuit analysis, simulation of mechanical systems and vibrating mass attached to an elastic bar. The time-delayed Burgers' equation, time-delayed generalized Burgers' equation and time-delayed Burgers'-Fisher equation [2] are well known delay differential equations which have an important feature in reaction-diffusion and convection-diffusion systems, forest fire, population growth models and Neolithic transitions.

Liu [3] established that the time-delayed Burgers' equation is exponentially stable if the delay parameter is sufficiently small using the Liapunov function method. Smauoi and Mekkaoui [4] have investigated the solutions of generalized Burgers' equation with and without time delay. They show that the generalized Burgers' equation without time-delay is globally asymptotically stable whereas the generalized Burgers' equation with time delay is exponentially stable for small delays. Fahmy et al. [5, 6] used the improved

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tanh-function method to construct exact multiple-soliton and triangular periodic solutions of the time-delayed Burgers' equation. They compared the exact solution with the numerical solutions obtained by the Adomian decomposition method and the variational iteration method.

Sakthivel et al. [7] employed the extended tanh method and the exponential function method to obtain solitary wave solutions for the time-delayed Burgers' equation. They have applied homotopy perturbation method to determine the numerical solution of the time-delayed Burgers' equation using initial conditions. Kim and Sakthivel [8] used the $(\frac{G'}{G})-$expansion method to get the travelling wave solutions for the time-delayed Burgers' equation and time-delayed Burgers'-Fisher equation. They have expressed the solutions in terms of hyperbolic and trigonometric functions. Roastamy and Karimi [9] investigated the analytical and numerical solutions of the time-delayed Burgers' equation using the commutative hypercomplex mathematics and the homotopy perturbation method respectively.

Zhang et al. [10] have derived the solitary wave solutions of the generalized time-delayed Burgers'-Fisher equation with positive fractal power terms and special type of generalized time-delayed Burgers'-Fisher equation. Vanani and Soleymani [11] have used homotopy perturbation method to solve generalized time-delayed Burgers'-Fisher equation. They have compared the numerical solution with the exact solution determined using Adomian decomposition method and variational iteration method. Lee and Sakthivel [12] solved the nonlinear time-delayed evolution equations and boundary value problems using differential transform method. Bhrawy et al. [2] have derived a Jacobi-Gauss-Lobatto collocation method in order to obtain a numerical solution for the nonlinear time-delayed Burgers'-type equations.

Jawad and Ali [13] have developed $\tan - \cot$ function method to obtain travelling wave solutions for nonlinear PDEs such as Burgers' equation, KdV Burgers' equation, coupled Burgers' equation and generalized time-delayed Burgers' equation. Haghani et al. [14] have extended the operational Tau method to find a numerical solution of time-delayed Burgers' equation. They have used Lagurre polynomials as basis functions to diminish the volume of calculations and runtime of the technique. Strani and Texier [15] established the solutions of the Burgers' equation in the torus with small viscosity and a complex forcing. Tang et al. [16] have obtained the travelling wave solutions of the time-delayed generalized Burgers'-type equations using the extended $(\frac{G'}{G})-$expansion method.

Wavelets are mathematical functions that decompose data into different frequency components and then each component is studied with a resolution matched to its scale. Wavelet theory is the result of a multidisciplinary effort that brought together mathematicians, physicists and engineers. This connection has created a flow of ideas that goes well beyond the construction of new bases or transforms. Wavelet theory has become an effective tool for the development of pure and applied mathematics. Wavelets are well-suited for approximating data with sharp discontinuities. Wavelet representation is more accurate and useful in data compression, noise removal, pattern classification and fast scientific computation.

In recent years, the wavelet approach for the solution of PDEs has become very popular. Multi-resolution analysis of wavelets capture local features efficiently as such enables to detect singularities, shocks, irregular structure and transient phenomena exhibited by the analyzed equations. Haar wavelets are based on the functions which were introduced by the Hungarian mathematician Alfred Haar in 1910. Haar's contribution to wavelets is very evident. The Haar wavelets are the simplest of the wavelet families [17].

Chen and Hsiao [18] recommended to expand into the Haar series the highest order derivatives appearing in the differential equation. This idea has been very prolific and it is being abundantly applied for the solution of differential equations. The wavelet coefficients appearing in the Haar series are calculated either using Collocation method or Galerkin method. Lepik [20, 21, 22] used uniform Haar wavelets to solve differential equations along with the segmentation technique, evolution equations and higher order differential equations. Lepik [23] solved differential equations using nonuniform Haar wavelets. Bujurke et al. [24] solved elliptic boundary value problems arising in mathematical physics using wavelet-based multigrid approach. Bujurke et al. [25, 26, 27] computed eigenvalues and solutions of regular Sturm-Liouville problems, numerical solutions of nonlinear stiff differential equations and oscillator equations using Haar wavelets.

Hariharan et al. [28, 29, 30, 31, 32] applied Haar wavelets to solve Cahn-Allen equation, Fisher's equation, FitzHugh-Nagumo equation, some nonlinear parabolic equations, Klein-Gordon equation and Sine-Gordon equation. Chang and Piau [33] applied Haar wavelets to solve ordinary differential equations with initial and boundary conditions. Islam et al. [34] obtained the numerical solution of second-order boundary-value problems occurring in many engineering problems using Haar wavelets. Haq and Ali [35] used uniform Haar wavelets to determine the numerical solution of multi-point fourth-order boundary value problems. Shi and Cao [36] applied Haar wavelets to solve eigenvalue problems of high order differential equations. Ray [37] obtained the numerical solution of fractional Bagley Torvik equation using Haar wavelet operational matrix.

Celik [38] used Haar wavelets to solve generalized Burgers'-Huxley equation. Khalid et al. [39] solved Airy differential equation using Haar wavelets. Sumana et al. [40, 41, 42, 43, 44] determined the numerical solution of two-dimensional hyperbolic, parabolic and elliptic PDEs, Fredholm integral equations and coupled Fredholm integral equations of second kind using two-dimensional Haar wavelets. Reddy [45, 46] used Haar wavelet approach to solve seventh and eight order boundary value problems. Recently, Sumana et. al. [47] obtained the numerical solution of non-planar Burgers' equation by Haar wavelet method.

In this paper, time-delayed Burgers' equation, time-delayed generalized Burgers' equation and time-delayed Burgers'-Fisher equation are solved using Haar wavelets.

2. Haar Wavelets

The Haar wavelet family for $x \in [0,1]$ is defined as follows [17],

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2) \\ -1 & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m} \tag{2}$$

In the above definition $m = 2^d, d = 0, 1, ..., J$ indicates the level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter. J is the maximum level of resolution. The index i in equation (1) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1, k = 0$ we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

In order to solve differential equations of any order, we need the following integrals.

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2) \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \dfrac{(x-\xi_1)^2}{2} & \text{for } x \in [\xi_1, \xi_2) \\ \dfrac{1}{4m^2} - \dfrac{(\xi_3 - x)^2}{2} & \text{for } x \in [\xi_2, \xi_3) \\ \dfrac{1}{4m^2} & \text{for } x \in [\xi_3, 1] \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

2.1. **Multi-resolution analysis.** The best way to understand wavelets is through multi-resolution analysis. A multi-resolution analysis (MRA) of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_d \in L^2(\mathbb{R})$, $d \in \mathbb{Z}$ with the following properties.

(i) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$

(ii) The spaces V_d satisfy $\bigcup_{d\in\mathbb{Z}} V_d$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{d\in\mathbb{Z}} V_d = 0$.

(iii) If $f(x) \in V_0$, then $f(2^d x) \in V_d$, i.e., the spaces V_d are scaled versions of the central space V_0.

(iv) If $f(x) \in V_0$, then $f(2^d x - k) \in V_d$, i.e., all the V_d are invariant under translation.

(v) There exists $\phi \in V_0$ such that $\{\phi(x-k); k \in \mathbb{Z}\}$ is a Riesz basis in V_0.

The space V_d is used to approximate general functions by defining approximate projection of these functions onto these spaces. Since the union of all the V_d is dense in $L^2(\mathbb{R})$, so it guarantees that any function in $L^2(\mathbb{R})$ can be approximated arbitrarily close by such projections. As an example, the space V_d can be defined like

$$V_d = W_{d-1} \oplus V_{d-1} = W_{d-1} \oplus W_{d-2} \oplus V_{d-2} = \ldots = \bigoplus_{d=1}^{J+1} W_d \oplus V_0,$$

then the scaling function $h_1(x)$ generates an MRA for the sequence of spaces $\{V_d; d \in \mathbb{Z}\}$ by translation and dilation as defined in equations (1), (3). For each d, the space W_d serves as an orthogonal complement of V_d in V_{d+1}. The space W_d includes all the functions in V_{d+1} that are orthogonal to all those in V_d under some chosen inner product. The set of functions which forms a basis for the space W_d are called wavelets.

Given a function $f \in L^2(\mathbb{R})$, the MRA of $L^2(\mathbb{R})$ produces a sequence of subspaces $V_d, V_{d+1}, \ldots$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $J \to \infty$ [19].

2.2. **Function Approximation.** Any function $f(x)$ which is square integrable on $(0,1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a(i) h_i(x), \tag{6}$$

where

$$a(i) = \int_0^1 f(x) h_i(x) dx. \tag{7}$$

If $f(x)$ is approximated as piecewise constant during each subinterval, then equation (6) will be terminated at finite terms, i.e.

$$f(x) = \sum_{i=1}^{2M} a(i) h_i(x), \tag{8}$$

where the wavelet coefficients $a(i),\ i = 1, 2, \ldots, 2M$ are to be determined

3. Method of Solution

In this section, we have given the outline of the Haar wavelet collocation method to solve a class of hyperbolic PDEs, namely, the time-delayed Burgers' equation, time-delayed generalized Burgers' equation and time-delayed Burgers'-Fisher equation.

3.1. **Time-delayed Burgers' equation.** Consider the time-delayed one-dimensional Burgers' equation

$$\tau\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\lambda u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=0,\ \ 0\le x\le 1, t\ge 0, \tag{9}$$

with initial and boundary conditions

$$\left.\begin{aligned} u(x,0)&=f_1(x)\\ \frac{\partial u}{\partial t}(x,0)&=f_2(x)\end{aligned}\right\} 0\le x\le 1, \tag{10}$$

$$\left.\begin{aligned} u(0,t)&=g_1(t)\\ u(1,t)&=g_2(t)\end{aligned}\right\} t\ge 0, \tag{11}$$

where $\tau>0$ is the time delay and λ is a real number.

We restrict the maximum value of t to T and then we divide the interval $[0,T]$ into N equal parts of length $\Delta t=\frac{T}{N}$. We denote $t_s=(s-1)\Delta t,\ s=1,2,\ldots,N$.

We take the Haar wavelet solution in the following form as the order of the PDE (9) is 2 w.r.t t as well as x,

$$\ddot{u}''(x,t)=\sum_{i=1}^{2M}a_s(i)h_i(x). \tag{12}$$

Integrating equation (12) w.r.t. t in the limits $[t_s,t]$ gives

$$\dot{u}''(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)h_i(x)+\dot{u}''(x,t_s). \tag{13}$$

Integrating equation (13) w.r.t. t in the limits $[t_s,t]$ leads to

$$u''(x,t)=\frac{1}{2}(t-t_s)^2\sum_{i=1}^{2M}a_s(i)h_i(x)+u''(x,t_s)+(t-t_s)\dot{u}''(x,t_s). \tag{14}$$

Integrating equation (14) w.r.t. x in the limits $[0,x]$, we obtain

(15)

$$\begin{aligned} u'(x,t)=\frac{1}{2}(t-t_s)^2\sum_{i=1}^{2M}a_s(i)p_i(x)+u'(x,t_s)+(t-t_s)\dot{u}'(x,t_s)+u'(0,t)\\ -u'(0,t_s)-(t-t_s)\dot{u}'(0,t_s).\end{aligned}$$

Integrating equation (15) w.r.t. x in the limits $[0, x]$, we arrive at

(16)
$$u(x,t) = \frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) + (t-t_s)\dot{u}(x,t_s) + u(0,t) - u(0,t_s) - (t-t_s)\dot{u}(0,t_s) + x[u'(0,t) - u'(0,t_s) - (t-t_s)\dot{u}'(0,t_s)].$$

From the boundary conditions (11), we have

(17)
$$u(0,t_s) = g_1(t_s),\ u(0,t_s) = g_2(t_s),$$
$$\dot{u}(0,t_s) = \dot{g}_1(t_s),\ \dot{u}(0,t_s) = \dot{g}_2(t_s).$$

Putting $x = 1$ in equation (16) and using the conditions in (11), (17), we get

(18)
$$u'(0,t) - u'(0,t_s) - (t-t_s)\dot{u}'(0,t_s) = -\frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i) q_i(1) + g_2(t) - g_2(t_s) - (t-t_s)\dot{g}_2(t_s) - g_1(t) + g_1(t_s) + (t-t_s)\dot{g}_1(t_s).$$

Substituting equations (11), (17) and (18) in equations (15) and (16) gives

(19)
$$u'(x,t) = \frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i)[p_i(x) - q_i(1)] + u'(x,t_s) + (t-t_s)\dot{u}'(x,t_s) + g_2(t) - g_2(t_s) - (t-t_s)\dot{g}_2(t_s) - g_1(t) + g_1(t_s) + (t-t_s)\dot{g}_1(t_s),$$

(20)
$$u(x,t) = \frac{1}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_s(i)[q_i(x) - xq_i(1)] + u(x,t_s) + (t-t_s)\dot{u}(x,t_s) + x[g_2(t) - g_2(t_s) - (t-t_s)\dot{g}_2(t_s)] + (1-x)[g_1(t) - g_1(t_s) - (t-t_s)\dot{g}_1(t_s)].$$

Differentiating equations (19) and (20) w.r.t. t leads to

(21)
$$\dot{u}'(x,t) = (t-t_s) \sum_{i=1}^{2M} a_s(i)[p_i(x) - q_i(1)] + \dot{u}'(x,t_s) + \dot{g}_2(t) - \dot{g}_2(t_s) - \dot{g}_1(t) + \dot{g}_1(t_s),$$

(22)
$$\dot{u}(x,t) = (t-t_s) \sum_{i=1}^{2M} a_s(i)[q_i(x) - xq_i(1)] + \dot{u}(x,t_s) + x[\dot{g}_2(t) - \dot{g}_2(t_s)] + (1-x)[\dot{g}_1(t) - \dot{g}_1(t_s)].$$

Differentiating equation (22) w.r.t. t, we obtain

(23)
$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_s(i)[q_i(x) - xq_i(1)] + x\ddot{g}_2(t) + (1-x)\ddot{g}_1(t).$$

Taking the wavelet collocation points $x \to x_l$ and $t \to t_{s+1}$ in equations (13), (14), (19)-(23), we arrive at

$$\dot{u}''(x_l, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i) h_i(x_l) + \dot{u}''(x_l, t_s), \tag{24}$$

$$u''(x_l, t_{s+1}) = \frac{1}{2}(\Delta t)^2 \sum_{i=1}^{2M} a_s(i) h_i(x_l) + u''(x_l, t_s) + \Delta t \dot{u}''(x_l, t_s), \tag{25}$$

(26)
$$\begin{aligned} u'(x_l, t_{s+1}) = {} & \frac{1}{2}(\Delta t)^2 \sum_{i=1}^{2M} a_s(i)[p_i(x_l) - q_i(1)] + u'(x_l, t_s) + \Delta t \dot{u}'(x_l, t_s) \\ & + g_2(t_{s+1}) - g_2(t_s) - \Delta t \dot{g}_2(t_s) - g_1(t_{s+1}) + g_1(t_s) + \Delta t \dot{g}_1(t_s), \end{aligned}$$

(27)
$$\begin{aligned} u(x_l, t_{s+1}) = {} & \frac{1}{2}(\Delta t)^2 \sum_{i=1}^{2M} a_s(i)[q_i(x_l) - x_l q_i(1)] + u(x_l, t_s) + \Delta t \dot{u}(x_l, t_s) \\ & + x_l[g_2(t_{s+1}) - g_2(t_s) - \Delta t \dot{g}_2(t_s)] + (1 - x_l)[g_1(t_{s+1}) - g_1(t_s) \\ & - \Delta t \dot{g}_1(t_s)], \end{aligned}$$

(28)
$$\begin{aligned} \dot{u}'(x_l, t_{s+1}) = {} & \Delta t \sum_{i=1}^{2M} a_s(i)[p_i(x_l) - q_i(1)] + \dot{u}'(x_l, t_s) + \dot{g}_2(t_{s+1}) - \dot{g}_2(t_s) \\ & - \dot{g}_1(t_{s+1}) + \dot{g}_1(t_s), \end{aligned}$$

(29)
$$\begin{aligned} \dot{u}(x_l, t_{s+1}) = {} & \Delta t \sum_{i=1}^{2M} a_s(i)[q_i(x_l) - x_l q_i(1)] + \dot{u}(x_l, t_s) + x_l[\dot{g}_2(t_{s+1}) - \dot{g}_2(t_s)] \\ & + (1 - x_l)[\dot{g}_1(t_{s+1}) - \dot{g}_1(t_s)], \end{aligned}$$

$$\ddot{u}(x_l, t_{s+1}) = \sum_{i=1}^{2M} a_s(i)[q_i(x_l) - x_l q_i(1)] + x_l \ddot{g}_2(t_{s+1}) + (1 - x_l)\ddot{g}_1(t_{s+1}). \tag{30}$$

Using the quasilinearization technique [48] to handle the nonlinear terms in equation (9), we have the following scheme

$$\begin{aligned} \tau \ddot{u}(x, t_{s+1}) + \dot{u}(x, t_{s+1}) + \lambda u'(x, t_s) u(x, t_{s+1}) + \lambda u(x, t_s) u'(x, t_{s+1}) \\ - u''(x, t_{s+1}) = \lambda u(x, t_s) u'(x, t_s), \ s = 0, 1, 2, \ldots, \end{aligned} \tag{31}$$

which leads us from the time level t_s to t_{s+1}.

Taking the collocation points $x \to x_l$ in equation (31) and using equations (24)-(30), we get

$$\sum_{i=1}^{2M} B_s(i, l) a_s(i) = \psi_s(x_l), \ l = 1, 2, \ldots, 2M, \ s = 0, 1, 2, \ldots, \tag{32}$$

where

(33)
$$\begin{aligned} B_s(i,l) = &\left[\tau + \Delta t + \frac{1}{2}\lambda(\Delta t)^2 u'(x_l,t_s)\right] q_i(x_l) + \frac{1}{2}\lambda(\Delta t)^2 u(x_l,t_s) p_i(x_l) \\ &- \frac{1}{2}(\Delta t)^2 h_i(x_l) - \left[x_l\left\{\tau + \Delta t + \frac{1}{2}\lambda(\Delta t)^2 u'(x_l,t_s)\right\}\right. \\ &\left. + \frac{1}{2}\lambda(\Delta t)^2 u(x_l,t_s)\right] q_i(1), \end{aligned}$$

(34)
$$\begin{aligned} \psi_s(x_l) = & u''(x_l,t_s) + \Delta t \dot{u}''(x_l,t_s) - \dot{u}(x_l,t_s) - \lambda u(x_l,t_s) u'(x_l,t_s) \\ &- \lambda \Delta t u'(x_l,t_s)\dot{u}(x_l,t_s) - \lambda \Delta t u(x_l,t_s)\dot{u}'(x_l,t_s) - x_l[\tau \ddot{g}_2(t_{s+1}) \\ &+ \dot{g}_2(t_{s+1}) - \dot{g}_2(t_s)] - (1-x_l)[\tau \ddot{g}_1(t_{s+1}) + \dot{g}_1(t_{s+1}) - \dot{g}_1(t_s)] \\ &- \lambda[\{x_l u'(x_l,t_s) + u(x_l,t_s)\}\{g_2(t_{s+1}) - g_2(t_s) - \Delta t \dot{g}_2(t_s)\} \\ &- \{(1-x_l)u'(x_l,t_s) - u(x_l,t_s)\}\{g_1(t_{s+1}) - g_1(t_s) - \Delta t \dot{g}_1(t_s)\}]. \end{aligned}$$

Using the initial conditions (10), we get

(35) $$u(x_l,0) = f_1(x_l), \quad u'(x_l,0) = f_1'(x_l), \quad u''(x_l,0) = f_1''(x_l),$$

(36) $$\dot{u}(x_l,0) = f_2(x_l), \quad \dot{u}'(x_l,0) = f_2'(x_l), \quad \dot{u}''(x_l,0) = f_2''(x_l).$$

The wavelet coefficients $a_s(i)$, $i = 1, 2, \ldots, 2M$ can be successively calculated from equation (9) using equations (35) and (36). These coefficients are then substituted in equations (24)-(29) to obtain the approximate solution of the time-delayed Burgers' equation (9) at different time levels.

The time-delayed Burgers' equation, time-delayed generalized Burgers' equation and time-delayed Burgers'-Fisher equation have the highest order of the derivative w.r.t. t and x as 2 and 2 respectively. Therefore, the Haar wavelet method outlined in (12)-(30) is common to all the equations of the same order.

3.2. **Time-delayed Generalized Burgers' equation.** Consider the time-delayed one-dimensional generalized Burgers' equation

(37) $$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \lambda u^\alpha \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \;\; 0 \le x \le 1, t \ge 0,$$

with initial and boundary conditions

(38) $$\left.\begin{aligned} u(x,0) &= f_1(x) \\ \frac{\partial u}{\partial t}(x,0) &= f_2(x) \end{aligned}\right\} 0 \le x \le 1,$$

(39) $$\left.\begin{aligned} u(0,t) &= g_1(t) \\ u(1,t) &= g_2(t) \end{aligned}\right\} t \ge 0,$$

where $\tau > 0$ is the time-delay and λ, α are real numbers.

Using the quasilinearization technique [48] to handle the nonlinear terms in equation (37), we have the following scheme
(40)
$$\tau\ddot{u}(x,t_{s+1})+\dot{u}(x,t_{s+1})+\lambda\alpha u^{\alpha-1}(x,t_s)u'(x,t_s)u(x,t_{s+1})$$
$$+\lambda u^{\alpha}(x,t_s)u'(x,t_{s+1})-u''(x,t_{s+1})=\lambda\alpha u^{\alpha}(x,t_s)u'(x,t_s),\ s=0,1,2,\ldots,$$
which leads us from the time level t_s to t_{s+1}.

Taking the collocation points $x\rightarrow x_l$ in equation (31) and using equations (24)-(30), we get

$$\sum_{i=1}^{2M}C_s(i,l)a_s(i)=\chi_s(x_l),\ l=1,2,\ldots,2M,\ s=0,1,2,\ldots, \tag{41}$$

where
(42)
$$\begin{aligned}C_s(i,l)=&\left[\tau+\Delta t+\frac{1}{2}\lambda\alpha(\Delta t)^2u^{\alpha-1}(x_l,t_s)u'(x_l,t_s)\right]q_i(x_l)\\&+\frac{1}{2}\lambda(\Delta t)^2u^{\alpha}(x_l,t_s)p_i(x_l)-\frac{1}{2}(\Delta t)^2h_i(x_l)-\left[x_l\left\{\tau+\Delta t\right.\right.\\&\left.\left.+\frac{1}{2}\lambda\alpha(\Delta t)^2u^{\alpha-1}(x_l,t_s)u'(x_l,t_s)\right\}+\frac{1}{2}\lambda(\Delta t)^2u^{\alpha}(x_l,t_s)\right]q_i(1),\end{aligned}$$

(43)
$$\begin{aligned}\chi_s(x_l)=&\,u''(x_l,t_s)+\Delta t\dot{u}''(x_l,t_s)-\dot{u}(x_l,t_s)-\lambda u^{\alpha}(x_l,t_s)u'(x_l,t_s)\\&-\lambda\alpha\Delta tu^{\alpha-1}(x_l,t_s)u'(x_l,t_s)\dot{u}(x_l,t_s)-\lambda\Delta tu^{\alpha}(x_l,t_s)\dot{u}'(x_l,t_s)\\&-x_l[\tau\ddot{g}_2(t_{s+1})+\dot{g}_2(t_{s+1})-\dot{g}_2(t_s)]-(1-x_l)[\tau\ddot{g}_1(t_{s+1})+\dot{g}_1(t_{s+1})\\&-\dot{g}_1(t_s)]-u^{\alpha-1}(x_l,t_s)[\{\alpha x_lu'(x_l,t_s)+u(x_l,t_s)\}\{g_2(t_{s+1})-g_2(t_s)\\&-\Delta t\dot{g}_2(t_s)\}-\{\alpha(1-x_l)u'(x_l,t_s)-u(x_l,t_s)\}\{g_1(t_{s+1})-g_1(t_s)\\&-\Delta t\dot{g}_1(t_s)\}].\end{aligned}$$

Using equations (35) and (36), the wavelet coefficients $a_s(i)$, $i=1,2,\ldots,2M$ can be determined from the system of equations in (41). The approximate solution of the time-delayed generalized Burgers' equation (37) can be obtained by substituting these coefficients in equations (24)-(29).

3.3. **Time-delayed Burgers'-Fisher equation.** Consider the time-delayed one-dimensional Burgers'-Fisher equation

$$\tau\frac{\partial^2u}{\partial t^2}+(1-\tau)\frac{\partial u}{\partial t}+2\tau u\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial^2u}{\partial x^2}-u+u^2=0,\ 0\le x\le 1,t\ge 0, \tag{44}$$

with initial and boundary conditions

$$\left.\begin{aligned}u(x,0)&=f_1(x)\\\frac{\partial u}{\partial t}(x,0)&=f_2(x)\end{aligned}\right\}0\le x\le 1, \tag{45}$$

$$\left.\begin{aligned}u(0,t)&=g_1(t)\\u(1,t)&=g_2(t)\end{aligned}\right\}t\ge 0, \tag{46}$$

where $\tau > 0$ is the time-delay.

Using the quasilinearization technique [48] to handle the nonlinear terms in equation (9), we have the following scheme

(47)
$$\begin{aligned}\tau\ddot{u}(x,t_{s+1}) + [1-\tau+2\tau u(x,t_s)]\dot{u}(x,t_{s+1}) + u(x,t_s)u'(x,t_{s+1}) + [2\tau\dot{u}(x,t_s)\\ +u'(x,t_s)+2u(x,t_s)-1]u(x,t_{s+1}) - u''(x,t_{s+1}) = 2\tau u(x,t_s)\dot{u}(x,t_s)\\ +u(x,t_s)u'(x,t_s)+u^2(x,t_s),\ s=0,1,2,\ldots,\end{aligned}$$

which leads us from the time level t_s to t_{s+1}.

Taking the collocation points $x \to x_l$ in equation (31) and using equations (24)-(30), we get

$$\sum_{i=1}^{2M} D_s(i,l)a_s(i) = \gamma_s(x_l),\ l=1,2,\ldots,2M,\ s=0,1,2,\ldots, \tag{48}$$

where

(49)
$$\begin{aligned}D_s(i,l) = {}& [\tau+\Delta t\{1-\tau+2\tau u(x_l,t_s)\} + \frac{1}{2}(\Delta t)^2\{2\tau\dot{u}(x_l,t_s)+u'(x_l,t_s)\\ &+2u(x_l,t_s)-1\}]q_i(x_l) + \frac{1}{2}(\Delta t)^2 u(x_l,t_s)p_i(x_l) - \frac{1}{2}(\Delta t)^2 h_i(x_l)\\ &-\Bigg[x_l\Bigg\{\tau+\Delta t[1-\tau+2\tau u(x_l,t_s)] + \frac{1}{2}(\Delta t)^2[2\tau\dot{u}(x_l,t_s)\\ &+u'(x_l,t_s)+2u(x_l,t_s)-1]\Bigg\} + \frac{1}{2}(\Delta t)^2 u(x_l,t_s)\Bigg]q_i(1),\end{aligned}$$

(50)
$$\begin{aligned}\gamma_s(x_l) = {}& u''(x_l,t_s) + \Delta t\dot{u}''(x_l,t_s) + u(x_l,t_s) - (1-\tau-\Delta t)\dot{u}(x_l,t_s)\\ &-u^2(x_l,t_s) - 2\tau\Delta t\dot{u}^2(x_l,t_s) - 2(\tau+\Delta t)u(x_l,t_s)\dot{u}(x_l,t_s)\\ &-\Delta t u'(x_l,t_s)\dot{u}(x_l,t_s) - \Delta t u(x_l,t_s)\dot{u}'(x_l,t_s) - u'(x_l,t_s)u(x_l,t_s)\\ &-x_l[\tau\ddot{g}_2(t_{s+1}) + \{1-\tau+2\tau u(x_l,t_s)\}\{\dot{g}_2(t_{s+1})-\dot{g}_2(t_s)\}]\\ &-(1-x_l)[\tau\ddot{g}_1(t_{s+1}) + \{1-\tau+2\tau u(x_l,t_s)\}\{\dot{g}_1(t_{s+1})-\dot{g}_1(t_s)\}]\\ &-[x_l-\{2\tau\dot{u}(x_l,t_s)+u'(x_l,t_s)+2u(x_l,t_s)-1\}+u(x_l,t_s)]\{g_2(t_{s+1})\\ &-g_2(t_s)-\Delta t\dot{g}_2(t_s)\} - [(1-x_l)\{2\tau\dot{u}(x_l,t_s)+u'(x_l,t_s)+2u(x_l,t_s)\\ &-1\}+u(x_l,t_s)]\{g_1(t_{s+1})-g_1(t_s)-\Delta t\dot{g}_1(t_s)\}.\end{aligned}$$

The approximate solution of the time-delayed Burgers'-Fisher equation (44) at different time levels can be obtained by solving the system of equations in (48) for wavelet coefficients $a_s(i)$, $i=1,2,\ldots,2M$ using equations (35) and (36), and then substituting these coefficients in equations (24)-(29).

4. Error Analysis

In this section, the error analysis of the Haar wavelet method has been discussed. To prove our theoretical arguments described as a theorem, we use the following lemma [47].

Lemma 4.1. *If* $f(x) \in L^2(\mathbb{R})$ *is a continuous function in* $(0,1)$ *with* $|f'(x)| \leq K\ \forall x \in (0,1);\ K > 0$ *and* $f(x) = \sum_{i=1}^{\infty} a_i h_i(x)$, *Then*

$$|a_i| \leq 2^{-(3d+2)/2} K.$$

Theorem 4.2. *If* $u(x, t_{s+1})$ *is the exact solution and* $u_{2M}(x, t_{s+1})$ *is the Haar wavelet solution at* $t = t_{s+1}$, *then*

$$\|E_J\| = \|u(x, t_{s+1}) - u_{2M}(x, t_{s+1})\| \leq \frac{2^{-\frac{1}{2}(J+5)}(\Delta t)^2 K\sqrt{C}}{1 - 2^{-\frac{1}{2}}},$$

where $C, K > 0$, J *is the level of resolution of the wavelet and* $M = 2^J$.

Proof. From equation (20), the Haar wavelet solution at $t = t_{s+1}$ is given by

(51)

$$u_{2M}(x, t_{s+1}) = \frac{1}{2}(\Delta t)^2 \sum_{i=1}^{2M} a_s(i)[q_i(x) - xq_i(1)] + u(x, t_s) + \Delta t \dot{u}(x, t_s) + x[g_2(t_{s+1}) - g_2(t_s) - \Delta t \dot{g}_2(t_s)] + (1-x)[g_1(t_{s+1}) - g_1(t_s) - \Delta t \dot{g}_1(t_s)].$$

Taking the asymptotic expansion of the above equation, we get

(52)

$$u(x, t_{s+1}) = \frac{1}{2}(\Delta t)^2 \sum_{i=1}^{\infty} a_s(i)[q_i(x) - xq_i(1)] + u(x, t_s) + \Delta t \dot{u}(x, t_s) + x[g_2(t_{s+1}) - g_2(t_s) - \Delta t \dot{g}_2(t_s)] + (1-x)[g_1(t_{s+1}) - g_1(t_s) - \Delta t \dot{g}_1(t_s)].$$

The error estimation at J^{th} level of resolution is

(53)

$$\|E_J\| = \|u(x, t_{s+1}) - u_{2M}(x, t_{s+1})\| = \left| \frac{1}{2}(\Delta t)^2 \sum_{i=2M+1}^{\infty} a_s(i)[q_i(x) - xq_i(1)] \right|.$$

$$\begin{aligned} \|E_J\|^2 &= \left| \int_{-\infty}^{\infty} \left\langle \frac{1}{2}(\Delta t)^2 \sum_{i=2M+1}^{\infty} a_s(i)[q_i(x) - xq_i(1)], \frac{1}{2}(\Delta t)^2 \sum_{l=2M+1}^{\infty} a_s(l)[q_l(x) - xq_l(1)] \right\rangle dx \right| \\ &= \left| \frac{1}{4}(\Delta t)^4 \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} a_s(i) a_s(l) \int_0^1 [q_i(x) - xq_i(1)][q_l(x) - xq_l(1)] dx \right| \end{aligned}$$

$$\leq \frac{1}{4}(\Delta t)^4 \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} |a_s(i)||a_s(l)|C$$

$$\text{where } C = \sup_{i,l} \int_0^1 [q_i(x) - xq_i(1)][q_l(x) - xq_l(1)]dx.$$

Thus, we obtain

$$\|E_J\|^2 \leq \frac{1}{4}C(\Delta t)^4 \sum_{i=2M+1}^{\infty} |a_s(i)| \sum_{l=2M+1}^{\infty} |a_s(l)|. \tag{54}$$

Using Lemma 1, we have

$$\sum_{i=2M+1}^{\infty} |a_s(i)| \leq \frac{2^{-\frac{1}{2}(J+3)}K}{1 - 2^{-\frac{1}{2}}}, \tag{55}$$

$$\sum_{l=2M+1}^{\infty} |a_s(l)| \leq \frac{2^{-\frac{1}{2}(J+3)}K}{1 - 2^{-\frac{1}{2}}}. \tag{56}$$

Substituting equations (55) and (56) in equation (54), we obtain

$$\|E_J\|^2 \leq \frac{2^{-(J+5)}(\Delta t)^4 K^2 C}{(1 - 2^{-\frac{1}{2}})^2}. \tag{57}$$

Therefore,

$$\|E_J\| \leq \frac{2^{-\frac{1}{2}(J+5)}(\Delta t)^2 K\sqrt{C}}{1 - 2^{-\frac{1}{2}}}. \tag{58}$$

It is clear from equation (58) that the error bound $\|E_J\| \to 0$ as $J \to \infty$. Hence the accuracy of the Haar wavelet method improves as the level of resolution J of the Haar wavelet is increased. □

Error Estimate: We define the error estimate at $t = t_s$ by

$$\mu(t_s) = \frac{1}{2M} \|u(x, t_s) - u_{ex}(x, t_s)\|, \tag{59}$$

where $u_{ex}(x, t_s)$ is the exact solution at $t = t_s$.

5. Examples and Discussions

In this section, one example each of time-delayed Burgers' equation, time-delayed generalized Burgers' equation, time-delayed Burgers'-Fisher equation are considered and the Haar wavelet collocation method (HWCM) is found to be very efficient and accurate method. Lagrange's interpolation is used to find the solution at specified points. The entire computational work has been done with the help of MATLAB software.

Example 1:

We consider the nonlinear time-delayed one-dimensional Burgers' equation (9) with initial and boundary conditions

$$\left.\begin{aligned} u(x,0) &= \frac{1}{2}\left[1-\tanh\left(\frac{\lambda x}{4-\tau\lambda^2}\right)\right] \\ \frac{\partial u}{\partial t}(x,0) &= \frac{\lambda^2}{4(4-\tau\lambda^2)}\operatorname{sech}^2\left(\frac{\lambda x}{4-\tau\lambda^2}\right) \end{aligned}\right\} 0 \le x \le 1, \tag{60}$$

$$\left.\begin{aligned} u(0,t) &= \frac{1}{2}\left[1+\tanh\left\{\frac{\lambda^2 t}{2(4-\tau\lambda^2)}\right\}\right] \\ u(1,t) &= \frac{1}{2}\left[1-\tanh\left\{\frac{\lambda(2-\lambda t)}{2(4-\tau\lambda^2)}\right\}\right] \end{aligned}\right\} t \ge 0. \tag{61}$$

The exact solution [2] is given by

$$u(x,t) = \frac{1}{2}\left[1-\tanh\left\{\frac{2\lambda}{2(4-\tau\lambda^2)}\left(x-\frac{\lambda}{2}t\right)\right\}\right]. \tag{62}$$

The HWCM solutions are obtained for $\tau = 0.5$ and $\lambda = 0.01$ with $\Delta t = 0.01$, $J = 2$. The results are compared with the exact solution and are presented in Table 1. Figure 1 represents the HWCM solution for $\tau = 0.5$ and $\lambda = 0.01$ at different times t, and its physical behaviour in contour and 3D are displayed in Figure 2. The error estimates obtained for $\tau = 0.5$ and $\lambda = 0.01$ at different J and t with $\Delta t = 0.01$ are presented in Table 2. We observe that the error values are negligibly small which indicate that the Haar wavelet solution is very close to the exact solution.

Example 2:

We consider the nonlinear time-delayed generalized one-dimensional Burgers' equation (37) with the initial and boundary conditions

$$\left.\begin{aligned} u(x,0) &= \sqrt{\frac{1}{2}\left[1-\tanh\left\{\frac{\lambda\alpha(\alpha+1)x}{2((\alpha+1)^2-\tau\lambda^2)}\right\}\right]} \\ \frac{\partial u}{\partial t}(x,0) &= \frac{\lambda^2\alpha\operatorname{sech}^2\left\{\frac{\lambda\alpha(\alpha+1)x}{2((\alpha+1)^2-\tau\lambda^2)}\right\}}{4((\alpha+1)^2-\tau\lambda^2)\sqrt{2\left[1-\tanh\left\{\frac{\lambda\alpha(\alpha+1)x}{2((\alpha+1)^2-\tau\lambda^2)}\right\}\right]}} \end{aligned}\right\} 0 \le x \le 1, \tag{63}$$

$$\left.\begin{aligned} u(0,t) &= \sqrt{\frac{1}{2}\left[1+\tanh\left\{\frac{\lambda^2\alpha t}{2((\alpha+1)^2-\tau\lambda^2)}\right\}\right]} \\ u(1,t) &= \sqrt{\frac{1}{2}\left[1-\tanh\left\{\frac{\lambda\alpha(\alpha+1)}{2((\alpha+1)^2-\tau\lambda^2)}\left(1-\frac{\lambda}{\alpha+1}t\right)\right\}\right]} \end{aligned}\right\} t \ge 0. \tag{64}$$

The exact solution [2] is given by

$$u(x,t) = \sqrt{\frac{1}{2}\left[1-\tanh\left\{\frac{\lambda\alpha(\alpha+1)}{2((\alpha+1)^2-\tau\lambda^2)}\left(x-\frac{\lambda}{\alpha+1}t\right)\right\}\right]}. \tag{65}$$

Tables 3-5 display the comparison of the HWCM solution and exact solution for $\tau = 0.5,\ \lambda = 0.01$ and $\alpha = 2, 3, 5$ with $\Delta t = 0.01,\ J = 2$. It is observed that the HWCM solution are in good agreement with the exact solution. The HWCM solution for $\tau = 0.5,\ \lambda = 0.01$ and $\alpha = 2, 3, 5$ at different times t are presented in Figures 3-5, and the physical behavior of the solution in contour and 3D are depicted in Figures 6-8. The error estimates obtained for $\tau = 0.5,\ \lambda = 0.01$ and $\alpha = 2, 3, 5$ at different J and t with $\Delta t = 0.01$ are presented in Tables 6-8.

Example 3:

We consider the nonlinear time-delayed one-dimensional Burgers'-Fisher equation (44) with the initial and boundary conditions

$$\left.\begin{aligned} u(x,0) &= \frac{1}{2}\left[1+\tanh\left\{\left(\frac{\tau+1}{\tau-4}\right)x\right\}\right] \\ \frac{\partial u}{\partial t}(x,0) &= -\frac{5}{4(\tau-4)}\,\mathrm{sech}^2\left\{\left(\frac{\tau+1}{\tau-4}\right)x\right\} \end{aligned}\right\} 0 \le x \le 1, \tag{66}$$

$$\left.\begin{aligned} u(0,t) &= \frac{1}{2}\left[1-\tanh\left\{\frac{5}{2(\tau-4)}t\right\}\right] \\ u(1,t) &= \frac{1}{2}\left[1+\tanh\left\{\frac{\tau+1}{\tau-4}\left(1-\frac{5}{2(\tau+1)}t\right)\right\}\right] \end{aligned}\right\} t \ge 0. \tag{67}$$

The exact solution [2] is given by

$$u(x,t) = \frac{1}{2}\left[1+\tanh\left\{\frac{\tau+1}{\tau-4}\left(x-\frac{5}{2(\tau+1)}t\right)\right\}\right]. \tag{68}$$

The comparison of the HWCM solution and the exact solution for $\tau = 0.5$ with $\Delta t = 0.01,\ J = 2$ are presented in Table 9. It is seen from the table that the HWCM solution matches approximately with the exact solution. The error estimates obtained for $\tau = 0.5$ at different J and t with $\Delta t = 0.01$ are displayed in Table 10. The HWCM solution for $\tau = 0.5$ at different t are shown in Figure 9. Figure 10 represents the physical behaviour of the HWCM solution in contour and 3D for $\tau = 0.5$.

6. Conclusion

In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve time-delayed Burgers' equations. The numerical scheme is tested for three examples. The obtained numerical results are compared with the exact solutions. We observe that the error values are negligibly small which indicate that the HWCM solution is very close to the exact solution. Thus the Haar wavelet method guarantees the necessary accuracy with a small number of grid points and a wide class of PDEs can be solved using this approach. This method takes care of boundary conditions automatically and hence it is the most convenient method for solving boundary value problems. The error analysis of the Haar wavelet method is carried out which shows that the accuracy of the method improves as the level of resolutions of the Haar wavelet are increased.

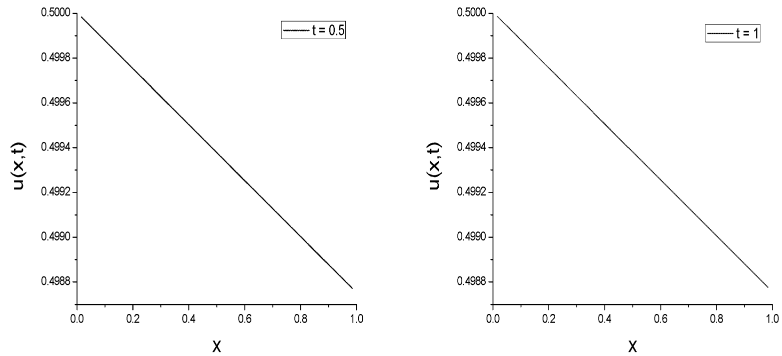

FIGURE 1: HWCM solution of Example 1 for $\tau = 0.5$ and $\lambda = 0.01$ at $t = 0.5, 1.0$.

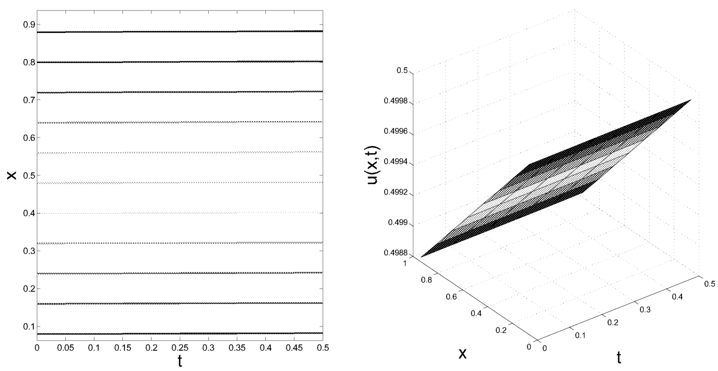

FIGURE 2: Physical behaviour of the HWCM solution of Example 1 for $\tau = 0.5$ and $\lambda = 0.01$.

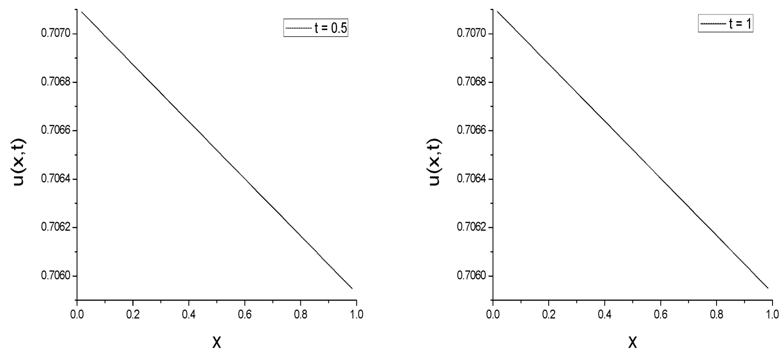

FIGURE 3: HWCM solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 2$ at $t = 0.5, 1.0$.

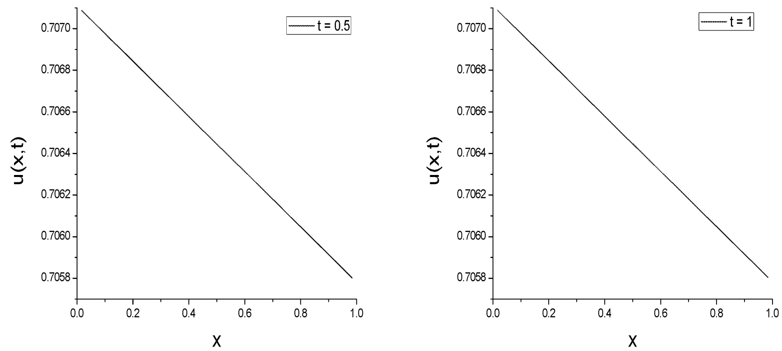

FIGURE 4: HWCM solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 3$ at $t = 0.5, 1.0$.

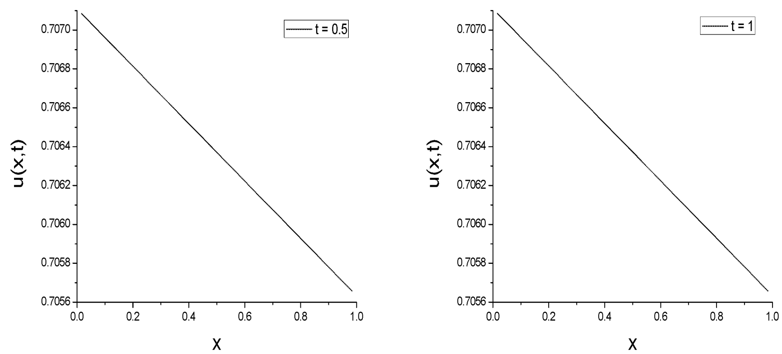

FIGURE 5: HWCM solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 5$ at $t = 0.5, 1.0$.

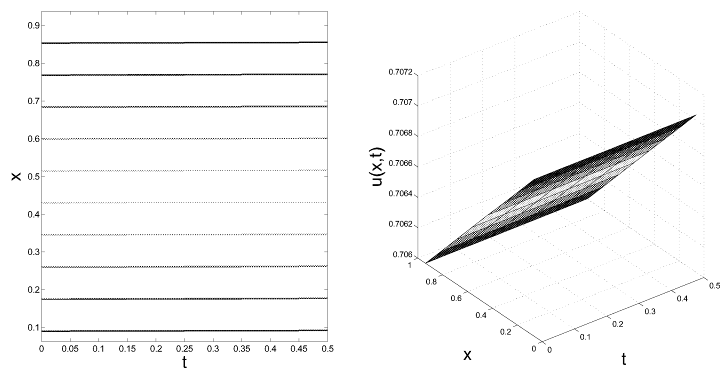

FIGURE 6: Physical behaviour of the HWCM solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 2$.

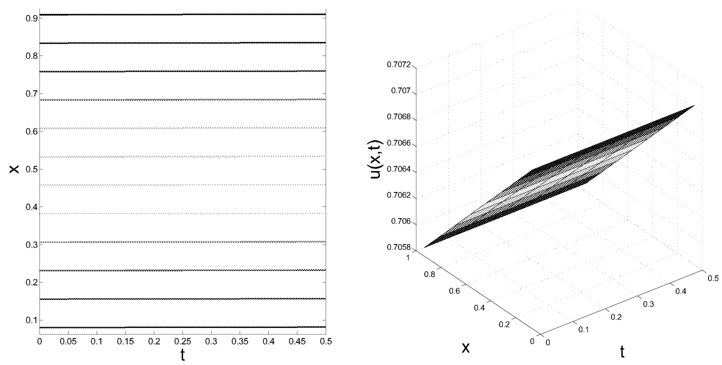

FIGURE 7: Physical behaviour of the HWCM solution of Example 2 for $\tau = 0.5,\ \lambda = 0.01$ and $\alpha = 3$.

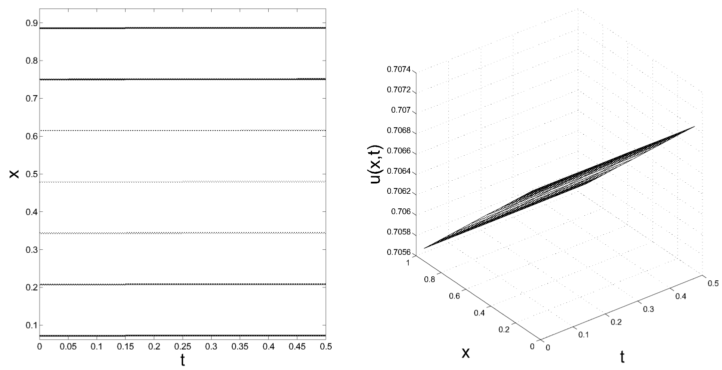

FIGURE 8: Physical behaviour of the HWCM solution of Example 2 for $\tau = 0.5,\ \lambda = 0.01$ and $\alpha = 5$.

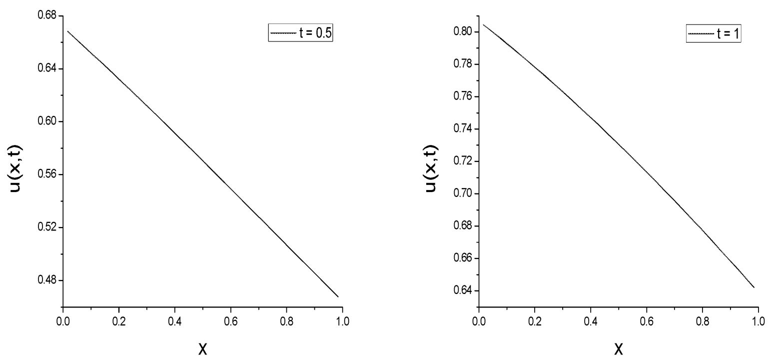

FIGURE 9: HWCM solution of Example 3 for $\tau = 0.5$ at $t = 0.5, 1.0$.

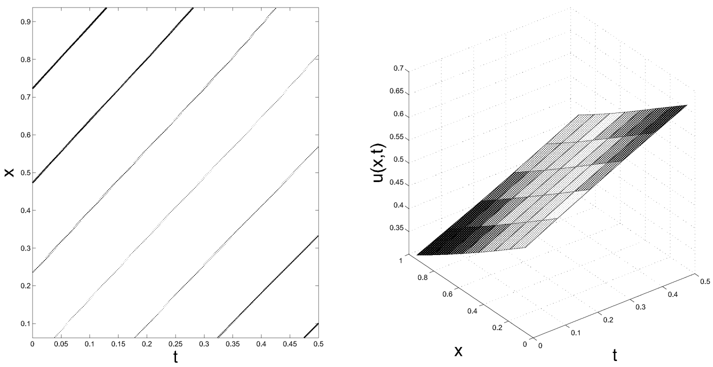

FIGURE 10: Physical behaviour of the HWCM solution of Example 3 for $\tau = 0.5$.

TABLE 1: Comparison of the HWCM solution and exact solution of Example 1 for $\tau = 0.5$ and $\lambda = 0.01$ at different t and x.

x	$u(x,t)$			
	HWCM	Exact	HWCM	Exact
	$t = 0.2$		$t = 0.6$	
0.2	0.49975124691150	0.49975124691111	0.49975374694678	0.49975374694175
0.4	0.49950124393138	0.49950124393097	0.49950374396708	0.49950374395975
0.6	0.49925124120062	0.49925124120022	0.49925374123322	0.49925374122588
0.8	0.49900123884425	0.49900123884386	0.49900373887020	0.49900373886516
	$t = 1.0$		$t = 3.0$	
0.2	0.49975624698432	0.49975624697240	0.49976874716212	0.49976874712583
0.4	0.49950624400641	0.49950624398855	0.49951874418725	0.49951874413292
0.6	0.49925624126945	0.49925624125158	0.49926874143499	0.49926874138064
0.8	0.49900623889844	0.49900623888651	0.49901873903033	0.49901873899400
	$t = 5.0$		$t = 8.0$	
0.2	0.49978124734081	0.49978124727955	0.49979999760939	0.49979999751064
0.4	0.49953124436968	0.49953124427789	0.49954999464449	0.49954999449643
0.6	0.49928124160243	0.49928124151061	0.49929999185534	0.49929999170724
0.8	0.49903123916403	0.49903123910272	0.49904998936691	0.49904998926806
	$t = 10.0$		$t = 15.0$	
0.2	0.49981249778875	0.49981249766501	0.49984374823815	0.49984374805194
0.4	0.49956249482839	0.49956249464284	0.49959374529050	0.49959374501121
0.6	0.49931249202504	0.49931249183943	0.49934374245300	0.49934374217362
0.8	0.49906248950364	0.49906248937978	0.49909373985056	0.49909373966416

TABLE 2: Error in the solution of Example 1 for $\tau = 0.5$ and $\lambda = 0.01$ at different t and J.

J	$\mu(t)$					
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	$t = 0.2$		$t = 0.6$		$t = 1.0$	
1	1.846E-13	1.024E-13	3.269E-12	1.926E-12	1.206E-11	7.159E-12
2	1.452E-13	7.414E-14	2.857E-12	1.816E-12	8.940E-12	4.380E-12
3	1.040E-13	4.291E-14	2.338E-12	9.825E-13	6.837E-12	3.699E-12
4	7.383E-14	2.310E-14	1.669E-12	5.088E-13	6.528E-12	1.951E-12
5	5.227E-14	1.201E-14	1.189E-12	2.615E-13	4.708E-12	1.013E-12
6	3.698E-14	6.136E-15	8.443E-13	1.330E-13	3.367E-12	5.180E-13
7	2.616E-14	3.105E-15	5.985E-13	6.711E-14	2.395E-12	2.621E-13
8	1.850E-14	1.562E-15	4.238E-13	3.372E-14	1.698E-12	1.319E-13
	$t = 5.0$		$t = 10.0$		$t = 15.0$	
1	1.941E-10	8.392E-11	5.526E-10	1.765E-10	9.979E-10	2.691E-10
2	1.834E-10	7.518E-11	5.117E-10	1.694E-10	9.787E-10	2.637E-10
3	1.743E-10	5.734E-11	5.111E-10	1.509E-10	8.816E-10	2.453E-10
4	1.413E-10	3.701E-11	4.211E-10	1.172E-10	8.312E-10	2.076E-10
5	1.404E-10	2.243E-11	4.134E-10	7.752E-11	7.048E-10	1.511E-10
6	1.063E-10	2.130E-11	3.227E-10	4.537E-11	6.434E-10	9.494E-11
7	7.786E-11	1.147E-11	2.972E-10	4.533E-11	4.530E-10	6.822E-11
8	3.501E-11	5.954E-12	7.074E-11	2.465E-11	1.065E-10	5.382E-11

TABLE 3: Comparison of the HWCM solution and exact solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 2$ at different t and x.

x	$u(x,t)$			
	HWCM	Exact	HWCM	Exact
	$t = 0.2$		$t = 0.6$	
0.2	0.70687182429426	0.70687182429418	0.70687339617223	0.70687339617118
0.4	0.70663600328607	0.70663600328598	0.70663757568700	0.70663757568547
0.6	0.70640010397135	0.70640010397126	0.70640167689389	0.70640167689235
0.8	0.70616412648142	0.70616412648133	0.70616569992422	0.70616569992316
	$t = 1.0$		$t = 3.0$	
0.2	0.70687496804719	0.70687496804469	0.70688282736757	0.70688282735996
0.4	0.70663914808521	0.70663914808147	0.70664701002066	0.70664701000927
0.6	0.70640324981372	0.70640324980997	0.70641111435733	0.70641111434593
0.8	0.70616727336402	0.70616727336151	0.70617514050888	0.70617514050126
	$t = 5.0$		$t = 8.0$	
0.2	0.70689068660091	0.70689068658806	0.70690247528749	0.70690247526679
0.4	0.70665487186930	0.70665487185006	0.70666666447912	0.70666666444808
0.6	0.70641897881428	0.70641897879503	0.70643077533684	0.70643077530578
0.8	0.70618300756714	0.70618300755428	0.70619480799193	0.70619480797120
	$t = 10.0$		$t = 15.0$	
0.2	0.70691033430292	0.70691033427697	0.70692998146007	0.70692998142103
0.4	0.70667452611021	0.70667452607131	0.70669417980717	0.70669417974862
0.6	0.70643863957660	0.70643863953769	0.70645829979589	0.70645829973731
0.8	0.70620267483338	0.70620267480740	0.70622234155749	0.70622234151839

TABLE 4: Comparison of the HWCM solution and exact solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 3$ at different t and x.

x	$u(x,t)$			
	HWCM	Exact	HWCM	Exact
	$t = 0.2$		$t = 0.6$	
0.2	0.7068421935	0.7068422288	0.7068434330	0.7068435551
0.4	0.7065768752	0.7065769142	0.7065780548	0.7065782410
0.6	0.7063114616	0.7063115006	0.7063126416	0.7063128279
0.8	0.7060459528	0.7060459881	0.7060471937	0.7060473159
	$t = 1.0$		$t = 3.0$	
0.2	0.7068447749	0.7068448814	0.7068514277	0.7068515130
0.4	0.7065794042	0.7065795679	0.7065860746	0.7065862019
0.6	0.7063139915	0.7063141552	0.7063206644	0.7063207917
0.8	0.7060485371	0.7060486437	0.7060551974	0.7060552827
	$t = 5.0$		$t = 8.0$	
0.2	0.7068580551	0.7068581445	0.7068680026	0.7068680917
0.4	0.7065927017	0.7065928359	0.7066026532	0.7066027868
0.6	0.7063272940	0.7063274282	0.7063372492	0.7063373828
0.8	0.7060618322	0.7060619216	0.7060717908	0.7060718799
	$t = 10.0$		$t = 15.0$	
0.2	0.7068746340	0.7068747230	0.7068912121	0.7068913011
0.4	0.7066092871	0.7066094206	0.7066258714	0.7066260049
0.6	0.7063438855	0.7063440191	0.7063604760	0.7063606096
0.8	0.7060784296	0.7060785187	0.7060950263	0.7060951154

TABLE 5: Comparison of the HWCM solution and exact solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 5$ at different t and x.

x	$u(x,t)$			
	HWCM	Exact	HWCM	Exact
	$t = 0.2$		$t = 0.6$	
0.2	0.7068124887	0.7068125829	0.7068132414	0.7068135654
0.4	0.7065176685	0.7065177710	0.7065182603	0.7065187539
0.6	0.7062227344	0.7062228369	0.7062233265	0.7062238202
0.8	0.7059276867	0.7059277807	0.7059284404	0.7059287645
	$t = 1.0$		$t = 3.0$	
0.2	0.7068142699	0.7068145479	0.7068192375	0.7068194603
0.4	0.7065193076	0.7065197368	0.7065243192	0.7065246513
0.6	0.7062243743	0.7062248035	0.7062293879	0.7062297201
0.8	0.7059294703	0.7059297482	0.7059344440	0.7059346668
	$t = 5.0$		$t = 8.0$	
0.2	0.7068241381	0.7068243728	0.7068315079	0.7068317414
0.4	0.7065292137	0.7065295658	0.7065365873	0.7065369375
0.6	0.7062342845	0.7062346366	0.7062416611	0.7062420113
0.8	0.7059393506	0.7059395853	0.7059467296	0.7059469631
	$t = 10.0$		$t = 15.0$	
0.2	0.7068364204	0.7068366538	0.7068487011	0.7068489345
0.4	0.7065415018	0.7065418519	0.7065537877	0.7065541377
0.6	0.7062465777	0.7062469277	0.7062588686	0.7062592187
0.8	0.7059516482	0.7059518816	0.7059639442	0.7059641776

TABLE 6: Error in the solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 2$ at different t and J.

J	$\mu(t)$					
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	$t = 0.2$		$t = 0.6$		$t = 1.0$	
1	3.867E-14	2.151E-14	5.992E-13	3.807E-13	1.434E-12	9.184E-13
2	2.663E-14	1.062E-14	4.178E-13	1.967E-13	1.008E-12	4.800E-13
3	1.876E-14	5.287E-15	2.944E-13	9.909E-14	7.118E-13	2.426E-13
4	1.324E-14	2.651E-15	2.080E-13	4.965E-14	5.032E-13	1.216E-13
5	9.350E-15	1.327E-15	1.471E-13	2.483E-14	3.558E-13	6.084E-14
6	6.609E-15	6.627E-16	1.040E-13	1.241E-14	2.516E-13	3.042E-14
7	4.675E-15	3.318E-16	7.354E-14	6.209E-15	1.779E-13	1.521E-14
8	3.306E-15	1.659E-16	5.200E-14	3.104E-15	1.258E-13	7.606E-15
	$t = 5.0$		$t = 10.0$		$t = 15.0$	
1	7.340E-12	4.702E-12	1.483E-11	9.504E-12	2.232E-11	1.430E-11
2	5.180E-12	2.467E-12	1.047E-11	4.987E-12	1.576E-11	7.507E-12
3	3.662E-12	1.248E-12	7.401E-12	2.523E-12	1.114E-11	3.798E-12
4	2.589E-12	6.258E-13	5.233E-12	1.265E-12	7.877E-12	1.905E-12
5	1.831E-12	3.131E-13	3.701E-12	6.330E-13	5.570E-12	9.529E-13
6	1.295E-12	1.566E-13	2.617E-12	3.166E-13	3.938E-12	4.765E-13
7	9.155E-13	7.829E-14	1.850E-12	1.583E-13	2.785E-12	2.383E-13
8	6.473E-13	3.915E-14	1.308E-12	7.915E-14	1.969E-12	1.191E-13

TABLE 7: Error in the solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 3$ at different t and J.

J	$\mu(t)$					
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	$t = 0.2$		$t = 0.6$		$t = 1.0$	
1	1.709E-08	9.899E-09	7.110E-08	4.577E-08	6.124E-08	3.947E-08
2	1.180E-08	4.879E-09	4.984E-08	2.391E-08	4.371E-08	2.104E-08
3	8.300E-09	2.428E-09	3.517E-08	1.209E-08	3.099E-08	1.070E-08
4	5.862E-09	1.212E-09	2.486E-08	6.064E-09	2.193E-08	5.374E-09
5	4.144E-09	6.062E-10	1.758E-08	3.034E-09	1.551E-08	2.690E-09
6	2.930E-09	3.031E-10	1.243E-08	1.517E-09	1.097E-08	1.345E-09
7	2.072E-09	1.516E-10	8.787E-09	7.586E-10	7.754E-09	6.727E-10
8	1.465E-09	7.578E-11	6.213E-09	3.793E-10	5.483E-09	3.363E-10
	$t = 5.0$		$t = 10.0$		$t = 15.0$	
1	5.115E-08	3.278E-08	5.088E-08	3.261E-08	5.087E-08	3.260E-08
2	3.609E-08	1.720E-08	3.592E-08	1.712E-08	3.592E-08	1.712E-08
3	2.551E-08	8.700E-09	2.540E-08	8.661E-09	2.539E-08	8.659E-09
4	1.804E-08	4.362E-09	1.796E-08	4.343E-09	1.796E-08	4.342E-09
5	1.275E-08	2.183E-09	1.270E-08	2.173E-09	1.270E-08	2.173E-09
6	9.019E-09	1.092E-09	8.979E-09	1.087E-09	8.978E-09	1.087E-09
7	6.377E-09	5.458E-10	6.349E-09	5.434E-10	6.348E-09	5.433E-10
8	4.509E-09	2.729E-10	4.490E-09	2.717E-10	4.489E-09	2.717E-10

TABLE 8: Error in the solution of Example 2 for $\tau = 0.5$, $\lambda = 0.01$ and $\alpha = 5$ at different t and J.

J	$\mu(t)$					
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	$t = 0.2$		$t = 0.6$		$t = 1.0$	
1	4.478E-08	2.595E-08	1.863E-07	1.199E-07	1.605E-07	1.034E-07
2	3.092E-08	1.279E-08	1.306E-07	6.265E-08	1.145E-07	5.513E-08
3	2.175E-08	6.364E-09	9.218E-08	3.169E-08	8.121E-08	2.805E-08
4	1.536E-08	3.178E-09	6.515E-08	1.589E-08	5.746E-08	1.408E-08
5	1.086E-08	1.589E-09	4.606E-08	7.951E-09	4.064E-08	7.049E-09
6	7.679E-09	7.944E-10	3.257E-08	3.976E-09	2.874E-08	3.526E-09
7	5.430E-09	3.972E-10	2.303E-08	1.988E-09	2.032E-08	1.763E-09
8	3.839E-09	1.986E-10	1.628E-08	9.941E-10	1.437E-08	8.815E-10
	$t = 5.0$		$t = 10.0$		$t = 15.0$	
1	1.341E-07	8.591E-08	1.334E-07	8.546E-08	1.334E-07	8.546E-08
2	9.459E-08	4.508E-08	9.416E-08	4.487E-08	9.416E-08	4.487E-08
3	6.686E-08	2.280E-08	6.658E-08	2.270E-08	6.657E-08	2.270E-08
4	4.728E-08	1.143E-08	4.708E-08	1.138E-08	4.708E-08	1.138E-08
5	3.343E-08	5.721E-09	3.329E-08	5.696E-09	3.329E-08	5.696E-09
6	2.364E-08	2.861E-09	2.354E-08	2.849E-09	2.354E-08	2.849E-09
7	1.671E-08	1.431E-09	1.664E-08	1.424E-09	1.664E-08	1.424E-09
8	1.182E-08	7.153E-10	1.177E-08	7.122E-10	1.177E-08	7.122E-10

TABLE 9: Comparison of the HWCM solution and exact solution of Example 3 for $\tau = 0.5$ at different t and x.

x	$u(x,t)$			
	HWCM	Exact	HWCM	Exact
	$t = 0.2$		$t = 0.6$	
0.2	0.5285287105	0.5285403710	0.6649772844	0.6650128711
0.4	0.4857053731	0.4857181717	0.6257523595	0.6258106073
0.6	0.4430921294	0.4431046351	0.5848222656	0.5848843897
0.8	0.4013012880	0.4013123399	0.5427095409	0.5427524941
	$t = 1.0$		$t = 3.0$	
0.2	0.7785086950	0.7785338745	0.9839252402	0.9839250207
0.4	0.7475315533	0.7475738349	0.9809763770	0.9809761957
0.6	0.7138310454	0.7138763513	0.9774986786	0.9774988033
0.8	0.6775891090	0.6776203185	0.9734026100	0.9734030064
	$t = 5.0$		$t = 8.0$	
0.2	0.9990652104	0.9990625681	0.9999915241	0.9999870854
0.4	0.9988915199	0.9988874640	0.9999915516	0.9999846705
0.6	0.9986838004	0.9986796951	0.9999888402	0.9999818039
0.8	0.9984359381	0.9984331858	0.9999831675	0.9999784013
	$t = 10.0$		$t = 15.0$	
0.2	1.0000048040	0.9999992583	1.0000083087	0.9999999994
0.4	1.0000077297	0.9999991196	1.0000129247	0.9999999993
0.6	1.0000077702	0.9999989549	1.0000132537	0.9999999992
0.8	1.0000047367	0.9999987595	1.0000089978	0.9999999990

TABLE 10: Error in the solution of Example 3 for $\tau = 0.5$ at different t and J.

J	$\mu(t)$					
	L_2	L_∞	L_2	L_∞	L_2	L_∞
	$t = 0.2$		$t = 0.6$		$t = 1.0$	
1	5.439E-05	3.225E-05	2.259E-04	1.513E-04	1.608E-04	1.087E-04
2	3.762E-05	1.591E-05	1.584E-04	7.744E-05	1.170E-04	5.751E-05
3	2.648E-05	7.923E-06	1.118E-04	3.880E-05	8.330E-05	2.907E-05
4	1.870E-05	3.955E-06	7.898E-05	1.939E-05	5.901E-05	1.456E-05
5	1.322E-05	1.977E-06	5.584E-05	9.697E-06	4.174E-05	7.286E-06
6	9.347E-06	9.884E-07	3.949E-05	4.848E-06	2.952E-05	3.643E-06
7	6.609E-06	4.942E-07	2.792E-05	2.424E-06	2.087E-05	1.822E-06
8	4.673E-06	2.471E-07	1.974E-05	1.212E-06	1.476E-05	9.109E-07
	$t = 5.0$		$t = 10.0$		$t = 15.0$	
1	9.872E-05	6.574E-05	1.995E-04	1.333E-04	2.995E-04	2.003E-04
2	1.771E-05	8.540E-06	3.689E-05	1.784E-05	5.541E-05	2.681E-05
3	2.803E-06	9.606E-07	6.594E-06	2.266E-06	9.906E-06	3.405E-06
4	2.431E-07	6.035E-08	1.169E-06	2.841E-07	1.756E-06	4.269E-07
5	1.402E-07	2.481E-08	2.066E-07	3.550E-08	3.106E-07	5.338E-08
6	1.533E-07	1.886E-08	3.645E-08	4.430E-09	5.492E-08	6.675E-09
7	1.180E-07	1.024E-08	6.388E-09	5.490E-10	9.710E-09	8.344E-10
8	8.515E-08	5.224E-09	1.090E-09	6.625E-11	1.716E-09	1.043E-10

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