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NOT NECESSARILY CONTINUOUS LOCALLY BOUNDED FINITE-DIMENSIONAL REPRESENTATIONS OF CONNECTED LIE GROUPS WITH INDECOMPOSABLE RESTRICTION TO THE RADICAL

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Abstract. Since an analog of the Weyl complete reducibility theorem for not necessarily continuous representations of connected Lie groups is now known, we consider the "opposite" case and find (in terms of representations of the radical and a Levi subgroup of the group) an explicit form for every (not necessarily continuous) locally bounded finite-dimensional representation of a connected Lie group whose restriction to the radical is indecomposable.

§ 1. Introduction

Since an analog of the Weyl complete reducibility theorem for not necessarily continuous representations of connected Lie groups is now known [1], we consider here the "opposite" case and find (in terms of representations of the radical and a Levi subgroup of the group) an explicit form for every (not necessarily continuous) locally bounded finite-dimensional representation, of a connected Lie group, whose restriction to the radical is indecomposable.

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§ 2. Preliminaries

Let us cite the theorem claiming a generalization of the well-known Lie theorem concerning continuous finite-dimensional representations of any solvable Lie group to the case of not necessary continuous representations (see Theorem 2.1 of [2]).

Theorem 1. *Let G be a solvable group, and let every commutative quotient group H in the composition series of G be generated by a divisible set. Let π be a representation of G in a finite-dimensional complex linear space E. If the representation π is irreducible, then the representation space E of π is one-dimensional.*

As usual, this assertion implies the following obvious corollary.

Corollary 1. *Let G be a solvable group, and let every commutative quotient group H in the composition series of G be generated by a divisible set. Let π be a representation of G in a finite-dimensional complex linear space E. There is a basis in the representation space E of π for which all operators of the representation π have upper triangular form.*

§ 3. Main theorem

Theorem 2. *Let G be a connected Lie group, let R be the radical of G, let L be a Levi subgroup of G, and let π be a (not necessarily continuous) locally bounded finite-dimensional representation of G in a space E whose restriction to the radical R is indecomposable. Then there is a (not necessarily continuous) character χ of R satisfying the condition*

$$\chi(grg^{-1}) = \chi(r) \quad \text{for every} \quad g \in G, r \in R, \tag{1}$$

an (automatically continuous) indecomposable unipotent representation θ of the radical R, and an (automatically continuous) representation ρ of L on E such that

$$\theta(l^{-1}rl) = \rho(l)^{-1}\theta(r)\rho(l) \quad \text{for every} \quad g = lr, \quad l \in L, \quad r \in R,$$

and

$$\pi(g) = \pi(lr) = \chi(r)\rho(l)\theta(r) \quad \text{for every} \quad g = lr, \quad l \in L, \quad r \in R. \tag{2}$$

Proof. It follows from the condition of the theorem that there are representations σ of R and ρ of L such that σ and ρ are the restrictions of π to R

and L, respectively. The representation σ has an irreducible subrepresentation. By Theorem 1, this irreducible subrepresentation is one-dimensional, and hence there is a one-dimensional subspace $L \subset E$ invariant with respect to the restriction σ of the representation π to R. Since G/R is a Lie group, it is generated by a neighborhood $U \subset G/R$ of the identity element, and U may be assumed to be so small that the exponential mapping is a homeomorphism of the local preimage of U in the Lie algebra of G/R onto U. Therefore, U is a divisible generating set for G/R. Thus, by the corollary to Theorem 1, there is a basis in E in which all operators $\pi(r) = \sigma(r)$, $r \in R$, have upper triangular form. Clearly, the diagonal matrix entries of the matrices of the operators $\sigma(r)$, $r \in R$, are (not necessarily continuous) characters χ of R. If there are at least two distinct characters, then the representation σ is automatically reducible, which contradicts the assumption; therefore, there is only one character of the form χ. Since the character $\chi^g(r) = \chi(grg^{-1})$, $r \in R$, $g \in G$, is to be another diagonal character, it follows that $\chi^g = \chi$ for every $g \in G$, i.e., $\chi(grg^{-1}) = \chi(r)$ for every $r \in R$ and $g \in G$. This means that χ is the so-called central character. In turn, this means that the mapping $\psi\colon G \to \mathbb{C}$ defined by the formula

$$\psi(g) = \psi(lr) = \chi(r), \qquad g \in G, \quad g = lr, \quad l \in L, \quad r \in R,$$

is a (one-dimensional) representation of G because

$$\begin{aligned} \psi(l_1 r_1 l_2 r_2) &= \psi(l_1 l_2 l_2^{-1} r_1 l_2 r_2) = \chi(l_2^{-1} r_1 l_2 r_2) = \chi(l_2^{-1} r_1 l_2)\chi(r_2) \\ &= \chi(r_1)\chi(r_2) = \chi(l_1 r_1)\chi(l_2 r_2), \qquad l_1, l_2 \in L, \quad r_1, r_2 \in R. \end{aligned} \tag{3}$$

The representation ψ need not be continuous. However, the representation

$$\theta(r) = \chi(r)^{-1}\sigma(r), \qquad r \in R,$$

is obviously unipotent. Since the θ-invariant subspaces and the σ-invariant subspaces coincide, it follows that θ is indecomposable. Since π is locally bounded, it follows that so is θ. However, a locally bounded finite-dimensional unipotent representation is automatically continuous [2–6].

Let us sum up. The above consideration proves that π is given by (2). Formula (1) is proved by the direct verification, as was done (in the one-dimensional situation) in (3). The representation ρ is automatically continuous by [2–6]. This completes the proof of the theorem.

§ 4. Concluding remarks

Thus, we have proved that formula (2) gives the general form of every (not necessarily continuous) locally bounded finite-dimensional representation of a connected Lie group with indecomposable restriction to the radical.

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