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ON GRADED COPRIMELY PACKED MODULES

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded coprimely packed modules and we give a number of results concerning such graded modules. In fact, our objective is to investigate graded coprimely packed modules and examine in particular when graded R-modules are graded coprimely packed. For example, we give a characterization of graded coprimely packed modules. Also, the relations between graded coprimely packed modules and graded compactly packed modules are studied.

1. Introduction

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, in particular, we are dealing with graded coprimely packed submodules.

Recently, Uregen, Tekir and Oral in [24] studied graded compactly packed rings and graded coprimely packed rings. We generalize their approach introducing and studying the concept of graded coprimely packed submodules.

Our article is organized as follows.

In Section 2 we recall important notions which will be used throughout the paper. In Section 3 we give a definition and a characterization of graded coprimely packed modules (see Theorem 3.2). We also study the behaviour of graded coprimely packed modules under graded homomorphisms. In Section 4 we establish sufficient conditions for being a graded coprimely packed module.

2. Preliminaries

Convention. Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [14], [18], [19] and [20] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group and e denote the identity element of G. A ring R is called a *graded ring* (or *G-graded ring*) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_k \subseteq R_{gk}$ for all $g, k \in G$. The elements of R_g are called *homogeneous*

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of degree g and all the homogeneous elements are denoted by $h(R)$, i.e., $h(R) = \cup_{g\in G}R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G}x_g$, where x_g is called a *homogeneous component of* x *in* R_g.

Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. An ideal I of R is said to be a *graded ideal* if $I = \oplus_{g\in G}I_g$ where $I_g = I \cap R_g$ for all $g \in G$. If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \underset{g\in G}{\oplus}(R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$.

A right R-module M is said to be a *graded R-module* (or *G-graded R-module*) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G}M_g$ and $M_gR_k \subseteq M_{gk}$ for all $g, k \in G$. Also if an element of M belongs to $\cup_{g\in G}M_g =: h(M)$, then it is called *homogeneous*.

Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module. A submodule N of M is said to be a *graded submodule* if $N = \oplus_{g\in G}N_g$ where $N_g = N \cap M_g$ for all $g \in G$. In this case, N_g is called the *g-component of* N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$.

We will also use the well known fact (see [12]) that if N, N_1, N_2 are graded submodules of a graded R-module M such that $N \subseteq N_1 \cup N_2$, then $N \subseteq N_1$ or $N \subseteq N_2$.

We call a graded R-module M *graded semisimple* if every graded submodule N of M is a direct summand such that $M = N \oplus P$ for some graded submodule P of M. M is called *faithful* in case $(\{0\} :_R M) = \{0\}$.

Graded prime ideals in a commutative graded ring have been introduced and studied in [22, 24]. A proper graded ideal P of R is said to be *a graded prime ideal of* R if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$. Also the finite union of graded prime ideals was studied in [24].

Similarly, graded prime submodules of graded modules over a graded commutative rings have been introduced and studied in [1–5, 7–9, 12, 21]. A proper graded submodule N of a graded module M over a G-graded ring R is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M)$ or $m \in N$. Recall that the finite union of graded prime submodules was studied in [12].

The *graded radical of a graded submodule N of a graded module M*, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M then $Gr_M(N) = M$, (see [8]). Recall that a graded module M is graded semisimple if and only if $Gr_M(M) = \{0\}$.

A graded submodule N of a graded R-module M is said to be *graded maximal* if $N \neq M$ and if there is a graded submodule K of M such that $N \subseteq K \subseteq M$, then $N = K$ or $K = M$. The set of all graded maximal submodules of M is denoted by $G\text{-}Max(M)$. In any graded multiplication module, every graded submodule is contained in a graded maximal submodule, (see [21]).

We call a graded R-module M *graded semilocal* if it has only finitely many graded maximal submodules.

Graded multiplication modules (gr-multiplication modules) over a commutative graded rings have been introduced and studied in [4, 6, 10, 11, 15–17, 25]. A graded R-module M over G-graded ring R is said to be *graded multiplication module* (*gr-multiplication module*) if for every graded submodule

N of M there exists a graded ideal I of R such that $N = IM$. It is clear that M is gr-multiplication R- module if and only if $N = (N :_R M)M$ for every graded submodule N of M.

The concept of graded compactly packed modules was introduced by Farzalipour and Ghiasvand in [12]. A proper graded submodule N of a graded R-module M is said to be *graded compactly packed by graded prime submodules* if for each family $\{N_\alpha\}_{\alpha\in\triangle}$ of graded prime submodules of M such that $N \subseteq \bigcup\limits_{\alpha\in\triangle} N_\alpha$, we have that $N \subseteq N_\beta$ for some $\beta \in \triangle$.

A graded module M over a G-graded ring R is called *graded compactly packed by graded prime submodules* if every graded submodule of M is a graded compactly packed submodule by graded prime submodules.

3. Characterization of graded coprimely packed modules

In [24], the concept of a graded coprimely packed ring was introduced. A proper graded ideal I of a G-graded ring R is said to be a *graded coprimely packed ideal* if whenever $I \subseteq \bigcup\limits_{\alpha\in\triangle} P_\alpha$ where P_α is a graded prime ideal of R for each $\alpha \in \triangle$, then there exists $\beta \in \triangle$ such that $I + P_\beta \neq R$. A G-graded ring R is called a *graded coprimely packed ring* if every proper graded ideal of R is a graded coprimely packed ideal.

In what follows we will generalize this concept to the concept of a graded coprimely packed module and give a number of its properties.

Definition 3.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be *a graded coprimely packed submodule* if whenever $N \subseteq \bigcup\limits_{\alpha\in\triangle} N_\alpha$ where N_α is a graded prime submodule of M for each $\alpha \in \triangle$, then there exists $\beta \in \triangle$ such that $N + N_\beta \neq M$. A graded R-module M is called *a graded coprimely packed module* if every proper graded submodule of M is a graded coprimely packed submodule.

The next theorem gives a characterization of being a graded coprimely packed module in terms of graded maximal submodules.

Theorem 3.2. *Let R be a G-graded ring and M a gr-multiplication R-module. Then the following statements are equivalent:*

(i) *M is a graded coprimely packed module.*
(ii) *If N is a proper graded submodule of M and $\{K_\alpha\}_{\alpha\in\triangle}$ is a family of graded maximal submodules of M such that $N \subseteq \bigcup\limits_{\alpha\in\triangle} K_\alpha$, then $N \subseteq K_\beta$ for some $\beta \in \triangle$.*

Proof. $(i) \Rightarrow (ii)$: Suppose that M is a graded coprimely packed module. Let $N \subseteq \bigcup\limits_{\alpha\in\triangle} K_\alpha$ where N is a proper graded submodule of M and K_α is a graded maximal submodule of M for each $\alpha \in \triangle$. Then K_α is a graded prime submodule by [8, Proposition 2.4]. As M is a graded coprimely packed module, it follows that there exists $\beta \in \triangle$ such that $N + K_\beta \neq M$. Since K_β is a graded maximal submodule and $K_\beta \subseteq N + K_\beta$ we obtain that $N + K_\beta = K_\beta$. It follows that $N \subseteq K_\beta$.

$(ii) \Rightarrow (i)$: Let N be a proper graded submodule of M and $\{N_\alpha\}_{\alpha\in\Delta}$ be a family of graded prime submodules of M such that $N + N_\alpha = M$ for all $\alpha \in \Delta$. Since M is a gr-multiplication module, for any $\alpha \in \Delta$, there exists a graded maximal submodule K_α that contains N_α. This yields that $N + K_\alpha = M$ for all $\alpha \in \Delta$. By our assumption we get $N \nsubseteq \bigcup\limits_{\alpha\in\triangle} K_\alpha$ and hence $N \nsubseteq \bigcup\limits_{\alpha\in\triangle} N_\alpha$. Therefore N is a graded coprimely packed submodule of M. □

Note that the condition (ii) of Theorem 3.2 was introduced in the non-graded case in [13, Definition 4.3] for an R-module M under the name *maximal compactly packed module.*

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a *graded homomorphism* if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$, (see [18]). The category of graded R-modules possesses direct sums, products injective and projective limits. A graded homomorphism that is an injective function will be referred to simply as a *monomorphism* and a graded homomorphism that is a surjective function will be called an *epimorphism.* If $\varphi : M \to M'$ is an epimorphism, then M' is said to be a graded homomorphic image of M.

In what follows we show that a graded homomorphic image of a graded coprimely packed module is again graded coprimely packed.

Theorem 3.3. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism. If M is a graded coprimely packed module, then so is M'.*

Proof. Assume that M is a graded coprimely packed module. Let N' be a proper graded submodule of M' and let $\{N'_\alpha\}_{\alpha\in\Delta}$ be a family of graded prime submodules of M' such that $N' \subseteq \bigcup\limits_{\alpha\in\triangle} N'_\alpha$. By [3, Lemma 2.4] or by a straightforward calculation, we have that $\varphi^{-1}(N'_\alpha)$ is a graded prime submodule of M for each $\alpha \in \Delta$.

Using the fact that φ is a graded epimorphism, we have $\varphi^{-1}(N') \subseteq \varphi^{-1}(\bigcup\limits_{\alpha\in\triangle} N'_\alpha) = \bigcup\limits_{\alpha\in\triangle} \varphi^{-1}(N'_\alpha)$. Since M is a graded coprimely packed module, $\varphi^{-1}(N') + \varphi^{-1}(N'_\beta) \neq M$ for some $\beta \in \Delta$. Suppose that $N' + N'_\beta = M'$ and let $m' \in h(M')$. Hence there exist $n' \in N' \cap h(M')$ and $t' \in N'_\beta \cap h(M')$ such that $m' = n' + t'$. Using again the fact that φ is a graded epimorphism, there exist $m, n, t \in h(M)$ such that $\varphi(m) = m'$, $\varphi(n) = n'$ and $\varphi(t) = t'$. This yields that $n \in \varphi^{-1}(N')$ and $t \in \varphi^{-1}(N'_\beta)$. Since $\varphi(m) = \varphi(n+t)$ it follows that $m-(n+t) \in Ker\varphi = \varphi^{-1}(\{0\}) \subseteq \varphi^{-1}(N'_\beta)$. Thus $m-n \in \varphi^{-1}(N'_\beta)$ and hence $m \in \varphi^{-1}(N') + \varphi^{-1}(N'_\beta)$. This implies $M = \varphi^{-1}(N') + \varphi^{-1}(N'_\beta)$ which is a contradiction, thus $N' + N'_\beta \neq M'$. Therefore M' is a graded coprimely packed module. □

Similarly, the property of being a graded prime submodule is preserved under graded homomorphic images.

Lemma 3.4. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism. If P is a graded prime submodule of M such that $Ker\varphi \subseteq P$, then $\varphi(P)$ is a graded prime submodule of M'.*

Proof. Assume that P is a graded prime submodule of M. Let $s \in h(R)$ and $y \in h(M')$ such that $sy \in \varphi(P)$ and $y \notin \varphi(P)$. Since $sy \in \varphi(P)$, there exists $n \in P \cap h(M)$ such that $\varphi(n) = sy$. Since $y \in h(M')$ and φ is an epimorphism, there exists $x \in h(M)$ such that $\varphi(x) = y$. Hence $\varphi(n) = s\varphi(x)$ this implies $\varphi(n - sx) = 0$. Thus $n - sx \in Ker\varphi \subseteq P$ and so $sx \in P$. Since P is a graded prime submodule of M and $x \notin P$, $s \in (P :_R M)$, i.e., $sM \subseteq P$ and so $sM' \subseteq \varphi(P)$. Therefore $\varphi(P)$ is a graded prime submodule of M'. □

We are now ready to show that a graded module is graded coprimely packed if and only if its homomorphic image is graded coprimely packed whenever the respective kernel is contained in any graded prime submodule.

Theorem 3.5. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism such that $Ker(\varphi) \subseteq Gr_M(\{0\})$. Then M is a graded coprimely packed module if and only if M' is a graded coprimely packed module.*

Proof. ($\Rightarrow$): By Theorem 3.3 we know that M' is a graded coprimely packed module.

($\Leftarrow$): Assume conversely that M' is a graded coprimely packed module. Let N be a proper graded submodule of M and let $\{N_\alpha\}_{\alpha\in\Delta}$ be a family of graded prime submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$. Hence $\varphi(N) \subseteq \varphi(\bigcup_{\alpha\in\triangle} N_\alpha) = \bigcup_{\alpha\in\triangle} \varphi(N_\alpha)$. Since $ker\varphi \subseteq N_\alpha$ for all $\alpha \in \triangle$ by Lemma 3.4, $\varphi(N_\alpha)$ is a graded prime submodule for all $\alpha \in \triangle$. Since M' is a graded coprimely packed module, $\varphi(N) + \varphi(N_\beta) \neq M'$ for some $\beta \in \Delta$. Suppose that $N + N_\beta = M$ and let $m' \in M'$. Since φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$ and there exist $n \in N \cap h(M)$ and $t \in N_\beta \cap h(M)$ such that $m = n + t$. Then $m' = \varphi(m) = \varphi(n + t) = \varphi(n) + \varphi(t)$. So $m' \in \varphi(N) + \varphi(N_\beta)$ this implies $M' \subseteq \varphi(N) + \varphi(N_\beta)$ which is a contradiction, thus $N + N_\beta \neq M$. Therefore M is a graded coprimely packed module. □

4. Some sufficient conditions

The results contained in this section can be seen as an attempt to transfer the results of [24] from graded rings to graded modules. In [24, Proposition 6] the authors proved that any semilocal graded u-ring is a graded coprimely packed ring. The relationship between the concepts 'a graded semilocal gr-multiplication R-module' and 'a graded coprimely packed module' is expressed in the following theorem.

Theorem 4.1. *Let R be a G-graded ring and M a graded semilocal gr-multiplication R-module. Then M is a graded coprimely packed module.*

Proof. Let us assume that M is a graded semilocal and G-$Max(M) = \{W_1, \ldots, W_n\}$. Let N be a proper graded submodule of M and let $\{N_\alpha\}_{\alpha\in\Delta}$

be a family of graded prime submodules of M such that $N + N_\alpha = M$ for all $\alpha \in \Delta$. Since M is a gr-multiplication R-module, there exists a subset $\{i_{1,} \ldots, i_k\}$ of $\{1, \ldots, n\}$ such that for all $\alpha \in \Delta$ there exists $i_j(\alpha) \in \{i_{1,} \ldots, i_k\}$ with $N_\alpha \subseteq W_{i_j(\alpha)}$ and, conversely, for all $j \in \{1, \ldots, k\}$ there exists $\alpha(k) \in \Delta$ such that $N_{\alpha(k)} \subseteq W_{i_j}$. It follows that $N + W_{i_j} = M$ for all $j = 1, \ldots, k$. Assume that $N \subseteq \bigcup_{j=1}^{k} W_{i_j}$. Since M is a gr-multiplication R-module by [12, Theorem 2.8], we have $N \subseteq W_{i_j(N)}$ for some $i_j(N) \in \{i_{1,} \ldots, i_k\}$. So $N + W_{i_j(N)} \subseteq W_{i_j(N)} + W_{i_j(N)} \subseteq W_{i_j(N)} \neq M$, a contradiction. Thus $N \nsubseteq \bigcup_{j=1}^{k} W_{i_j}$. But $\bigcup_{\alpha \in \triangle} N_\alpha \subseteq \bigcup_{j=1}^{k} W_{i_j}$ thus $N \nsubseteq \bigcup_{\alpha \in \triangle} N_\alpha$. Therefore M is a graded coprimely packed module. $\square$

From [24, Proposition 7] we know that every graded compactly packed ring is a graded coprimely packed ring. We will show a generalization of this result.

Theorem 4.2. *Let R be a G-graded ring and M a graded R-module. If M is graded compactly packed by graded prime submodules, then M is a graded coprimely packed module.*

Proof. Suppose that M is a graded compactly packed by graded prime submodules and let $N \subseteq \bigcup_{\alpha \in \triangle} N_\alpha$ where N is a proper graded submodule of M and N_α is a graded prime submodule of M for each $\alpha \in \triangle$. Since M is a graded compactly packed by graded prime submodules, there exists $\beta \in \triangle$ such that $N \subseteq N_\beta$. Then $N + N_\beta \neq M$. Therefore M is a graded coprimely packed submodule. $\square$

In what follows we will use [23, Example 3.10] to show that there is a G-graded ring R and a graded gr-multiplication R-module M that is a graded compactly packed by graded prime submodules (and hence graded coprimely packed) which is not semilocal.

Example 4.3. Let $G = \{0\}$ be a trivial group. Let F be any field and let R denote the collection of all sequences $(f_1, f_2, f_3, \ldots)$ of elements of F with the property that there exists a positive integer n (depending on the particular sequence) such that $f_n = f_{n+1} = f_{n+2} = \ldots$. Then $R \subseteq F^{\mathbb{N}}$ and R is a commutative von Neumann regular ring with respect to the induced operations $+$ and $\cdot$. Moreover, R is clearly a G-graded module. Let S consist of all sequences $(f_1, f_2, f_3, \ldots)$ of elements of F such that $0 = f_n = f_{n+1} = f_{n+2} = \ldots$ for some positive integer n. We denote, for any $f = (f_k)_{k=1}^{\infty}$, $n(f) = \min\{n \in \mathbb{N} \mid x_k = 0 \text{ for all } k \text{ such that } k \geq n\}$. We put $S_n = \{(f_k)_{k=1}^{\infty} \in F^{\mathbb{N}} \mid 0 = f_n = f_{n+1} = f_{n+2} = \ldots\}$. Hence $S_1 = \{0\}$ and any S_n is a submodule of S. Then by [23, Example 3.10] S is a graded faithful multiplication module such that $S = \bigcup_{n=1}^{\infty} S_n$. For any positive integer n, we put $M_n = \{(x_k)_{k=1}^{\infty} \in R \mid x_n = 0\}$. Then M_n is a maximal ideal of R, i.e., the set of maximal ideals of R is infinite. Hence the set of maximal submodules of S is infinite and therefore S is not semilocal. Moreover, let P be a (graded) prime submodule, $P \neq \{0\}$. Let us show that $P = M_n \cdot S$

for some $n \in \mathbb{N}$, i.e., P is a (graded) maximal submodule. Clearly, there is $n \in \mathbb{N}$ such that $x_n = 0$ for all $x \in P$ (otherwise we would get that $P = S$, a contradiction). We put $\overline{x}_n = 1$ and $\overline{x}_k = x_k$ for all $k \in \mathbb{N}$ such that $k \neq n$. It follows that $(d_n)_k = \begin{cases} 0 & \text{if } n = k \\ 1 & \text{otherwise} \end{cases}$ satisfies $d_n \cdot \overline{x} = x \in P$. It follows that $\overline{x} \notin P$ and hence $d_n \in (P :_R S)$. Hence $S_n = d_n S \subseteq P$, i.e.. $P = S_n$. In particular, any maximal submodule P is of the form $P = S_n$.

Let use verify that S is graded compactly packed by graded prime submodules. Let N be a proper (graded) submodule of S and let $\{N_\alpha\}_{\alpha\in\Delta}$ be a family of (graded) prime submodules of S such that that $N \subseteq \bigcup\limits_{\alpha\in\triangle} N_\alpha$. Assume that $N \nsubseteq N_\alpha$ for all $\alpha \in \triangle$. We put $n_N = \min\{n(f) \mid \exists\alpha \in \triangle \text{ such that } f \in N \text{ and } f \notin N_\alpha\} > 1$. We denote the respective f by f_N and the respective α by α_N. Then $f_N \in N$, $f \notin N_{\alpha_N}$ and $f \in N_\beta$ for some $\beta \in \triangle$. One can easily check that we may assume that $(f_N)_{n_N-1} = 1$. We put $q_N = (\underbrace{0,\dots,0,}_{(n_N-1)\text{-times}} 1,0,\dots,0,\dots)$. Then $q_N \in N$. Assume that $q_N \in N_{\alpha_N}$. Then $f_N - q_N \notin N_{\alpha_N}$ (otherwise we would have $f_N \in N_{\alpha_N}$) but this is a contradiction with the definition of n_N. Hence $q_N \notin N_{\alpha_N}$.

Now we put $c_N = (\underbrace{1,\dots,1,}_{(n_N-2)\text{-times}} 0,1,\dots,1,\dots)$. It follows that $c_N \in R$ and $c_N \cdot q_N = (0,0,0,\dots) \in N_{\alpha_N}$. Since N_{α_N} is prime and $q_N \notin N_{\alpha_N}$, we obtain that $c_N \in (N_{\alpha_N} :_R S)$. From $N_{\alpha_N} = (N_{\alpha_N} :_R S)S$ we get that $S_{n_N} = c_N \cdot S \subseteq N_{\alpha_N}$.

Using the fact that $q_N \in N$ we infer that there is $\beta \in \triangle$ such that $q_N \in N_\beta$. Since $N \nsubseteq N_\beta$ there is $x_\beta \in N \setminus N_\beta$ and we may assume that there is a minimal $k \in \mathbb{N}$ such that $0 = (x_\beta)_k = (x_\beta)_{k+1} = (x_\beta)_{k+2} = \dots$ and $(x_\beta)_{k-1} \neq 0$. Evidently, $x_\beta \notin S_{n_N}$, otherwise we would have by the same considerations as above that $q_N \notin N_\beta$, a contradiction. It follows that the element $q_\beta = (\underbrace{0,\dots,0,}_{(k-1)\text{-times}} 1,0,\dots,0,\dots) \in N \setminus N_\beta$ and $k > n_N$.

By a *chain of graded prime submodules* of a graded R-module M we mean a finite strictly increasing sequence $N_0 \subseteq \ \dots \subseteq N_n$; the *length* of this chain is n. We define the *graded dimension of* M to be the supremum of the lengths of all finite chains of graded prime submodules in M. We denote the graded dimension of M by $Gdim(M)$, (see [12, Definition 2.9].)

Recall that a graded R-module M is said to be *a graded torsion-free R-module* whenever $r \in h(R)$ and $m \in h(M)$ with $rm = 0$ implies that either $r = 0$ or $m = 0$, (see [5, Definition 2.2].)

In [24, Theorem 8] the authors were able to prove that a graded integral domain with Krull dimension 1 is a graded compactly packed ring if and only if it is a graded coprimely packed ring. Analogous to the above statement we get the following theorem.

Theorem 4.4. *Let M be a graded torsion-free gr-multiplication R-module with $Gdim(M) = 1$. Then M is a graded compactly packed by graded prime submodules if and only if M is a graded coprimely packed module.*

Proof. ($\Rightarrow$) Theorem 4.2.

($\Leftarrow$) Since M is a graded torsion free, $\{0\}$ is a graded prime submodule by [7, Proposition 2.6]. Let N be a proper graded submodule of M and let $\{N_\alpha\}_{\alpha\in\triangle}$ be a family of graded prime submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$. We may assume that $N_\alpha \neq \{0\}$ for all $\alpha \in \triangle$. Since M is a graded coprimely packed module, there exists $\beta \in \triangle$ such that $N + N_\beta \neq M$. Then, since M is a gr-multiplication R-module, there exists a graded maximal submodule L of M such that $\{0\} \subseteq N_\beta \subseteq N + N_\beta \subseteq L$. Since $Gdim(M) = 1$ we have that $L = N_\beta \subseteq N + N_\beta \subseteq L$. It follows that $N \subseteq N + N_\beta = N_\beta$. Therefore M is a graded coprimely packed module. $\square$

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