Advanced Studies in Contemporary Mathematics
29 (2019), No. 2, pp. 257 - 262
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.2.257

AN EXPRESSION FOR CENTRAL BELL POLYNOMIALS

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Abstract. Recently, the central Bell polynomials were introduced by Kim-Kim in [12]. In this paper, we derive an expression for these polynomials involving the central factorial numbers of the second kind.

2010 Mathematics Subject Classification. 11T23, 20G40, 94B05.

Keywords and phrases. central Bell polynomials, central factorial numbers of the second kind.

1. Introduction

For $n \geq 0$, the central factorial sequence $x^{[n]}$ is defined by

$$x^{[0]} = 1,\ x^{[n]} = x\big(x + \frac{n}{2} - 1\big)\big(x + \frac{n}{2} - 2\big) \cdots \big(x - \frac{n}{2} + 1\big),\ \text{(see [4,5,6])}, \tag{1.1}$$

where $n \geq 1$.

For all nonnegative integers n, k $(k \leq n)$, the central factorial numbers of the second kind are defined by the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, \quad \text{(see [1,4,5,14])}. \tag{1.2}$$

From (1.2), we can derive the following generating function of the central factorial numbers of the second kind

$$\frac{1}{k!}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}, \quad (k \geq 0). \tag{1.3}$$

Recently, Kim-Kim introduced the central Bell polynomials which are given by the generating function

$$e^{x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} = \sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{1.4}$$

From (1.3) and (1.4), we note that

$$B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k) x^k, \quad (n \geq 0), \quad \text{(see [12])}. \tag{1.5}$$

When $x = 1$, $B_n^{(c)} = B_n^{(c)}(1)$ are called the central Bell numbers.

It is well known that the Bell number B_m is the number of ways to partition a set of m objects. For example, $B_3 = 5$ because there are five ways to partition the set $\{1,2,3\}$ (see [3, 16]):

$$\{1,2,3\};\ \ \{1,2\}\cup\{3\};\ \ \{1,3\}\cup\{2\};\ \ \{2,3\}\cup\{1\};\ \ \{1\}\cup\{2\}\cup\{3\}.$$

The two most well known expressions for the Bell numbers are the following (see [16]):

$$B_m = \sum_{j=0}^{m} S_2(m,j), \tag{1.6}$$

and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad (n,m \geq 0), \tag{1.7}$$

where $S_2(m,j)$ is the Stirling number of the second kind.

In [16], Spivey gave a combinatorial proof of the following addition formula for Bell numbers which generalizes both (1.6) and (1.7):

$$B_{n+m} = \sum_{k=0}^{n}\sum_{j=0}^{m} j^{n-k} S_2(m,j) B_k \binom{n}{k}. \tag{1.8}$$

Here n,m are nonnegative integers. It is known that the Bell polynomials are given by

$$B_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k, \quad (\text{see } [3,15]). \tag{1.9}$$

In [2], Gould and Quaintance extended the equation (1.8) to the case of Bell polynomials as follows:

$$B_{m+n}(x) = \sum_{k=0}^{n}\sum_{j=0}^{m} j^{n-k}\binom{n}{k} S_2(m,j) B_k(x) x^j. \tag{1.10}$$

In view of (1.8) and (1.10), we consider the central Bell polynomials and derive an expression for these polynomials involving the central factorial numbers of the second kind.

Let $F(t)$ be the generating function for the central Bell polynomials. Then we have

$$F(t) = e^{x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} = \sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!}. \tag{1.11}$$

From (1.11), we note that

$$\begin{aligned} F(s+t) &= e^{x\left(e^{\frac{s+t}{2}}-e^{-\frac{s+t}{2}}\right)} = e^{x\left(e^{-\frac{t}{2}}(e^{\frac{s}{2}}-e^{-\frac{s}{2}})+e^{\frac{s}{2}}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})\right)} \\ &= e^{xe^{-\frac{t}{2}}\left(e^{\frac{s}{2}}-e^{-\frac{s}{2}}\right)} e^{xe^{\frac{s}{2}}(e^{\frac{t}{2}}-e^{-\frac{t}{2}}))} \\ &= \Big(\sum_{k_1=0}^{\infty} x^{k_1} e^{-\frac{k_1}{2}t}\frac{1}{k_1!}\big(e^{\frac{s}{2}}-e^{-\frac{s}{2}}\big)^{k_1}\Big)\Big(\sum_{j_1=0}^{\infty} x^{j_1} e^{\frac{j_1}{2}s}\frac{1}{j_1!}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^{j_1}\Big). \end{aligned} \tag{1.12}$$

Now, we observe that

$$\begin{aligned} &\sum_{k_1=0}^{\infty} x^{k_1} e^{-\frac{k_1}{2}t}\frac{1}{k_1!}\big(e^{\frac{s}{2}}-e^{-\frac{s}{2}}\big)^{k_1} \\ &\quad= \sum_{k_1=0}^{\infty} x^{k_1}\sum_{k_2=k_1}^{\infty} T(k_2,k_1)\frac{s^{k_2}}{k_2!}\sum_{k_3=0}^{\infty}\big(-\frac{k_1}{2}\big)^{k_3}\frac{t^{k_3}}{k_3!} \\ &\quad= \sum_{k_2=0}^{\infty}\big(\sum_{k_1=0}^{k_2} x^{k_1}T(k_2,k_1)\big)\frac{s^{k_2}}{k_2!}\sum_{k_3=0}^{\infty}\big(-\frac{k_1}{2}\big)^{k_3}\frac{t^{k_3}}{k_3!}. \end{aligned} \tag{1.13}$$

On the other hand

$$\begin{aligned} &\sum_{j_1=0}^{\infty} x^{j_1} e^{\frac{j_1}{2}s}\frac{1}{j_1!}\big(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\big)^{j_1} \\ &\quad= \sum_{j_1=0}^{\infty} x^{j_1}\sum_{j_2=j_1}^{\infty} T(j_2,j_1)\frac{t^{j_2}}{j_2!}\sum_{j_3=0}^{\infty}\big(\frac{j_1}{2}\big)^{j_3}\frac{s^{j_3}}{j_3!} \\ &\quad= \sum_{j_2=0}^{\infty}\big(\sum_{j_1=0}^{j_2} x^{j_1}T(j_2,j_1)\big)\frac{t^{j_2}}{j_2!}\sum_{j_3=0}^{\infty}\big(\frac{j_1}{2}\big)^{j_3}\frac{s^{j_3}}{j_3!}. \end{aligned} \tag{1.14}$$

From (1.12),(1.13) and (1.14), we have

$$\begin{aligned} F(s+t) &= e^{x\left(e^{\frac{s+t}{2}}-e^{-\frac{s+t}{2}}\right)} \\ &= \Big(\sum_{k_2=0}^{\infty}\big(\sum_{k_1=0}^{k_2} x^{k_1}T(k_2,k_1)\big)\frac{s^{k_2}}{k_2!}\Big)\Big(\sum_{k_3=0}^{\infty}\big(-\frac{k_1}{2}\big)^{k_3}\frac{t^{k_3}}{k_3!}\Big) \\ &\times\Big(\sum_{j_2=0}^{\infty}\big(\sum_{j_1=0}^{j_2} x^{j_1}T(j_2,j_1)\big)\frac{t^{j_2}}{j_2!}\Big)\Big(\sum_{j_3=0}^{\infty}\big(\frac{j_1}{2}\big)^{j_3}\frac{s^{j_3}}{j_3!}\Big) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_2=0}^{n_1}\sum_{k_1=0}^{k_2}\sum_{j_2=0}^{n_2}\sum_{j_1=0}^{j_2} x^{k_1}T(k_2,k_1)\binom{n_1}{k_2}\big(\frac{j_1}{2}\big)^{n_1-k_2}x^{j_1}T(j_2,j_1) \\ &\times\binom{n_2}{j_2}\big(-\frac{j_1}{2}\big)^{n_2-j_2}\frac{s^{n_1}}{n_1!}\frac{t^{n_2}}{n_2!} \end{aligned}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big(\sum_{k_2=0}^{n_1}\sum_{k_1=0}^{k_2}\sum_{j_2=0}^{n_2}\sum_{j_1=0}^{j_2}\binom{n_1}{k_2}\binom{n_2}{j_2}x^{k_1+j_1}T(k_2,k_1)T(j_2,j_1) \times (-1)^{n_2-j_2}\big(\frac{j_1}{2}\big)^{n_1+n_2-j_2-k_2}\Big)\frac{s^{n_1}}{n_1!}\frac{t^{n_2}}{n_2!}. \tag{1.15}$$

By (1.11), we get

$$\begin{aligned} F(s+t) = e^{x\left(e^{\frac{s+t}{2}}-e^{-\frac{s+t}{2}}\right)} &= \sum_{n_1=0}^{\infty}B_{n_1}^{(c)}(x)\frac{1}{n_1!}(s+t)^{n_1} \\ &= \sum_{n_1=0}^{\infty}B_{n_1}^{(c)}(x)\frac{1}{n_1!}\sum_{n_2=0}^{n_1}\binom{n_1}{n_2}s^{n_2}t^{n_1-n_2} \\ &= \sum_{n_2=0}^{\infty}\sum_{n_1=n_2}^{\infty}B_{n_1}^{(c)}(x)\frac{1}{n_1!}\binom{n_1}{n_2}s^{n_2}t^{n_1-n_2} \\ &= \sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}B_{n_1+n_2}^{(c)}(x)\frac{1}{(n_1+n_2)!}\binom{n_1+n_2}{n_2}s^{n_2}t^{n_1} \\ &= \sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}B_{n_1+n_2}^{(c)}(x)\frac{t^{n_1}}{n_1!}\frac{s^{n_2}}{n_2!}. \end{aligned} \tag{1.16}$$

Therefore, by (1.15) and (1.16), we obtain the following theorem.

Theorem 1.1. *Let n_1, n_2 be nonnegative integers. Then we have*

$$B_{n_1+n_2}^{(c)}(x) = \sum_{k_2=0}^{n_1}\sum_{k_1=0}^{k_2}\sum_{j_2=0}^{n_2}\sum_{j_1=0}^{j_2}\binom{n_1}{k_2}\binom{n_2}{j_2}x^{k_1+j_1}T(k_2,k_1)T(j_2,j_1) \times (-1)^{n_2-j_2}\big(\frac{j_1}{2}\big)^{n_1+n_2-j_2-k_2}.$$

In particular,

$$B_{n_1+n_2}^{(c)} = \sum_{k_2=0}^{n_1}\sum_{k_1=0}^{k_2}\sum_{j_2=0}^{n_2}\sum_{j_1=0}^{j_2}\binom{n_1}{k_2}\binom{n_2}{j_2}T(k_2,k_1)T(j_2,j_1) \times (-1)^{n_2-j_2}\big(\frac{j_1}{2}\big)^{n_1+n_2-j_2-k_2}.$$

Let us take $n_2 = 1$ in Theorem 2.1. Then we have the following corollary.

Corollary 1.2. *Let n be a nonnegative integer. Then we have*

$$B_{n+1}^{(c)} = \sum_{k=0}^{n}\sum_{i=0}^{k}T(k,i)\binom{n}{k}\big(\frac{1}{2}\big)^{n-k}.$$

2. Further remark and an open question

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad \text{(see [9, 10])}. \tag{2.1}$$

By (2.1), we easily get

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [7, 10, 11, 13])}, \tag{2.2}$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big), \quad (n \geq 1).$$

In [8], Kim defined the degenerate Stirling number of the second kind as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \quad (n \geq 0), \tag{2.3}$$

where

$$(x)_0 = 1, \quad (x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad (n \geq 1).$$

From (2.3), we note that

$$\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}} - 1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [7, 8])}. \tag{2.4}$$

Note that $\lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k), \quad (n,k \geq 0)$.
In [7], Kim-Kim-Dolgy defined the degenerate Bell polynomials by

$$e^{x\left(e_\lambda(t)-1\right)} = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.5}$$

Now, we observe that

$$\begin{aligned} e^{x\left(e_\lambda(s+t)-1\right)} &= \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{1}{n!}(s+t)^n = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}s^j t^{n-j} \\ &= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{B_{n,\lambda}(x)}{n!}\binom{n}{j}s^j t^{n-j} = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{B_{n+j,\lambda}(x)}{(n+j)!}\binom{n+j}{j}s^j t^n \\ &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} B_{n+j,\lambda}(x)\frac{s^j}{j!}\frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Here, we have the following open question.
Question. Can we get a formula similar to equation (1.10) for $B_{n+j,\lambda}(x)$?

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