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SOME IDENTITIES OF PARTIALLY DEGENERATE BELL POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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ABSTRACT. Recently, several authors have studied partially degenerate Bell numbers and polynomials. In this paper, we find the inversion formula of differential equation arising from the generating function of partially degenerate Bell polynomials already shown in [6] and we derive some new interesting identities and properties from this differential equation.

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1. Introduction

The Bell polynomials are defined by the following generating function:

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad (see \quad [4,5,7]). \tag{1.1}$$

In [8], introduced the partially degenerate Bell polynomials, which are given by the following generating function:

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $\lambda \to 0$, the partially degenerate Bell polynomials reduces to the Bell polynomials.

$$\lim_{\lambda\to 0} bel_{n,\lambda}(x) = Bel_n(x).$$

That is (1.2) $\to$ (1.1). It is well known that the generating function of $S_1(n,l)$ and $S_2(n,l)$ are given by

$$\frac{1}{l!}\big(\log(1+t)\big)^l = \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}, \tag{1.3}$$

and

$$\frac{1}{l!}(e^t-1)^l = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, \quad (see \quad [2,3]). \tag{1.4}$$

Now, we observe that

$$
\begin{aligned}
e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} &= e^{x\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-1\right)} \\
&= \sum_{m=0}^{\infty} Bel_m(x)\frac{1}{m!}\big(\frac{1}{\lambda}\log(1+\lambda t)\big)^m \\
&= \sum_{m=0}^{\infty} Bel_m(x)\lambda^{-m}\sum_{n=m}^{\infty}\lambda^n S_1(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\big(\sum_{m=0}^{n} Bel_m(x)S_1(n,m)\lambda^{n-m}\big)\frac{t^n}{n!}.
\end{aligned} \tag{1.5}
$$

From (1.2) and (1.5), we get

$$
bel_{n,\lambda}(x) = \sum_{m=0}^{n} Bel_m(x)S_1(n,m)\lambda^{n-m}. \tag{1.6}
$$

Also, by replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (1.2), we get

$$
\begin{aligned}
e^{x(e^t-1)} &= \sum_{m=0}^{\infty} bel_{m,\lambda}(x)\frac{1}{m!}\big(\frac{1}{\lambda}e^{\lambda t}-1\big)^m \\
&= \sum_{m=0}^{\infty} bel_{m,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)\lambda^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\big(\sum_{m=0}^{n} bel_{m,\lambda}(x)\lambda^{n-m}S_2(n,m)\big)\frac{t^n}{n!}.
\end{aligned} \tag{1.7}
$$

From (1.1) and (1.7), we have

$$
Bel_n(x) = \sum_{m=0}^{n} bel_{m,\lambda}(x)\lambda^{n-m}S_2(n,m). \tag{1.8}
$$

In [6], introduced the differential equations associated with partially degenerate Bell polynomials(Touchard polynomials) is given by

$$
F^{(N)} = \Big(\sum_{i=1}^{N} b_i(N,\lambda)x^i(1+\lambda t)^{\frac{i}{\lambda}-N}\Big)F, \tag{1.9}
$$

where $F = F(t,x;\lambda) = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}$ and $F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x;\lambda)$.

Recently, several authors have studied partially degenerate Bell numbers and polynomials. In this paper, we find the inversion formula of (1.9) and we derive some new interesting identities and properties from this differential equation.

2. Some identities of partially degenerate Bell polynomials arising from differential equation

Let

$$
F = F(t,x;\lambda) = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}. \tag{2.1}
$$

Then by taking the derivative with respect to t of (2.1), we get

$$\begin{aligned} F^{(1)} &= \frac{\partial}{\partial t}F(t,x;\lambda) = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}x(1+\lambda t)^{\frac{1}{\lambda}-1} \\ &= x(1+\lambda t)^{\frac{1}{\lambda}-1}F. \end{aligned} \tag{2.2}$$

From (2.2), we have

$$xF = (1+\lambda t)^{1-\frac{1}{\lambda}}F^{(1)}. \tag{2.3}$$

Let us take the derivative with respect to t of (2.3), we get

$$xF^{(1)} = (\lambda-1)(1+\lambda t)^{-\frac{1}{\lambda}}F^{(1)} + (1+\lambda t)^{1-\frac{1}{\lambda}}F^{(2)}. \tag{2.4}$$

Multiplying by $(1+\lambda t)^{1-\frac{1}{\lambda}}$ both sides of (2.4) and by (2.3), we get

$$x^2F = (\lambda-1)(1+\lambda t)^{1-\frac{2}{\lambda}}F^{(1)} + (1+\lambda t)^{2-\frac{2}{\lambda}}F^{(2)}. \tag{2.5}$$

From (2.5), we have

$$\begin{aligned} x^2F^{(1)} &= (\lambda-1)(\lambda-2)(1+\lambda t)^{-\frac{2}{\lambda}}F^{(1)} + (\lambda-1)(1+\lambda t)^{1-\frac{2}{\lambda}}F^{(2)} \\ &\quad + (2\lambda-2)(1+\lambda t)^{1-\frac{2}{\lambda}}F^{(2)} + (1+\lambda t)^{2-\frac{2}{\lambda}}F^{(3)} \\ &= (\lambda-1)(\lambda-2)(1+\lambda t)^{-\frac{2}{\lambda}}F^{(1)} + (3\lambda-3)(1+\lambda t)^{1-\frac{2}{\lambda}}F^{(2)} \\ &\quad + (1+\lambda t)^{2-\frac{2}{\lambda}}F^{(3)}. \end{aligned} \tag{2.6}$$

Multiplying by $(1+\lambda t)^{1-\frac{1}{\lambda}}$ both sides of (2.6) and by (2.3), we have

$$\begin{aligned} x^3F &= (\lambda-1)(\lambda-2)(1+\lambda t)^{1-\frac{3}{\lambda}}F^{(1)} + (3\lambda-3)(1+\lambda t)^{2-\frac{3}{\lambda}}F^{(2)} \\ &\quad + (1+\lambda t)^{3-\frac{3}{\lambda}}F^{(3)}. \end{aligned} \tag{2.7}$$

Continuing this process, we get

$$x^NF = \sum_{k=1}^{N} a_k(N)(1+\lambda t)^{k-\frac{N}{\lambda}}F^{(k)}, \tag{2.8}$$

where $N \in \mathbb{N}$. To determine the coefficients $a_k(N)$ in (2.8), let us take the derivative with respect to t on both sides of (2.8), we get

$$\begin{aligned} x^NF^{(1)} &= \sum_{k=1}^{N}(k\lambda-N)a_k(N)(1+\lambda t)^{k-\frac{N}{\lambda}-1}F^{(k)} \\ &\quad + \sum_{k=1}^{N} a_k(N)(1+\lambda t)^{k-\frac{N}{\lambda}}F^{(k+1)}. \end{aligned} \tag{2.9}$$

Multiplying by $(1+\lambda t)^{1-\frac{1}{\lambda}}$ both sides of (2.9) and by (2.3), we get

$$\begin{aligned} x^{N+1}F &= \sum_{k=1}^{N}(k\lambda - N)a_k(N)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)} \\ &+ \sum_{k=1}^{N} a_k(N)(1+\lambda t)^{k+1-\frac{N+1}{\lambda}}F^{(k+1)} \\ &= \sum_{k=1}^{N}(k\lambda - N)a_k(N)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)} \\ &+ \sum_{k=2}^{N+1} a_{k-1}(N)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)} \\ &= (\lambda - N)a_1(N)(1+\lambda t)^{1-\frac{N+1}{\lambda}}F^{(1)} \\ &+ a_N(N)(1+\lambda t)^{N+1-\frac{N+1}{\lambda}}F^{(N+1)} \\ &+ \sum_{k=2}^{N}\big((k\lambda - N)a_k(N) + a_{k-1}(N)\big)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)}. \end{aligned} \tag{2.10}$$

By replacing N by $N+1$ in (2.8), we get

$$\begin{aligned} x^{N+1}F &= \sum_{k=1}^{N+1} a_k(N+1)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)} \\ &= a_1(N+1)(1+\lambda t)^{1-\frac{N+1}{\lambda}}F^{(1)} \\ &+ a_{N+1}(N+1)(1+\lambda t)^{N+1-\frac{N+1}{\lambda}}F^{(N+1)} \\ &+ \sum_{k=2}^{N} a_k(N+1)(1+\lambda t)^{k-\frac{N+1}{\lambda}}F^{(k)}. \end{aligned} \tag{2.11}$$

Comparing the coefficients on the both sides of (2.10) and (2.11), we get

$$a_1(N+1) = (\lambda - N)a_1(N), \quad a_{N+1}(N+1) = a_N(N), \tag{2.12}$$

and

$$a_k(N+1) = (k\lambda - N)a_k(N) + a_{k-1}(N), \tag{2.13}$$

where $2 \le k \le N$. From (2.3) and (2.8), we get

$$\begin{aligned} xF &= \sum_{k=1}^{1} a_k(1)(1+\lambda t)^{k-\frac{1}{\lambda}}F^{(k)} \\ &= a_1(1)(1+\lambda t)^{1-\frac{1}{\lambda}}F^{(1)} \\ &= (1+\lambda t)^{1-\frac{1}{\lambda}}F^{(1)}. \end{aligned} \tag{2.14}$$

Thus, by (2.14), we get

$$a_1(1) = 1. \tag{2.15}$$

It is easily get

$$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1, \tag{2.16}$$

and

$$\begin{aligned} a_1(N+1) &= (\lambda - N)a_1(N) = (\lambda - N)(\lambda - N + 1)a_1(N-1) = \cdots \\ &= (\lambda - N)(\lambda - N + 1)\cdots(\lambda - 1)a_1(1) =< \lambda - N >_N, \end{aligned} \tag{2.17}$$

where $< x >_0 = 1$, $< x >_n = x(x+1)\cdots(x+(n-1))$, $(n \geq 1)$.

Therefore, we obtain the following theorem.

Theorem 2.1. *For $N \in \mathbb{N}$, the following differential equations*

$$x^N F = \sum_{k=1}^{N} a_k(N)(1+\lambda t)^{k-\frac{N}{\lambda}} F^{(k)},$$

have the solution $F = F(t, x; \lambda) = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}$, where

$$a_1(N) =< \lambda - N + 1 >_{N-1},$$

$$a_k(N) = (k\lambda - N + 1)a_k(N-1) + a_{k-1}(N-1), \quad (2 \leq k \leq N),$$

$$a_N(N) = 1.$$

From right hand sides of (2.8), we get

$$\begin{aligned} \sum_{k=1}^{N} b_k(N)(1+\lambda t)^{k-\frac{N}{\lambda}} F^{(k)} &= \sum_{k=1}^{N} a_k(N)\Big(\sum_{l=0}^{\infty}(k - \frac{N}{\lambda})_l \lambda^l \frac{t^l}{l!}\Big) \\ &\times \Big((\frac{\partial}{\partial t})^k \sum_{m=0}^{\infty} bel_{m,\lambda}(x)\frac{t^m}{m!}\Big) \\ &= \sum_{k=1}^{N} a_k(N)\Big(\sum_{l=0}^{\infty}(k\lambda - N)_{l,\lambda}\frac{t^l}{l!}\Big) \\ &\times \Big(\sum_{m=0}^{\infty} bel_{m+k,\lambda}(x)\frac{t^m}{m!}\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\sum_{k=1}^{N}\binom{n}{m} a_k(N) bel_{m+k,\lambda}(x) \\ &\times (k\lambda - N)_{n-m,\lambda}\Big)\frac{t^n}{n!}, \end{aligned} \tag{2.18}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$. Thus, from (2.18), we get the following theorem

Theorem 2.2. *Let $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$x^N bel_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=1}^{N}\binom{n}{m} a_k(N) bel_{m+k,\lambda}(x)(k\lambda - N)_{n-m,\lambda}.$$

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