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SOME RESULTS ON THE BOUNDEDNESS OF SOLUTIONS OF A CERTAIN THIRD ORDER NON-AUTONOMOUS DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. This paper considers the boundedness of solutions for a third order non-autonomous delay differential equations of the form

$$x'''+(a(t,x,x',x'',x''(t-r))+m(t,x,x',x'',x''(t-r)))x''+b(t)g(x'(t-r))$$
$$+c(t,x,x',x'',x''(t-r))h(x)=p(t,x,x',x''),$$

where a, m, b, g, c, h and p are real valued functions which depend on the argument displayed explicitly. Some criteria on the regularity of solutions for the same equation were also investigated. The results obtained improved and extend some earlier results.

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1. Introduction

In this paper, we consider the boundedness of solutions of a certain third order nonlinear non-autonomous delay differential equations of the form

$$\begin{aligned} &x''' + (a(t,x,x',x'',x''(t-r)) + m(t,x,x',x'',x''(t-r)))x'' \\ &+b(t)g(x'(t-r)) + c(t,x,x',x'',x''(t-r))h(x) = p(t,x,x',x'') \end{aligned} \tag{1}$$

and its equivalent system

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -(a(t,x,y,z,z(t-r)) + m(t,x,y,z,z(t-r)))z - b(t)g(y) \\ &\quad + b(t)\int_{t-r}^{t} g'(y(s))z(s)ds - c(t,x,y,z,z(t-r))h(x) + p(t,x,y,z). \end{aligned} \tag{2}$$

where r is a constant delay, $r \geq 0$ and a, m, b, g, c, h and p are real valued functions which depend on the argument displayed explicitly. The primes in equation (1) denote differentiation with respect to t. It is assumed that $b(t)$ is continuous on $\mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$ and $a(t,x,y,z,z(t-r))$, $m(t,x,y,z,z(t-r))$, $c(t,x,y,z,z(t-r))$, $g(y)$ and $h(x)$ are continuous in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^4$ and $\mathbb{R}$ respectively. The derivatives $b'(t) \equiv \dfrac{db(t)}{dt}$ and $g'(y) \equiv \dfrac{dg(y)}{dy}$ exist and are continuous for all t and y respectively.

It is worth mentioning that there exist numerous papers on the qualitative properties of solutions of certain nonlinear differential equations of third order without delay (see for instance Reissig et.al [20], Adams [2] as a survey and the papers Adams et.al [1], Ademola et.al [3], Mehri and Shadman [14], Ogundare [15], Tunç [24] and some references cited therein). While some results achieved on the same subject of qualitative behaviour with respect to certain nonlinear differential equations of third order with delay include but not limited to Graef and Tunç [7], Olutimo [16], Olutimo and Adams [17], Oudjedi and Remili [19], Tunç [22], [26].

In the article published in 2009, Omeike [18] considered the third order non-autonomous nonlinear differential equations with delay:

$$\dddot{x} + a(t)\,\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = 0 \tag{3}$$

and

$$\dddot{x} + a(t)\,\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = p(t), \tag{4}$$

where r is a positive constant $a(t), b(t), c(t), g(t), h(x)$ are real valued functions continuous in their respective argument. The author gave sufficient conditions for the asymptotic stability and boundedness of solutions for the above equations. Equations of the form (3) and (4) in which $a(t)$, $b(t)$ and $c(t)$ are constants had earlier been studied by Sadek [21] and Zhu [28]. They obtained conditions that ensure the uniform boundedness and uniformly ultimate boundedness of solutions. Tunç [23] considered real third order nonlinear delay differential equation of the form

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x}, \ddot{x}\,(t-r)) + b(t)g(\dot{x}\,(t-r)) + c(t)h(x) = e(t)$$

and established sufficient criteria for boundedness of solutions.

However, the motivation for this paper come from the papers of Tunç ([24], [23]) and Adams et.al [1]. These authors made use of energy function to establish results for the nonlinear differential equations being considered.

Non-autonomous differential equations with delay can exhibit highly complicated dynamical behaviour especially in after effect, nonlinear oscillation and differential equations with deviating arguments (see [17]). By employing energy function, we establish the boundedness and regularity of solutions of equation (1). Our results in this paper extend and improve on Tunç ([24], [23]) and some earlier results.

2. Preliminaries and Notations

We will have to give some basic notations and definitions for the general non-autonomous delay differential equation (see Burton [4] and also Tunç [23], Èl'sgol'ts [5], Èl'sgol'ts and Norkin [6], Kolmanovskii and Myšhkis [9], Kolmanovskii and Nosov [10], Krasovskiĭ [11], Makay [12], Mohammed [13], Yoshizawa [27] and the references cited therein). Consider the general non-autonomous delay differential system

$$\dot{x} = f(t, x),\ x_t(\theta) = x(t+\theta),\ -r \le \theta \le 0,\ t \ge 0, \tag{5}$$

where $f : [0, \infty) \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, and suppose that f takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \|.\|)$

is the Banach space of continuous function $\phi : [-r, 0] \longrightarrow \mathbb{R}^n$ with supremum norm, $r > 0$, C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \|\phi\| < H\}$. Standard existence theory (see [4]) shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, x_0, \phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying equation (5) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote the norm in $\mathbb{R}^n$ with $|x| = \max_{1 \leq j \leq n} |x_j|$.

Definition 1. (See [4].) A continuous positive definite function $W : \mathbb{R}^n \longrightarrow [0, \infty)$ is called a Wedge.
Definition 2. (See [4].) A continuous function $W : [0, \infty) \longrightarrow [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W is strictly increasing is a Wedge. (We denote wedges by W or W_i, where i is an integer.)
Definition 3. (See [4].) Let D be an open set in $\mathbb{R}^n$ with $0 \in D$. A function $V : [0, \infty) \times D \longrightarrow [0, \infty)$ is called
(a) positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$,
(b) decrescent if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.
Definition 4. (See [4].) Let $V(t, \phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of $V(t_0, \phi)$ along solutions of (5) will be denoted by $\dot{V}_{(5)}\,(t_0, \phi)$ and is defined by the following relation

$$\dot{V}_{(5)}\,(t, \phi) = \lim_{h \longrightarrow 0} \sup \frac{V(t + h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ is the solution of (5) with $x_{t_0}(t_0, \phi) = \phi$.

3. Main Results

Theorem 3.1. *In addition to the basic assumptions imposed on functions b, c, f, g, h, and p in equation (1), we assume that the following conditions hold:*

(i) $b(t) > 0,\ b'(t) > 0$ *for all* $t \in \mathbb{R}^+$;
(ii) $\dfrac{a(t, x, y, z, z(t-r))}{z} \geq \delta_0$ *for all* $t \in \mathbb{R}^+$ *and* $x, y, z, z(t-r) \in \mathbb{R}$ $(z \neq 0)$, *where* δ_0 *is a positive constant;*
(iii) $m(t, x, y, z, z(t-r)) \geq 0$ *for all* $t \in \mathbb{R}^+$ *and* $x, y, z, z(t-r) \in \mathbb{R}$;
(iv) $g(0) = 0$, $yg(y) \geq 0$ *and* $0 < g'(y) \leq \lambda$ *for all* $y \in \mathbb{R}$, *where* λ *is a positive constant, and* $\lim_{s \longrightarrow \pm\infty} G(s) = +\infty$ $\left(G(s) = \int_0^s g(\tau) d\tau\right)$;
(v) $|c(t, x, y, z, z(t-r))| \leq \varphi(t)$ *where* $\varphi \in L^1(0, \infty)$ *and* $t \in \mathbb{R}^+$;
(vi) $0 < \dfrac{h(x)}{x} \leq \delta_1$ *for all* $x \in \mathbb{R}$ $(x \neq 0)$, *where* δ_1 *is a positive constant;*
(vii) $|p(t, x, y, z)| \leq e(t)$ *for all* $t \in \mathbb{R}^+$ *and* $x, y, z \in \mathbb{R}$;
(viii) *There are arbitrary continuous functions* α_0, α_1, β *and* φ *on* $\mathbb{R}^+ = (0, \infty)$ *such that* α_0, α_1 *and* φ *are positive and decreasing functions and* β *is positive and increasing for all* $t \in \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$ *and*

(ix) $\dfrac{e(t)}{\sqrt{b(t)}}, \Big[\dfrac{\alpha_0(t)}{\alpha_1(t)}\Big]^{\frac{1}{2}}, \Big[\dfrac{\alpha_1(t)b(t)}{\beta(t)}\Big]^{\frac{1}{2}}, \varphi(t)\Big[\dfrac{\beta(t)}{\alpha_0(t)b(t)}\Big]^{\frac{1}{2}} \in L^1(0,\infty)$,
where $L^1(0,\infty)$ *is the space of integrable Lebesgue functions.*

Then, for every solution of equation (1), $\dfrac{x}{\sqrt{\beta/\alpha_0}}, \dfrac{x'}{\sqrt{\beta/\alpha_1}}, \dfrac{x''}{\sqrt{b(t)}}$ *are bounded for all* $t \in \mathbb{R}^+$ *provided that*

$$r \leq \Big[\frac{2\delta_0}{\lambda + \mu b(t)} + \frac{b'(t)}{b(t)(\lambda + \mu b(t))}\Big]$$

where μ *is a positive constant.*

Remark. If $a(t,x,x',x'',x''(t-r)) = 0$ and $c(t,x,x',x'',x''(t-r)) = c(t)$ then equation (1) reduces to the nonlinear delay differential equation by Tunç [23] in which the assumptions (i), (ii), (iv), (vi) - (ix) remain valid.

Now, throughout all the main results established here, our main tool is the continuous differentiable energy function $E = E(t,x,y,z)$ defined by:

$$E := \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 + 2G(y) + \mu\int_{-r}^{0}\int_{t+s}^{t} z^2(u)duds, \tag{6}$$

where μ is a positive constant; α_0, α_1, β and b are positive functions, both α_0 and α_1 are decreasing functions while β and b are increasing functions for all $t \in \mathbb{R}^+$, $\mathbb{R}^+ = (0,\infty)$. Clearly, the term $\mu\displaystyle\int_{-r}^{0}\int_{t+s}^{t} z^2(u)duds$ is non-negative.

Proof. Let $(x,y,z) = (x(t),y(t),z(t))$ be an arbitrary solutions of equation (2). Differentiating the functional $E = E(t,x,y,z)$ along the equation (2) and making use of the assumptions of Theorem 1, we have as follows:

$$\begin{aligned}\frac{dE}{dt} &= \Big[\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big]x^2 + \Big[\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big]y^2 + 2\frac{\alpha_0(t)}{\beta(t)}xy\\ &+ 2\frac{\alpha_1(t)}{\beta(t)}yz - \frac{b'(t)}{b^2(t)}z^2 - 2\frac{a(t,x,y,z,z(t-r))}{b(t)}z^2\\ &- 2\frac{m(t,x,y,z,z(t-r))}{b(t)}z^2 - 2\frac{c(t,x,y,z,z(t-r))}{b(t)}h(x)z\\ &+ 2z\int_{t-r}^{t} g'(y(s))z(s)ds - 2\frac{p(t,x,y,z)}{b(t)}z + \mu r z^2 - \mu\int_{t-r}^{t} z^2(s)ds.\end{aligned} \tag{7}$$

Obviously, using the conditions imposed on the functions α_0, α_1, β, b for all $t \in \mathbb{R}^+$ and the assumption (iii), we have

$$\Big[\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'}{\beta^2(t)}\Big] < 0,\ \Big[\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'}{\beta^2(t)}\Big] < 0,\ -\frac{b'(t)}{b^2(t)} < 0$$

and $-2\dfrac{m(t,x,y,z,z(t-r))}{b(t)} \leq 0.$

Then

$$\begin{aligned}\frac{dE}{dt} \leq & -2\frac{a(t,x,y,z,z(t-r))}{b(t)}z^2 + 2\frac{\alpha_0(t)}{\beta(t)}xy + 2\frac{\alpha_1(t)}{\beta(t)}yz \\ & - 2\frac{c(t,x,y,z,z(t-r))}{b(t)}h(x)z + 2z\int_{t-r}^{t} g'(y(s))z(s)ds \\ & - 2\frac{p(t,x,y,z)}{b(t)}z + \mu r z^2 - \mu\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

By applying the assumptions (ii), (v) - (vii), we obtain

$$\begin{aligned}\frac{dE}{dt} \leq & -2\frac{\delta_0}{b(t)}z^2 + 2\frac{\alpha_0(t)}{\beta(t)}|x||y| + 2\frac{\alpha_1(t)}{\beta(t)}|y||z| + 2\frac{\varphi(t)}{b(t)}\delta_1|z| \\ & + 2z\int_{t-r}^{t} g'(y(s))z(s)ds - 2\frac{p(t,x,y,z)}{b(t)}z + \mu r z^2 \\ & - \mu\int_{t-r}^{t} z^2(s)ds. \end{aligned} \tag{8}$$

In view of the inequality $2|uv| \leq u^2 + v^2$, it can be shown from inequality (8) that

$$2z\int_{t-r}^{t} g'(y(s))z(s)ds \leq \lambda r z^2 + \lambda\int_{t-r}^{t} z^2(s)ds. \tag{9}$$

And in view of the functional $E = E(t,x,y,z)$, it is obvious from (8) that

$$\begin{aligned}& |x| \leq E^{\frac{1}{2}}\Big(\frac{\beta(t)}{\alpha_0(t)}\Big)^{\frac{1}{2}}, \\ & |y| \leq E^{\frac{1}{2}}\Big(\frac{\beta(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}} \\ & \textit{and} \\ & |z| \leq b^{\frac{1}{2}}(t)E^{\frac{1}{2}} \leq b^{\frac{1}{2}}(t)\Big(\frac{1}{2} + \frac{E}{2}\Big).\end{aligned}$$

Therefore, the following terms from inequality (8) become

$$\begin{aligned} 2\frac{\alpha_0(t)}{\beta(t)}|x||y| &\leq 2\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E \\ 2\frac{\alpha_1(t)}{\beta(t)}|y||z| &\leq 2\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}E \\ 2\frac{|e(t)|}{b(t)}|z| &\leq \frac{|e(t)|}{b^{\frac{1}{2}}(t)} + \frac{|e(t)|}{b^{\frac{1}{2}}(t)}E \\ 2\frac{\delta_1(t)\varphi(t)}{b(t)}|x||z| &\leq 2\delta_1(t)\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}E. \end{aligned} \tag{10}$$

Subject to the above inequalities (9) and (10), inequality (8) now gives

$$\begin{aligned}\frac{dE}{dt} &\le \Big[-2\frac{\delta_0}{b(t)}-\frac{b'(t)}{b^2(t)}+\lambda r+\mu r\Big]z^2+2\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E\\ &+2\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}E+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}E+2\delta_1(t)\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}E\\ &-\Big[\mu-\lambda\Big]\int_{t-r}^{t}z^2(s)ds. \qquad (11)\end{aligned}$$

Let $\mu=\lambda$. Hence, we obtain from inequality (11) that

$$\begin{aligned}\frac{dE}{dt} &\le 2\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E+2\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}E+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}E\\ &+2\delta_1(t)\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}E, \qquad (12)\end{aligned}$$

provided

$$r\le\frac{2\delta_0 b(t)+b'(t)}{2\lambda b^2(t)}.$$

Now, let

$$\Phi(t)=2\Big[\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}+\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}+\frac{|e(t)|}{2b^{\frac{1}{2}}(t)}+\delta_1(t)\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]. \qquad (13)$$

It follows from (12) and (13) that

$$\frac{dE}{dt}\le\Big[\frac{|e(t)|}{b^{\frac{1}{2}}(t)}\Big]+\Phi(t)E. \qquad (14)$$

Integrating (14) between 0 and t, we obtain

$$\begin{aligned}E(t,x,y,z)-E(0,x(0),y(0),z(0)) &= \int_0^t\frac{|e(s)|}{\sqrt{b(s)}}ds\\ &+\int_0^t\Phi(s)E(s,x(s),y(s),z(s))ds.\end{aligned}$$

By using assumption (ix) of Theorem 3.1 and the Gronwall-Ried-Bellman inequality, we have

$$E(t,x,y,z)\le A\exp\Big(\int_0^t\Phi(s)ds\Big)$$

for a positive constant A, where $A=E(0,x(0),y(0),z(0))+\displaystyle\int_0^\infty\frac{|e(s)|}{\sqrt{b(s)}}ds.$
Then, the Theorem 3.1 yields $\Phi(t)\in L^1(0,\infty)$, hence the boundedness of the function E. That is, it can be easily concluded that

$$\frac{\alpha_0(t)}{\beta(t)}x^2,\ \frac{\alpha_1(t)}{\beta(t)}y^2\ and\ \frac{1}{b(t)}z^2$$

are bounded. Thus, this result guarantees the boundedness of

$$\frac{x}{\sqrt{\beta/\alpha_0}},\ \frac{x'}{\sqrt{\beta/\alpha_1}}\ and\ \frac{x''}{\sqrt{b(t)}}$$

which proves the theorem. □

Theorem 3.2. *Let the assumptions (i), (ii), (iv) - (vii) of Theorem 3.1 remain valid. Thus:*

(x) *there exist a positive constant H such that $m(t,x,y,z,z(t-r)) \geq H$ for all $t \in \mathbb{R}^+$ and x,y,z,$z(t-r) \in \mathbb{R}$ $(z \pm 0)$ and $b'(t) + Hb(t) > 0$ for all $t \in \mathbb{R}$;*

(xi) *there are arbitrary continuous functions α_0, α_1, β and φ on $\mathbb{R}^+ = (0,\infty)$ such that α_0, α_1 and φ are positive and decreasing, β is positive and increasing for all $t \in \mathbb{R}^+$, and*

$$\frac{e^2(t)}{b'(t)+2Hb(t)},\ \frac{e(t)}{\sqrt{b(t)}},\ \alpha_2(t),\ \Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}},\ \Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}},$$

$$\varphi(t)\Big(\frac{\beta(t)}{\alpha_1(t)b(t)}\Big)^{\frac{1}{2}} \in L^1(0,\infty).$$

Then, the conclusion of Theorem 3.1 holds.

Proof. As known, the function E defined in equation (6) is positive definite. Now subject to the assumption of Theorem 3.2, a calculation from equations (6) along with (2) shows that

$$\begin{aligned}\frac{dE}{dt} &= \Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2 + \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2 - \frac{b'(t)}{b^2(t)}z^2 \\ &\quad - \frac{2}{b(t)}(a(t,x,y,z,z(t-r)) + m(t,x,y,z,z(t-r)))z^2 \\ &\quad - \frac{2}{b(t)}c(t,x,y,z,z(t-r))h(x)z + 2z\int_{t-r}^{t} g'(y(s))z(s)ds \\ &\quad + 2\frac{\alpha_0(t)}{\beta(t)}xy + 2\frac{\alpha_1(t)}{\beta(t)}yz + \frac{2}{b(t)}p(t,x,y,z)z + \mu r z^2 - \mu\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{dE}{dt} &\leq -\frac{b'(t)}{b^2(t)}z^2 - \frac{2}{b(t)}m(t,x,y,z,z(t-r)))z^2 - \frac{2}{b(t)}c(t,x,y,z,z(t-r))h(x)z \\ &\quad + 2z\int_{t-r}^{t} g'(y(s))z(s)ds + 2\frac{\alpha_0(t)}{\beta(t)}xy + 2\frac{\alpha_1(t)}{\beta(t)}yz + \frac{2}{b(t)}p(t,x,y,z)z + \mu r z^2 \\ &\quad - \mu\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

Now, using the assumptions (iv) - (vii) and (x), we have

$$\begin{aligned}\frac{dE}{dt} &\leq -2\frac{H}{b(t)}z^2 + \mu r z^2 + \lambda r z^2 - \frac{b'(t)}{b^2(t)}z^2 + 2\frac{\alpha_0(t)}{\beta(t)}xy + 2\frac{\alpha_1(t)}{\beta(t)}yz \\ &\quad + 2\frac{\varphi(t)}{b(t)}\delta_1 xz + 2\frac{|e(t)|}{b(t)}z - \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{t-r}^{t} z^2(s)ds \\ &= -\Big[\frac{H}{b(t)} - \mu r - \lambda r\Big]z^2 - \frac{H}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + 2\frac{\alpha_0(t)}{\beta(t)}|x||y| \\ &\quad + 2\frac{\alpha_1(t)}{\beta(t)}|y||z| + 2\frac{\varphi(t)}{b(t)}\delta_1|x||z| + 2\frac{|e(t)|}{b(t)}|z| - \Big[\mu - \lambda\Big]\int_{t-r}^{t} z^2(s)ds. \qquad (15)\end{aligned}$$

Choosing $\mu = \lambda$, it follows from (15) that

$$\begin{aligned}\frac{dE}{dt} &\leq -\frac{H}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + 2\frac{\alpha_0(t)}{\beta(t)}|x||y| + 2\frac{\alpha_1(t)}{\beta(t)}|y||z| \\ &\quad + 2\frac{\varphi(t)}{b(t)}\delta_1|x||z| + 2\frac{|e(t)|}{b(t)}|z| \end{aligned} \tag{16}$$

provided that $r \leq \Big(\frac{H}{2\lambda b(t)}\Big)$ which we can now assume. Thus, we have from inequality (16) that

$$\begin{aligned}\frac{dE}{dt} &\leq -(b'(t) + Hb(t))\Big(\frac{|z|}{b(t)} - \frac{|e(t)|}{b'(t) + Hb(t)}\Big)^2 + \frac{e^2(t)}{b'(t) + Hb(t)} \\ &\quad + 2\frac{\alpha_0(t)}{\beta(t)}|x||y| + 2\frac{\alpha_1(t)}{\beta}|y||z| + 2\frac{\varphi(t)}{b(t)}\delta_1|z||x| \\ &\leq \frac{e^2(t)}{b'(t) + Hb(t)} + 2\Big[\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}} + \Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}} + \delta_1\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]E.\end{aligned}$$

Therefore,

$$\frac{dE}{dt} \leq \frac{e^2(t)}{b'(t) + Hb(t)} + \Big[\Phi(t) - \frac{|e(t)|}{b^{\frac{1}{2}}(t)}\Big]E, \tag{17}$$

where $\Phi(t)$ is defined in equation (13). Now, as in the proof of Theorem 3.1, integrating (17) from 0 to t, later using the assumption (xi) of Theorem 3.2 and the Gronwall-Ried-Bellman inequality, one can deduce the boundedness of the function E. The proof of Theorem 3.2 is now complete. □

The next is concerned with the regularity of solutions of the non-linear delay differential equation (1).

Theorem 3.3. *Let all assumptions of Theorem 3.2 hold. Then every solution of the equation (1) satisfies*

$$\Big(\frac{|\alpha_0'(t)|}{\beta(t)}\Big)^{\frac{1}{2}} x \in L^2(0,\infty) \text{ and } \Big(\frac{|\alpha_1'(t)|}{\beta(t)}\Big)^{\frac{1}{2}} x' \in L^2(0,\infty).$$

If in addition, we now assume

$$l.u.b\ \frac{b^2(t)}{b'(t) + 2Hb(t)} = \mu < \infty,\ t \geq 0,$$

then

$$x'' \in L^2(0,\infty).$$

Proof. Now, following the procedure used in the proof of Theorem 3.1 and Theorem 3.2 above, except for some minor modification, we obtain as follows:

$$\begin{aligned}
\frac{dE}{dt} &\leq \Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2 + \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2 - \frac{b'(t)}{b^2(t)}z^2 - 2\frac{H}{b(t)}z^2 \\
&+ \frac{|e(t)|}{b^{\frac{1}{2}}(t)} + 2\Big[\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}} + \Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}} + \frac{|e(t)|}{2b^{\frac{1}{2}}(t)} + \delta_1\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]E \\
&= \Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2 + \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2 - \frac{b'(t)}{b^2(t)}z^2 - 2\frac{H}{b(t)}z^2 \\
&+ \frac{|e(t)|}{2b^{\frac{1}{2}}(t)} + \Phi(t)E,
\end{aligned}$$

where $\Phi(t)$ is given by equation (13).
Hence, it follows that

$$\begin{aligned}
&\Big(\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)} - \frac{\alpha_0'(t)}{\beta(t)}\Big)x^2 + \Big(\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)} - \frac{\alpha_1'(t)}{\beta(t)}\Big)y^2 \\
&+ \frac{b'(t)+2Hb(t)}{b^2(t)}z^2 \leq -\frac{dE}{dt} + \frac{|e(t)|}{\sqrt{b(t)}} + \Phi(t)E. \qquad (18)
\end{aligned}$$

Integrating both sides of inequality (18) from 0 to t, we have

$$\begin{aligned}
&\int_0^t \Big[\Big(\frac{\alpha_0(\tau)\beta'(\tau)}{\beta^2(\tau)} - \frac{\alpha_0'(\tau)}{\beta(\tau)}\Big)x^2 + \Big(\frac{\alpha_1(\tau)\beta'(\tau)}{\beta^2(\tau)} - \frac{\alpha_1'(\tau)}{\beta(\tau)}\Big)y^2 \\
&+ \frac{b'(\tau)+2Hb(\tau)}{b^2(\tau)}z^2\Big]d\tau \leq E(0) - E(t) + \int_0^t \frac{|e(\tau)|}{\sqrt{b(\tau)}}d\tau + k\int_0^t \Phi(\tau)d\tau,
\end{aligned}$$

where it is assumed that $E(t,x,y,z) \equiv E \leq k$, $t > 0$. In view of the assumptions of Theorem 3.3, and the boundedness of the function E, we can conclude that

$$\int_0^t \frac{|\alpha_0'(\tau)|}{\beta(\tau)}x^2 d\tau < \infty, \int_0^t \frac{|\alpha_1'(\tau)|}{\beta(\tau)}y^2 d\tau < \infty \text{ and } \int_0^t z^2 d\tau < \infty, \; t \geq 0.$$

The proof of Theorem 3.3 is now complete. □

Example. Here we consider a suitable application to Theorem 3.1, the third order nonlinear differential equation

$$\begin{aligned}
&x''' + [-(1+t^2+x^2+(x')^2+(x'')^2+(x''(t-r))^2)^{-1} \\
&+ (2+t^2+x^2+(x')^2+(x'')^2+(x''(t-r))^2)]x'' + (t^2+1)^5 x' \\
&+ \frac{1}{(t^2+1)^9}x = e(t). \qquad (19)
\end{aligned}$$

Obviously, equation (19) is a special case of (1), and its equivalent system become

$$
\begin{aligned}
x' &= y,\\
y' &= z,\\
z' &= (1+t^2+x^2+y^2+z^2+z^2(t-r))^{-1}z\\
&\quad -(2+t^2+x^2+y^2+z^2+z^2(t-r))z\\
&\quad -(t^2+1)^5 y - \frac{1}{(t^2+1)^9}x + e(t).
\end{aligned}
\tag{20}
$$

Let
$\alpha_0 = \dfrac{1}{(t^2+1)^7}$, $\alpha_1 = \dfrac{1}{(t^2+1)^5}$, $\beta = (t^2+1)^6$ and $\varphi = \dfrac{1}{(t^2+1)^9}$.

Clearly, α_0, α_1, and φ are positive and decreasing functions while β is postive and increasing for all $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty]$ and

$$
\begin{aligned}
\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}} &= \frac{1}{t^2+1} \in L^1(0,\infty)\\
\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}} &= \frac{1}{(t^2+1)^3} \in L^1(0,\infty)\\
\varphi(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}} &= \frac{1}{(t^2+1)^5} \in L^1(0,\infty).
\end{aligned}
$$

Now, if we take $e(t)$ such that $\dfrac{e(t)}{\sqrt{b(t)}} = \dfrac{e(t)}{(b(t))^{1/2}} = \dfrac{e(t)}{(t^2+1)^{5/2}} \in L^1(0,\infty)$ then every $x(t)$ of equation (19), we can have the following conclusion

$$
\frac{x}{(t^2+1)^{13/2}}, \ \frac{x'}{(t^2+1)^{11/2}} \ \textit{and} \ \frac{x''}{(t^2+1)^{5/2}}
$$

are bounded for all $t \geq 0$. In view of the above choice, we have from (6) that

$$
\begin{aligned}
E(t,x,y,z) &= \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 + 2G(y) + \mu\int_{-r}^{0}\int_{t+s}^{t} z^2(u)duds\\
&= \frac{1}{(t^2+1)^{13}}x^2 + \frac{1}{(t^2+1)^{11}}y^2 + \frac{1}{(t^2+1)^5}z^2 + y^2\\
&\quad + \mu\int_{-r}^{0}\int_{t+s}^{t} z^2(u)duds.
\end{aligned}
\tag{21}
$$

Then, the function $E = E(t,x,y,z)$ is a positive definite function.

Now, differentiating the function E in (21) along the above system (20), we obtain as follows

$$\begin{aligned}\frac{dE}{dt} &= -\frac{26t}{(t^2+1)^{14}}x^2 - \frac{22t}{(t^2+1)^{12}}y^2 - \frac{10t}{(t^2+1)^5}z^2 \\ &- \frac{2}{(t^2+1)^5}(2+t^2+x^2+y^2+z^2+z^2(t-r))z^2 \\ &+ \frac{2}{(t^2+1)^5}(1+t^2+x^2+y^2+z^2+z^2(t-r))^{-1}z^2 \\ &+ \frac{2}{(t^2+1)^{13}}xy + \frac{2}{(t^2+1)^{11}}yz - \frac{2}{(t^2+1)^{14}}xz + \frac{2e(t)}{(t^2+1)^5}z \\ &+ \mu r z^2 - \mu\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

Following the assumptions imposed on the functions α_0, α_1, β, b and the assumption (iii) as in the proof of Theorem 3.1, we now achieve the following inequality

$$\begin{aligned}\frac{dE}{dt} &\leq \left[\frac{2}{(t^2+1)^5}(1+t^2+x^2+y^2+z^2+z^2(t-r))^{-1} + \mu r\right]z^2 \\ &+ \frac{2}{(t^2+1)^{13}}|x||y| + \frac{2}{(t^2+1)^{11}}|y||z| - \frac{2}{(t^2+1)^{14}}|x||z| + \frac{2e(t)}{(t^2+1)^5}|z|. \qquad (22)\end{aligned}$$

Clearly, inequality (22) implies that

$$\begin{aligned}\frac{dE}{dt} &\leq \frac{2}{(t^2+1)^{13}}|x||y| + \frac{2}{(t^2+1)^{11}}|y||z| - \frac{2}{(t^2+1)^{14}}|x||z| \\ &+ \frac{2e(t)}{(t^2+1)^5}|z| \qquad (23)\end{aligned}$$

provided that the assumption

$$r \leq \left[\frac{-2(1+t^2+x^2+y^2+z^2+z^2(t-r))^{-1}}{\mu(t^2+1)^5}\right].$$

Comparing the inequalities (10), (12) with the equation (13), we can now deduce from inequality (23) that

$$\frac{dE}{dt} \leq \Phi(t)E + \frac{e(t)}{(t^2+1)^{5/2}}, \qquad (24)$$

where $\Phi(t) = 2\left[\frac{2}{(t^2+1)} + \frac{1}{(t^2+1)^3} + \frac{e(t)}{2(t^2+1)^{5/2}} + \frac{1}{(t^2+1)^5}\right]$.

Integrating inequality (24) from 0 to t, we have

$$E(t) - E(0) \leq \int_0^t \Phi(\tau)E(\tau)d\tau + \int_0^t \frac{e(\tau)}{(\tau^2+1)^{5/2}}d\tau.$$

Now, applying the Gronwall-Ried-Bellman inequality, we have

$$E(t) \leq Q\exp\left(\int_0^t \Phi(\tau)d\tau\right),$$

where $Q = E(0) + \int_0^t \frac{e(\tau)}{(\tau^2+1)^{5/2}} d\tau$. Thus, $\Phi(t) \in L^1(0,\infty)$ implies the boundedness of E and hence the boundedness of

$$\frac{x}{(t^2+1)^{13/2}}, \ \frac{x'}{(t^2+1)^{11/2}} \ and \ \frac{x''}{(t^2+1)^{5/2}}.$$

Hence, this shows the useful application of Theorem 3.1.

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