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A NOTE ON q-FUBINI POLYNOMIALS

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Abstract. Motivated by the construction of the generating functions of q-Bernoulli polynomials and q-Euler polynomials satisfying with their important results, we define a new q-class of the Fubini polynomials. We give some new properties including correlations with the number $S_{2,q}(n,k)$ given in the paper. We also define two types q-Fubini polynomials with three parameters and then provide several correlations and identities.

1. Introduction

Special numbers and polynomials play an extremely important role in the development of several branches of mathematics, physics and engineering. The problems arising in the mathematical physics and engineering fields are mathematically framed in terms of differential equations. Most of them can only be solved using families of special functions which provide new means of mathematical analysis. They are also useful under algebraic operations including addition, multiplication, differentiation, integration, and composition. They are also richly utilized in computational models of scientific and engineering problems. Such families can be described in various ways, for example, by orthogonality conditions, as solutions to differential equations, by generating functions, by recurrence relations, by operational formulas, and by integral transforms. For more information about the applications of special functions and polynomials, see [1, 2, 3, 4, 5].

The Fubini polynomials are defined by

$$F_n(y)=\sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!y^k$$

where $\left\{ {n \atop k} \right\}$ is known as Stirling numbers of the second kind. These polynomials can be defined by means of the following generating function:

$$\frac{1}{1-y\left(e^t-1\right)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}.$$

Because of the relation

$$\sum_{k=0}^{\infty}k^m y^k=\frac{1}{1-y}F_m\left(\frac{y}{1-y}\right), \qquad (|y|<1)$$

Fubini polynomials are called geometric polynomials ([6, 7]).

The n-th Fubini number is also defined by

$$F_n(1):=F_n=\sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!y^k$$

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and counts all the possible set partitions of an n element set such that the order of the blocks matters ([6, 7]).

Recent works including a class of the new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials by [8], degenerate Fubini polynomials by [9], symmetric identities for Fubini polynomials by [10], degenerate ordered Bell numbers and polynomials by [11], q-generalized Euler numbers and polynomials by [12], two variable higher-order Fubini polynomials by [13], Fourier series of functions related to two variable higher-order Fubini polynomials by [14], sums of products of two variable higher-order Fubini functions arising from Fourier series by [15], symmetric identities involving Fubini polynomials and Euler numbers by [16], have been studied extensively.

We begin with the definitions of the following notations that will be useful in this paper:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

As usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Let q be chosen as a fixed real number satisfying $|q| < 1$. A q-analogue of a real number n is defined by

$$[n]_q := \frac{q^n - 1}{q - 1} \quad (q \neq 1).$$

Obviously that

$$\lim_{q \to 1^-} [n]_q = n.$$

The q-derivative operator D_q of a function f is given as

$$(D_q f)(x) = D_{q:x} f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1-q)x} & \text{if } q \neq 1 \text{ and } x \neq 0 \\ f'(0) & \text{if } \quad x = 0 \end{cases} \tag{1.1}$$

provided $f'(0)$ exists.

For any two functions $f(x)$ and $g(x)$, the product rule and quotient rule of the q-derivative operator are provided by

$$D_q (f(x) g(x)) = g(qx) D_q f(x) + f(x) D_q g(x) \tag{1.2}$$

and

$$D_q \left(\frac{g(x)}{f(x)} \right) = \frac{f(qx) D_q g(x) - g(qx) D_q f(x)}{f(x) f(qx)} = \frac{f(x) D_q g(x) - g(x) D_q f(x)}{f(x) f(qx)}. \tag{1.3}$$

The q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and q-factorial $[n]_q!$ are, respectively, defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} \quad (n \geq k; \; n, k \in \mathbb{N}_0)$$

and

$$[n]_q! = [n]_q [n-1]_q [n-2]_q \cdots [1]_q; \quad [0]_q! = 1 \qquad (n \in \mathbb{N}).$$

The q-generalization of $(\lambda + \mu)^n$ is defined by

$$(\lambda \oplus \mu)_q^n = (\lambda + \mu)(\lambda + q\mu) \cdots \left(\lambda + q^{n-1}\mu\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \lambda^k \mu^{n-k}.$$

where the notation $\oplus$ is borrowed from ref. [17], [18].

The two different types of q-exponential functions are given by

$$e_q(t)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\ \textit{and}\ E_q(t)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{t^n}{[n]_q!}, \tag{1.4}$$

satisfying $e_q(x+y)=e_q(x)e_q(y)$ for the q-commuting variables x and y such as $yx=qxy$ and

$$e_q(x)E_q(y)=e_q\left((x\oplus y)_q\right).$$

Note that

$$\lim_{q\to 1^-}e_q(t)=\lim_{q\to 1^-}E_q(t)=e^t=\sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

We also have

$$D_{q:t}e_q(t)=e_q(t)\ \text{and}\ D_{q:t}E_q(t)=E_q(qt),$$

where the operator $D_{q:t}$ means q-derivative with respect to parameter t.

The tools related to q-calculus shown in this part can be found in $[12,17-22]$.

By using q-exponential function, the generating function of q-Bernoulli polynomials $B_{n,q}(x)$ are defined by Kupershmidt [21] as follows:

$$\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{[n]_q!}=\frac{t}{e_q(t)-1}e_q(xt) \tag{1.5}$$

and Kim also gave q-Euler polynomials $E_{n,q}(x)$ by the following generating function:

$$\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}=\frac{2}{e_q(t)+1}e_q(xt). \tag{1.6}$$

Since $\lim_{q\to 1^-}e_q(t)=e^t$, it is obvious that

$$\lim_{q\to 1^-}B_{n,q}(x):=B_n(x)\ \textit{and}\ \lim_{q\to 1^-}E_{n,q}(x):=E_n(x)$$

calling the classical Bernoulli polynomials and the classical Euler polynomials, respectively.

The family of a number denoted by $S_{2,q}(n,k)$ are defined by means of the following generating function to be (see [19])

$$\sum_{n=k}^{\infty}S_{2,q}(n,k)\frac{t^n}{[n]_q!}=\frac{(e_q(t)-1)^k}{[k]_q!}. \tag{1.7}$$

In the following section, motivated by the generating functions of q-Bernoulli polynomials and q-Euler polynomials, we consider a new q-class of the generating function of the Fubini polynomials. By making use of the generating function of q-Fubini polynomials, we derive some of their basic properties including correlations with the number $S_{2,q}(n,k)$ We also define two types q-Fubini polynomials with three parameters and then provide several correlations and identities.

2. Construction of the q-Fubini polynomials and their applications

In this section, we begin with the following generating series:

$$\frac{1}{1-z(e^t-1)}=\sum_{n=0}^{\infty}F_n(z)\frac{t^n}{n!} \tag{2.1}$$

where $F_n(z)$ is known as the classical Fubini polynomials (or known as geometric polynomials).

Motivated by (1.5) and (1.6), we extend (2.1) to q-class of Fubini polynomials by using q-exponential function as follows.

Definition 1. *q-Fubini polynomials via q-exponential function as in (1.4) are defined by the following generating function to be*

$$\frac{1}{1-z\left(e_q\left(t\right)-1\right)}=\sum_{n=0}^{\infty}F_{n,q}\left(z\right)\frac{t^n}{\left[n\right]_q!}. \tag{2.2}$$

Remark 1. *When $z=1$ in (2.2), one may write $F_{n,q}\left(1\right):=F_{n,q}$ that can be called n-th q-Fubini number as in the classical Fubini number.*

Remark 2. *In the case $q\to 1^-$ in (2.2), one may get*

$$\lim_{q\to 1^-}F_{n,q}\left(z\right)=F_n\left(z\right)$$

are the classical Fubini polynomials.

We give the derivative property with respect to the variable z for q-Fubini polynomials as follows.

Theorem 1. *We have the derivative of q-Fubini polynomial as follows:*

$$\frac{d}{dz}F_{n,q}\left(z\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\sum_{k=0}^{j}\begin{bmatrix}j\\k\end{bmatrix}_qF_{k,q}\left(z\right)F_{j-k,q}\left(z\right)-\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_qF_{l,q}\left(z\right)F_{n-l,q}\left(z\right). \tag{2.3}$$

Proof. By (2.2), we derive

$$\begin{aligned}
\frac{d}{dz}\left(\sum_{n=0}^{\infty}F_{n,q}\left(z\right)\frac{t^n}{\left[n\right]_q!}\right) &= \frac{d}{dz}\left(\frac{1}{1-z\left(e_q\left(t\right)-1\right)}\right)\\
&= \frac{e_q\left(t\right)-1}{\left(1-z\left(e_q\left(t\right)-1\right)\right)^2}\\
&= \frac{1}{\left(1-z\left(e_q\left(t\right)-1\right)\right)^2}e_q\left(t\right)-\frac{1}{\left(1-z\left(e_q\left(t\right)-1\right)\right)^2}\\
&= \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_qF_{k,q}\left(z\right)F_{n-k,q}\left(z\right)\right)\frac{t^n}{\left[n\right]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{\left[n\right]_q!}\right)\\
&\quad -\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_qF_{l,q}\left(z\right)F_{n-l,q}\left(z\right)\right)\frac{t^n}{\left[n\right]_q!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\sum_{k=0}^{j}\begin{bmatrix}j\\k\end{bmatrix}_qF_{k,q}\left(z\right)F_{j-k,q}\left(z\right)-\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_qF_{l,q}\left(z\right)F_{n-l,q}\left(z\right)\right)\frac{t^n}{\left[n\right]_q!}
\end{aligned}$$

which means the desired result (2.3). □

We now state the following theorem.

Theorem 2. *The following relation*

$$F_{n,q}\left(z\right)=\sum_{k=0}^{n}z^k\left[k\right]_q!S_{2,q}\left(n,k\right). \tag{2.4}$$

holds true for $n\in\mathbb{N}_0$.

Proof. In a suitable interval of convergence, we use geometric series expansion for $\frac{1}{1-z(e_q(t)-1)}$ as follows:

$$\begin{aligned}
\frac{1}{1-z\left(e_q(t)-1\right)} &= \sum_{n=0}^{\infty} F_{n,q}(z) \frac{t^n}{[n]_q!} \\
&= \sum_{k=0}^{\infty} z^k \left(e_q(t)-1\right)^k \\
&= \sum_{k=0}^{\infty} z^k [k]_q! \sum_{n=k}^{\infty} S_{2,q}(n,k) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} z^k [k]_q! S_{2,q}(n,k) \right) \frac{t^n}{[n]_q!}
\end{aligned}$$

which means the asserted result (2.4). □

From the special case $z = 1$ in Theorem 2, we have the following corollary.

Corollary 1. *The q-Fubini numbers satisfy the following relation*

$$F_{n,q} = \sum_{k=0}^{n} [k]_q! S_{2,q}(n,k). \tag{2.5}$$

We here give the following theorem for the q-Fubini polynomials.

Theorem 3. *The following relation*

$$\frac{1}{1-z} F_{n,q}\left(\frac{z}{1-z}\right) = \sum_{m=0}^{\infty} z^m \sum_{k=0}^{\min(n,m)} \binom{m}{k} [k]_q! S_{2,q}(n,k). \tag{2.6}$$

holds true for $n \in \mathbb{N}_0$ and $z \neq 1$.

Proof. By (1.7) and (2.2), we observe

$$\begin{aligned}
\frac{1}{1-z} \sum_{n=0}^{\infty} F_{n,q}\left(\frac{z}{1-z}\right) \frac{t^n}{[n]_q!} &= \frac{1}{1-z} \left(\frac{1}{1-\frac{z}{1-z}\left(e_q(t)-1\right)} \right) \\
&= \frac{1}{1-ze_q(t)} \\
&= \sum_{m=0}^{\infty} z^m \left(e_q(t)-1+1\right)^m \\
&= \sum_{m=0}^{\infty} z^m \sum_{k=0}^{m} \binom{m}{k} \left(e_q(t)-1\right)^k \\
&= \sum_{m=0}^{\infty} z^m \sum_{k=0}^{m} \binom{m}{k} [k]_q! \sum_{n=k}^{\infty} S_{2,q}(n,k) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} z^m \sum_{k=0}^{\min(n,m)} \binom{m}{k} [k]_q! S_{2,q}(n,k) \right) \frac{t^n}{[n]_q!}
\end{aligned}$$

which imply the desired result (2.6). □

From the special case $z=\frac{1}{2}$ in Theorem 3, we obtain the following corollary.

Corollary 2. *The q-Fubini numbers holds the following relation*

$$F_{n,q}=\sum_{m=0}^{\infty}\frac{1}{2^{m+1}}\sum_{k=0}^{\min(n,m)}\binom{m}{k}[k]_q!S_{2,q}(n,k). \tag{2.7}$$

We give a relationship for q-Fubini polynomials as follows.

Theorem 4. *We have the following recurrence relation for q-Fubini polynomial:*

$$F_{n+1,q}(z)=z\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q F_{k,q}(z)F_{n-m,q}(z)q^{n-m}$$

Proof. By (1.3) and (2.2), we get

$$D_{q:t}\left(\sum_{n=0}^{\infty}F_{n,q}(z)\frac{t^n}{[n]_q!}\right)=\sum_{n=0}^{\infty}F_{n+1,q}(z)\frac{t^n}{[n]_q!}$$

and on the other hand

$$\begin{aligned}
D_{q:t}\left(\frac{1}{1-z(e_q(t)-1)}\right) &= \frac{-D_q(1-z(e_q(t)-1))}{(1-z(e_q(t)-1))(1-z(e_q(qt)-1))}\\
&= z\frac{e_q(t)}{(1-z(e_q(t)-1))(1-z(e_q(qt)-1))}\\
&= z\frac{e_q(t)}{(1-z(e_q(t)-1))}\frac{1}{(1-z(e_q(qt)-1))}\\
&= z\left(\sum_{n=0}^{\infty}F_{n,q}(z)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}F_{n,q}(z)q^n\frac{t^n}{[n]_q!}\right)\\
&= z\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q F_{k,q}(z)\right)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}F_{n,q}(z)q^n\frac{t^n}{[n]_q!}\right)\\
&= \sum_{n=0}^{\infty}\left(z\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q F_{k,q}(z)F_{n-m,q}(z)q^{n-m}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Hence, the proof of this theorem is completed. □

We now consider the following definition related to q-Fubini polynomials with three parameters as follows.

Definition 2. *We introduce the q-Fubini polynomials with three parameters via q-exponential function as in (1.4) by means of the following generating function*

$$\frac{1}{1-z(e_q(t)-1)}e_q(xt)E_q(yt)=\sum_{n=0}^{\infty}F_{n,q}(x,y;z)\frac{t^n}{[n]_q!}. \tag{2.8}$$

Remark 3. *Taking $x=y=0$ in (2.8) gives $F_{n,q}(0,0;z):=F_{n,q}(z)$ given in (2.2). Also, upon setting $x=y=0$ and $z=1$, we have $F_{n,q}(0,0;1)=F_{n,q}$ given in Remark 1.*

The immediate result for the q-Fubini polynomials $F_{n,q}(x,y;z)$ with three parameters are related to the classical q-Fubini polynomials $F_{n,q}(z)$ as stated below.

Theorem 5. *We have the following binomial formula for q-Fubini polynomial:*

$$F_{n,q}(x,y;z)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(z)\,(x\oplus y)_q^{n-k}. \tag{2.9}$$

Proof. In view of (2.2) and (2.8), using Cauchy product, we see

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,q}(x,y;z)\frac{t^n}{[n]_q!} &= \frac{1}{1-z\,(e_q(t)-1)}e_q(xt)\,E_q(yt)\\ &= \left(\sum_{n=0}^{\infty}F_{n,q}(z)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}(x\oplus y)_q^n\frac{t^n}{[n]_q!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(z)\,(x\oplus y)_q^{n-k}\right)\frac{t^n}{[n]_q!}\end{aligned}$$

which is the claimed result (2.9). □

By Theorem 5, we easily attain the following Corollary.

Corollary 3. *q-Fubini polynomial with three variables can be written by q-Fubini polynomial with two variables as follows:*

$$F_{n,q}(x,y;z)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(0,y;z)\,x^{n-k} \tag{2.10}$$

and

$$F_{n,q}(x,y;z)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}F_{k,q}(x,0;z)\,y^{n-k}. \tag{2.11}$$

We present the following theorem.

Theorem 6. *The following relation holds true:*

$$zF_{n,q}(x+1,y;z)=(z+1)\,F_{n,q}(x,y;z)-(x\oplus y)_q^n\ . \tag{2.12}$$

Proof. Using Definition 2, we consider

$$\begin{aligned}\sum_{n=0}^{\infty}\left(F_{n,q}(x+1,y;z)-F_{n,q}(x,y;z)\right)\frac{t^n}{[n]_q!} &= \frac{e_q((x+1)\,t)-e_q(xt)}{1-z\,(e_q(t)-1)}E_q(yt)\\ &= \frac{1}{z}\frac{z\,(e_q(t)-1)}{1-z\,(e_q(t)-1)}e_q(xt)\,E_q(yt)\\ &= \frac{1}{z}\left(\frac{e_q(xt)\,E_q(yt)}{1-z\,(e_q(t)-1)}-e_q(xt)\,E_q(yt)\right)\\ &= \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(F_{n,q}(x,y;z)-(x\oplus y)_q^n\right)\frac{t^n}{[n]_q!}\right)\end{aligned}$$

for the q-commuting variables x and t such as $tx=qxt$, which yields

$$zF_{n,q}(x+1,y;z)=(z+1)\,F_{n,q}(x,y;z)-(x\oplus y)_q^n \qquad (n\in\mathbb{N}_0).$$

So, the proof is completed. □

Some special cases of the Theorem 6 seems nice consequences as follows.

Corollary 4. *Each of the following identities*

$$\begin{gathered} zF_{n,q}(1,y;z)=(z+1)F_{n,q}(0,y;z)-q^{n(n-1)/2}y^{n}, \\ zF_{n,q}(1,0;z)=(z+1)F_{n,q}(z), \\ zF_{n,q}(0,y;z)=(z+1)F_{n,q}(-1,y;z)-(-1\oplus y)_{q}^{n}\ , \\ zF_{n,q}(0,0;z)=(z+1)F_{n,q}(-1,0;z)-(-1)^{n}. \end{gathered}$$

hold true for $n\in\mathbb{N}$.

We provide the q-derivative properties of the q-Fubini polynomials $F_{n,q}(x,y;z)$ with three parameters with respect to x and y as follows.

Theorem 7. *The following q-derivatives for q-Fubini polynomials with three variables*

$$D_{q:x}F_{n,q}(x,y;z)=[n]_{q}F_{n-1,q}(x,y;z)\ \text{ and } D_{q:y}F_{n,q}(x,y;z)=[n]_{q}F_{n-1,q}(x,qy;z). \tag{2.13}$$

hold true.

Proof. By means of (1.1) and (1.4), we derive

$$\begin{aligned} \sum_{n=0}^{\infty}D_{q:x}F_{n,q}(x,y;z)\frac{t^{n}}{[n]_{q}!} &= D_{q:x}\sum_{n=0}^{\infty}F_{n,q}(x,y;z)\frac{t^{n}}{[n]_{q}!} \\ &= \frac{1}{1-z(e_{q}(t)-1)}D_{q:x}e_{q}(xt)E_{q}(yt) \\ &= \frac{t}{1-z(e_{q}(t)-1)}e_{q}(xt)E_{q}(yt) \\ &= \sum_{n=0}^{\infty}F_{n,q}(x,y;z)\frac{t^{n+1}}{[n]_{q}!}. \end{aligned}$$

By comparing the coefficients $\frac{t^{n}}{[n]_{q}!}$ of the both sides, we can readily acquire the first asserted result in (2.13). The other can be derived utilizing the similar method used above. □

The q-definite integral is defined by (*cf.* [17]):

$$\int_{0}^{\xi}f(x)\,d_{q}x=(1-q)\,\xi\sum_{k=0}^{\infty}q^{k}f\left(q^{k}\xi\right)\ \text{ with } \int_{\xi}^{\varpi}f(x)\,d_{q}x=\int_{0}^{\varpi}f(x)\,d_{q}x-\int_{0}^{\xi}f(x)\,d_{q}x. \tag{2.14}$$

We provide the definite q-integral propeties of the q-Fubini polynomials $F_{n,q}(x,y;z)$ with three parameters with respect to x and y as follows.

Theorem 8. *The following definite q-integrals are valid:*

$$\int_{\eta}^{\xi}F_{n,q}(x,y;z)\,d_{q}x=\frac{F_{n+1,q}(\xi,y;z)-F_{n+1,q}(\eta,y;z)}{[n+1]_{q}} \tag{2.15}$$

and

$$\int_{\varkappa}^{\varsigma}F_{n,q}(x,y;z)\,d_{q}y=\frac{F_{n+1,q}\left(x,\frac{\varsigma}{q};z\right)-F_{n+1,q}\left(x,\frac{\chi}{q};z\right)}{[n+1]_{q}}.$$

hold true.

Proof. Since

$$\int_{\alpha}^{\beta} D_q f(x)\, d_q x = f(\beta) - f(\alpha) \text{ (see [17])}$$

using Eqs. (1.1), (1.4) and (2.8), we obtain

$$\begin{aligned}
\int_{\varkappa}^{\varsigma} F_{n,q}(x,y;z)\, d_q x &= \frac{1}{[n+1]_q} \int_{\varkappa}^{\varsigma} D_q F_{n+1,q}(x,y;z)\, d_q x \\
&= \frac{1}{[n+1]_q} \left(F_{n+1,q}\left(x, \frac{\varsigma}{q}; z\right) - F_{n+1,q}\left(x, \frac{\chi}{q}; z\right) \right).
\end{aligned}$$

Thus, we have the first desired result (2.15). Also, the other q-integral representation can be shown in a like manner. □

We give the followig theorem.

Theorem 9. *q-Fubini polynomials with three variables of degree $n+1$ can be written by q-Fubini polynomials with three variables of degree n as follows:*

$$\begin{aligned}
F_{n+1,q}(x,y;z) &= xF_{n,q}\left(\frac{x}{q}, y; z\right) q^n + yF_{n,q}\left(\frac{x}{q}, y; z\right) q^n \\
&\quad + z\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x,y;z)\, F_{k,q}\left(q^{-1}, 0; z\right) q^k.
\end{aligned}$$

Proof. By (1.2), (1.3) and (2.8), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+1,q}(x,y;z) \frac{t^n}{[n]_q!} &= D_{q:t}\left(\sum_{n=0}^{\infty} F_{n,q}(x,y;z) \frac{t^n}{[n]_q!} \right) = D_{q:t}\left(\frac{e_q(xt)\, E_q(yt)}{1 - z(e_q(t) - 1)} \right) \\
&= \frac{xe_q(xt)\, E_q(yqt)}{(1 - z(e_q(qt) - 1))} + \frac{ye_q(xt)\, E_q(yqt)}{(1 - z(e_q(qt) - 1))} \\
&\quad + \frac{ze_q(xt)\, E_q(yt)}{(1 - z(e_q(t) - 1))} \frac{e_q(t)}{(1 - z(e_q(qt) - 1))} \\
&= x\sum_{n=0}^{\infty} F_{n,q}\left(\frac{x}{q}, y; z\right) q^n \frac{t^n}{[n]_q!} + y\sum_{n=0}^{\infty} F_{n,q}\left(\frac{x}{q}, y; z\right) q^n \frac{t^n}{[n]_q!} \\
&\quad + z\left(\sum_{n=0}^{\infty} F_{n,q}(x,y;z) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} F_{n,q}\left(q^{-1}, 0; z\right) q^n \frac{t^n}{[n]_q!} \right) \\
&= \sum_{n=0}^{\infty} \left(xF_{n,q}\left(\frac{x}{q}, y; z\right) q^n + yF_{n,q}\left(\frac{x}{q}, y; z\right) q^n \right. \\
&\quad \left. + z\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x,y;z)\, F_{k,q}\left(q^{-1}, 0; z\right) q^k \right) \frac{t^n}{[n]_q!},
\end{aligned}$$

which yields to the wanted result. □

We state the following theorem including the symmetric property.

Theorem 10. *For $n \in \mathbb{N}_0$ and $z_1 \neq z_2$, we have the following symmetric identities for q-Fubini polynomials with three variables as follows:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(x_1, y_1; z_1) F_{n-k,q}(x_2, y_2; z_2)$$
$$= \frac{z_2}{z_2 - z_1} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x_1 \oplus y_1, y_2; z_2) x_2^k - \frac{z_1}{z_2 - z_1} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x_2 \oplus y_2, y_1; z_1) x_1^k.$$

Proof. It is observed that

$$\left(\frac{1}{1 - z_1 (e_q(t) - 1)} e_q(x_1 t) E_q(y_1 t) \right) \left(\frac{1}{1 - z_2 (e_q(t) - 1)} e_q(x_2 t) E_q(y_2 t) \right)$$
$$= \frac{z_2}{z_2 - z_1} \left(\frac{e_q(x_1 t) E_q(y_1 t) e_q(x_2 t) E_q(y_2 t)}{1 - z_2 (e_q(t) - 1)} \right) - \frac{z_1}{z_2 - z_1} \left(\frac{e_q(x_1 t) E_q(y_1 t) e_q(x_2 t) E_q(y_2 t)}{1 - z_1 (e_q(t) - 1)} \right),$$

then, by (2.8), we get

$$\left(\sum_{n=0}^{\infty} F_{n,q}(x_1, y_1; z_1) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} F_{n,q}(x_2, y_2; z_2) \frac{t^n}{[n]_q!} \right)$$
$$= \frac{1}{z_2 - z_1} \left(z_2 \frac{e_q((x_1 \oplus y_1) t) E_q(y_2 t)}{1 - z_2 (e_q(t) - 1)} e_q(x_2 t) - z_1 \frac{e_q((x_2 \oplus y_2) t) E_q(y_1 t)}{1 - z_1 (e_q(t) - 1)} e_q(x_1 t) \right)$$
$$= \frac{1}{z_2 - z_1} \sum_{n=0}^{\infty} \left(z_2 \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x_1 \oplus y_1, y_2; z_2) x_2^k - z_1 \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{n-k,q}(x_2 \oplus y_2, y_1; z_1) x_1^k \right) \frac{t^n}{[n]_q!}.$$

Thus, the proof of this theorem is completed. □

The Theorem 10 provides several pretty outcomes and some of them are presented below.

Corollary 5. *We have the following difference relations for q-Fubini polynomials:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(z_1) F_{n-k,q}(x_2, y_2; z_2) = \frac{z_2 F_{n,q}(x_2, y_2; z_2) - z_1 F_{n,q}(x_2, y_2; z_1)}{z_2 - z_1},$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(x_1, y_1; z_1) F_{n-k,q}(z_2) = \frac{z_2 F_{n,q}(x_1, y_1; z_2) - z_1 F_{n,q}(x_1, y_1; z_1)}{z_2 - z_1},$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(z_1) F_{n-k,q}(z_2) = \frac{z_2 F_{n,q}(z_2) - z_1 F_{n,q}(z_1)}{z_2 - z_1},$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{k,q}(0, y_1; z_1) F_{n-k,q}(x_2, y_2; z_2) = \frac{z_2 F_{n,q}(x_2 \oplus y_1, y_2; z_2) - z_1 F_{n,q}(x_2 \oplus y_2, y_1; z_1)}{z_2 - z_1}.$$

We here define the q-Fubini polynomials with three parameters of the second kind as follows:

$$\frac{1}{1 - z (E_q(t) - 1)} e_q(xt) E_q(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,q}(x, y; z) \frac{t^n}{[n]_q!}. \tag{2.16}$$

When $x = 0 = y$, $\mathcal{F}_{n,q}(0, 0; z) := \mathcal{F}_{n,q}(z)$ is called the q-Fubini numbers with three parameters of the second kind.

Some basic properties of the polynomial $\mathcal{F}_{n,q}(x,y;z)$ are listed below:

$$\begin{aligned}
\mathcal{F}_{n,q}(x,y;z) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{F}_{k,q}(z)(x\oplus y)_q^{n-k} \\
\mathcal{F}_{n,q}(x,y;z) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{F}_{k,q}(0,y;z)\,x^{n-k} \\
\mathcal{F}_{n,q}(x,y;z) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} \mathcal{F}_{k,q}(x,0;z)\,y^{n-k} \\
\frac{d}{dz}\mathcal{F}_{n,q}(x,y;z) &= \sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathcal{F}_{n,q}(x,y;z)
\end{aligned}$$

We now introduce a new family of numbers as $\mathcal{S}_{2,q}(n,k)$ as follows:

$$\sum_{n=k}^{\infty}\mathcal{S}_{2,q}(n,k)\frac{t^n}{[n]_q!} = \frac{(E_q(t)-1)^k}{[k]_q!}. \tag{2.17}$$

We give the following theorem.

Theorem 11. *The following relation*

$$\mathcal{F}_{n,q}(z) = \sum_{k=0}^{n} z^k [k]_q! \mathcal{S}_{2,q}(n,k). \tag{2.18}$$

holds true for $n \in \mathbb{N}_0$.

Proof. By (1.7) and (2.16), we obtain

$$\begin{aligned}
\frac{1}{1-z(E_q(t)-1)} &= \sum_{k=0}^{\infty} z^k (E_q(t)-1)^k \\
&= \sum_{k=0}^{\infty} z^k [k]_q! \sum_{n=k}^{\infty}\mathcal{S}_{2,q}(n,k)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} z^k [k]_q! \mathcal{S}_{2,q}(n,k)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

which means the asserted result (2.4). □

Two correlations between the both q-Fubini polynomials are given by the following theorem.

Theorem 12. *We have the following identity for q-Fubini polynomial as follows:*

$$F_{n,q}(x,y;z-1)(-1)^n = \mathcal{F}_{n,q}\left((-x\oplus 1)_q, -y; -z\right) \tag{2.19}$$

and

$$F_{n,q}(x,y;z-1)(-1)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} \mathcal{F}_{k,q}(-x,-y;-z). \tag{2.20}$$

Proof. In view of (2.8) and (2.16), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,q}(x,y;z-1)(-1)^n \frac{t^n}{[n]_q!} &= \frac{1}{1-(z-1)(e_q(-t)-1)} e_q(-xt)E_q(-yt)\\ &= \frac{e_q(-xt)E_q(-yt)}{1-z(e_q(-t)-1)+e_q(-t)-1}\\ &= \frac{e_q\left((-x\oplus 1)_q t\right)E_q(-yt)}{1+z(E_q(t)-1)}\\ &= \sum_{n=0}^{\infty} \mathcal{F}_{n,q}\left((-x\oplus 1)_q, -y; -z\right)\frac{t^n}{[n]_q!}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,q}(x,y;z-1)(-1)^n \frac{t^n}{[n]_q!} &= \frac{1}{1-(z-1)(e_q(-t)-1)} e_q(-xt)E_q(-yt)\\ &= \frac{e_q(-xt)E_q(-yt)}{1-z(e_q(-t)-1)+e_q(-t)-1}\\ &= \frac{e_q(-xt)E_q(-yt)}{1+z(E_q(t)-1)}E_q(t)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\mathcal{F}_{k,q}(-x,-y;-z)\right)\frac{t^n}{[n]_q!}\end{aligned}$$

which give the claimed results (2.19) and (2.20). □

Taking $x = 0 = y$ in (2.19) and (2.20) yields to the following results below:

$$F_{n,q}(z) = (-1)^n \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\mathcal{F}_{k,q}(-z-1). \tag{2.21}$$

From (2.21), we deduce that

$$F_{n,q}(z) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\sum_{j=0}^{k}(-1)^{n+j}(z+1)^j [j]_q!\mathcal{S}_{2,q}(k,j).$$

Thus, we state the following theorem.

Theorem 13. *q-Fubini polynomial can be written in terms of a number $\mathcal{S}_{2,q}(k,j)$ given in (2.17) as follows:*

$$F_{n,q}(z) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\sum_{j=0}^{k}(-1)^{n+j}(z+1)^j [j]_q!\mathcal{S}_{2,q}(k,j).$$

3. **Concluding Remarks and Observation**

Kupershmidt defined q-Bernoulli polynomials by the generating function:

$$\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1}e_q(xt). \tag{3.1}$$

and Kim defined q-Euler polynomials by means of the following generating function:

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1} e_q(xt). \tag{3.2}$$

Motivated by the generating functions (3.1) and (3.2), we have introduced q-family of Fubini polynomials by means of the following generating function:

$$\frac{1}{1-z(e_q(t)-1)} = \sum_{n=0}^{\infty} F_{n,q}(z) \frac{t^n}{[n]_q!} \tag{3.3}$$

where $F_{n,q}(z)$ is called q-Fubini polynomials. One can see that

$$\lim_{q \to 1^-} F_{n,q}(z) = F_n(z)$$

are Fubini polynomials. By (3.3), we have derived some new and interesting relations. After that, we have defined q-Fubini polynomials with three parameters

$$\frac{1}{1-z(E_q(t)-1)} e_q(xt) E_q(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,q}(x,y;z) \frac{t^n}{[n]_q!}. \tag{3.4}$$

By (3.4), we have derived some new applications. In [13], Kim applied tools of umbral calculus to the classical Fubini polynomials, and derived many interesting identities. By making use of the technique of Kim given in [13], we will try to derive some new interesting formulae for q-Fubini polynomials arising from q-umbral calculus.

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