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BOUNDS AND COLOR ENERGY OF DERIVED GRAPHS

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ABSTRACT. Let G be a finite connected simple graph. The color energy of a graph G is defined as the sum of absolute values of color eigenvalues of G. The derived graph of a simple graph G, denoted by $G^{\dagger}$, is a graph having same vertex set as G, in which two vertices are adjacent if and only if their distance in G is two. In this paper, we establish an upper and lower bounds for color energy of a graph and obtain color energy of derived graphs of some families of graphs.

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1. INTRODUCTION

Let $G = (V, E)$ be a graph with n vertices and m edges and let $A(G)$ be its adjacency matrix. Since $A(G)$ is symmetric, its eigenvalues are real. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are assumed in non increasing order. The energy of a graph G was first defined by Ivan Gutman [10] in 1978 as sum of absolute eigenvalues of G. i. e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on energy of a graph refer [3, 4, 5, 9, 11, 14, 16, 17].

A coloring of a graph G [12] is coloring of its vertices such that no two adjacent vertices share the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number of G and is denoted by $\chi(G)$.

In 2013, C. Adiga, E. Sampathkumar, M. A. Sriraj and A. S. Shrikanth, [1] have introduced the energy of colored graph. The entries of color adjacency matrix $A_c(G)$ are as follows: If $c(v_i)$ is the color of vertex v_i, then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i)=c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $A_c(G)$ are called color eigenvalues of G. The color energy of a graph denoted by $E_c(G)$ is defined as sum of absolute values of color eigenvalues of G, i.e., $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$. Also, these authors have introduced a concept of complement of a colored graph, denoted as $\overline{G_c}$ has same vertex set and same coloring of G with the following conditions:

(i) v_i and v_j are adjacent in $\overline{G_c}$, whenever v_i and v_j are non-adjacent in G with $c(v_i) \neq c(v_j)$.
(ii) v_i and v_j are non-adjacent in $\overline{G_c}$, whenever v_i and v_j are non-adjacent in G with $c(v_i) = c(v_j)$ or if v_i and v_j are adjacent in G.

The matrix of $\overline{G_c}$ of order n is denoted by $A(\overline{G_c})$, whose entries are

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } \overline{G_c} \text{ with } c(v_i) \neq c(v_j), \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent in } \overline{G_c} \text{ with } c(v_i)=c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

Some well known properties of graph color eigenvalues are

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{i=1}^{n} \lambda_i^2 = 2(m + m_c^{'})$$

where $m_c^{'}$ is the number of pairs of non-adjacent vertices receiving same color in G.

And

$$\det(A_c) = \prod_{i=1}^{n} \lambda_i.$$

For recent mathematical works on color energy of a graph see [2, 6, 7].

This paper is organized as follows. In section 2, we present important results which are used in subsequent sections. In Section 3, we obtain some upper bounds for $E_c(G)$. In Section 4, we establish color energy of derived graphs.

2. Preliminaries

Lemma 2.1. [8] *(Cauchy interlace theorem) Let B be a $n \times n$ symmetric matrix and let B_k be its leading $k \times k$ sub matrix (that is, B_k is a matrix obtained from B by deleting its last $n-k$ rows and columns). Then for $i = 1, 2, \ldots, k$,*

$$\rho_{n-i+1}(B) \leq \rho_{k-i+1}(B_k) \leq \rho_{k-i+1}(B)$$

where $\rho_i(B)$ is the i^{th} largest eigenvalue of B.

Lemma 2.2. [13] *Let $x_1, x_2, \ldots, x_N$ be non-negative numbers and let*

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i \quad and \quad \gamma = \left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}}$$

be their arithmetic and geometric means. Then

$$\frac{1}{N(N-1)}\sum_{i<j}(\sqrt{x_i} - \sqrt{x_j})^2 \leq \alpha - \gamma \leq \frac{1}{N}\sum_{i<j}(\sqrt{x_i} - \sqrt{x_j})^2.$$

Moreover equality holds if and only if $x_1 = x_2 = \cdots = x_N$.

Definition 2.3. [12] *Let G be a simple graph with vertex set $V(G)$. The derived graph of G, denoted by $G^{\dagger}$ is graph having same vertex set as G, in which two vertices are adjacent if and only if their distance in G is two.*

Definition 2.4. [15] *A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called root of G. Let G be a rooted graph. The graph G^m obtained by identifying roots of m copies of G is called a one-point union of m copies of G.*

3. Upper and lower bounds for the color energy of a graph

Theorem 3.1. *Let G be a connected color graph with n vertices and m edges. Let $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ be the color eigenvalues of G. Let p_1 and p_2 be a set of positive and negative eigenvalues of G respectively. Then for $1 \le r \le n$*

$$-\sqrt{\frac{2(m+m_c^{'})p_1}{(n-r+1)(n-r+1+p_1)}} \le \lambda_r \le \sqrt{\frac{2(m+m_c^{'})p_2}{r(r+p_2)}}.$$

Proof. Consider right inequality and it is true for $\lambda_r \le 0$. Assume that $\lambda_r > 0$. From the known equality $2(m+m_c^{'}) = \lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2$, we obtain

$$\lambda_r^2 = 2(m+m_c^{'}) - \sum_{\substack{\lambda_i>0\\ i\neq r}} \lambda_i^2 - \sum_{\lambda_i<0} \lambda_i^2.$$

Since, it is well known that $\sum_{i=1}^{n} \lambda_i = 0$ and right side of above relation is maximized when $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ and for $\lambda_i < 0$, $\lambda_i = -\frac{\lambda_r r}{p_2}$.
Thus

$$\lambda_r^2 \le 2(m+m_c^{'}) + (r-1)\lambda_r^2 - \frac{\lambda_r^2 r^2}{p_2} \quad or \quad \lambda_r \le \sqrt{\frac{2(m+m_c^{'})p_2}{r(r+p_2)}}.$$

The left side inequality is obvious when $\lambda_r \ge 0$. In similar manner, when $\lambda_r < 0$,

$$\lambda_r^2 \le 2(m+m_c^{'}) - \frac{\lambda_r^2(n-r+1)^2}{p_1} - (n-r)\lambda_r^2$$

or

$$\lambda_r^2 \le \frac{2(m+m_c^{'})p_1}{(n-r+1)(n-r+1+p_1)}.$$

Since $\lambda_r < 0$,

$$\lambda_r \ge -\sqrt{\frac{2(m+m_c^{'})p_1}{(n-r+1)(n-r+1+p_1)}}.$$

□

In Theorem 3.1, as p_1 and p_2 are unknown values, whenever $\lambda_r > 0$, the value of $p_2 \le n-r$ and whenever $\lambda_r < 0$, the value of $p_1 \le r-1$.

Corollary 3.2. *For a colored graph G and for $1 \le r \le n$*

$$-\sqrt{\frac{2(m+m_c^{'})(r-1)}{n(n-r+1)}} \le \lambda_r \le \sqrt{\frac{2(m+m_c^{'})(n-r)}{nr}}.$$

Theorem 3.3. *Let G be a colored graph of order $n > 2$ with m edges and $n \le 2(m+m_c^{'})$ and $\lambda_1 \ge \frac{2(m+m_c^{'})}{n}$. Then*

$$E_c(G) \ge \sqrt{2(m+m_c^{'}) + n(n-1)|\det A_c|^{\frac{2}{n}} + \frac{4}{(n+1)(n+2)}\left[\sqrt{\left(\frac{2(m+m_c^{'})}{n}\right)} - \left(\frac{2(m+m_c^{'})}{n}\right)^{\frac{1}{4}}\right]^2}.$$

Equality holds if $G = \overline{(K_n)_c}$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the color eigenvalues of G.
By Lemma 2.2, we have

$$\sum_{i=1}^{N} x_i \geq N\left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}} + \frac{1}{N-1}\sum_{i<j}(\sqrt{x_i}-\sqrt{x_j})^2. \tag{1}$$

Putting $N = \dfrac{n(n-1)}{2}$ and taking

$(x_1, x_2, \ldots, x_n) = (|\lambda_1||\lambda_2|, |\lambda_1||\lambda_3|, \ldots, |\lambda_1||\lambda_n|, |\lambda_2||\lambda_3|, \ldots, |\lambda_2||\lambda_n|, \ldots, |\lambda_{n-1}||\lambda_n|)$

in inequality (1), we get

$$\sum_{1\leq i<j\leq n} |\lambda_i||\lambda_j| \geq \frac{n(n-1)}{2}\left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}} + \frac{2}{n^2-n-2}\sum_{\substack{i<j,k<l\\(i,j)\neq(k,l)}}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_l|}\right)^2$$

(2)

$$2\sum_{1\leq i<j\leq n} |\lambda_i||\lambda_j| \geq n(n-1)|\det A_c|^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}\sum_{\substack{i<j,k<l\\(i,j)\neq(k,l)}}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_l|}\right)^2.$$

From Corollary 3.2,

$$\lambda_{\frac{n}{2}} \leq \sqrt{\frac{2(m+m_c^{'})}{n}} \quad \text{and} \quad \lambda_{\frac{(n+1)}{2}} \leq \sqrt{\frac{2(m+m_c^{'}(n-1))}{n(n+1)}} < \frac{2(m+m_c^{'})}{n}$$

for even and odd n, respectively.

Since, $n = 2(m+m_c^{'})$ and $\lambda_1 \geq \dfrac{2(m+m_c^{'})}{n}$,

$$\lambda_1 \geq \frac{2(m+m_c^{'})}{n} \quad \text{and} \quad \lambda_{\lceil\frac{n}{2}\rceil} \leq \sqrt{\frac{2(m+m_c^{'})}{n}} \quad for \quad n \geq 3.$$

Since $m \geq 1$, by Lemma 2.1.

$$\lambda_n \leq \lambda_2(A_2) = -1.$$

Hence $|\lambda_n| \geq 1$. Since $n \geq 3$ and $m \geq 2$,

$$\begin{aligned}\sum_{\substack{i<j,k<l\\(i,j)\neq(k,l)}}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_l|}\right)^2 &\geq \left(\sqrt{|\lambda_1||\lambda_n|}-\sqrt{|\lambda_{\lceil\frac{n}{2}\rceil}||\lambda_n|}\right) + \\ &\quad \sum_{\substack{i<j,k<l\\(i,j)\neq(1,n)\\(k,l)\neq(\lceil\frac{n}{2},n\rceil)}}\left(\sqrt{|\lambda_i||\lambda_j|}-\sqrt{|\lambda_k||\lambda_l|}\right)^2 \\ &> |\lambda_n|\left(\sqrt{|\lambda_1|}-\sqrt{|\lambda_{\lceil\frac{n}{2}\rceil}|}\right)^2 \\ &\geq \left[\sqrt{\tfrac{2(m+m_c^{'})}{n}}-\left(\tfrac{2(m+m_c^{'})}{n}\right)^{\frac{1}{4}}\right]^2. \end{aligned} \tag{3}$$

Using inequality (3) in inequality(2), we get

$$2\sum_{1\le i<j\le n}|\lambda_i||\lambda_j| > n(n-1)|\det A_c|^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}\left[\sqrt{\frac{2(m+m_c^{'})}{n}} - \left(\frac{2(m+m_c^{'})}{n}\right)^{\frac{1}{4}}\right]^2.$$

Adding $\sum_{i=1}^{n}\lambda_i^2 = 2(m+m_c^{'})$ to both sides,

$$E_c(G)^2 > 2(m+m_c^{'}) + n(n-1)|\det A_c|^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}\left[\sqrt{\frac{2(m+m_c^{'})}{n}} - \left(\frac{2(m+m_c^{'})}{n}\right)^{\frac{1}{4}}\right]^2.$$

Equality holds if $G = \overline{(K_n)_c}$. □

Theorem 3.4. *Let G be a connected nonsingular colored graph of order n with m edges and $n \le 2(m+m_c^{'})$ and $\lambda_1 \ge \frac{2(m+m_c^{'})}{n}$. Then*

$$E_c(G) \le 2(m+m_c^{'}) - \frac{2(m+m_c^{'})}{n}\left(\frac{2(m+m_c^{'})}{n} - 1\right) - \ln\left(\frac{n|\det A_c|}{2(m+m_c^{'})}\right).$$

Equality holds if $G = (K_n)_c$.

Proof. Since G is nonsingular, we have $|\lambda_i| > 0$, $i = 1, 2, \ldots, n$. Thus

$$|\det A_c| = \prod_{i=1}^{n}|\lambda_i| > 0.$$

Consider the function

$$f(x) = x^2 - x - \ln x, \quad x > 0$$

for which

$$f^{'}(x) = 2x - 1 - \frac{1}{x}.$$

Thus $f(x)$ is an increasing function on $x \ge 1$ and $f(x)$ is decreasing function on $0 < x \le 1$. Thus $f(x) \ge f(1) = 0$ implies $x \le x^2 - \ln x$ for $x > 0$, equality holds if and only if $x = 1$.

$$\begin{aligned} E(G) &= \lambda_1 + \sum_{i=2}^{n}|\lambda_i| \\ &\le \lambda_1 + \sum_{i=2}^{n}(\lambda_i^2 - \ln|\lambda_i|) \\ &= \lambda_1 + 2(m+m_c^{'}) - \lambda_1^2 - \ln\prod_{i=1}^{n}|\lambda_i| + \ln\lambda_1 \end{aligned}$$

$$(4) \qquad = 2(m+m_c^{'}) + \lambda_1 - \lambda_1^2 - \ln|\det A_c| + \ln\lambda_1$$

Since, $\lambda_1 \ge \frac{2(m+m_c^{'})}{n}$.

Consider the function

$$g(x) = 2(m+m_c^{'}) + x - x^2 - \ln|\det A_c| + \ln x.$$

$g(x)$ is an increasing function in $0 < x \leq 1$ and decreasing function for $x \geq 1$.
Since,

$$x \geq \frac{2(m+m_c^{'})}{n} \geq \frac{2m}{n} \geq 1.$$

We have

$$\begin{aligned} g(x) \geq g\left(\frac{2(m+m_c^{'})}{n}\right) &= 2(m+m_c^{'}) + \frac{2(m+m_c^{'})}{n} - \left(\frac{2(m+m_c^{'})}{n}\right)^2 - \\ &\ln|\det A_c| + \ln\left(\frac{2(m+m_c^{'})}{n}\right) \\ &= 2(m+m_c^{'}) - \left(\frac{2(m+m_c^{'})}{n}\right)\left(\frac{2(m+m_c^{'})}{n} - 1\right) \\ &- \ln\left(\frac{|\det A_c|n}{2(m+m_c^{'})}\right). \end{aligned} \tag{5}$$

In view of inequality (5), equation (4) reduces to

$$E_c(G) \leq 2(m+m_c^{'}) - \frac{2(m+m_c^{'})}{n}\left(\frac{2(m+m_c^{'})}{n} - 1\right) - \ln\left(\frac{n|\det A_c|}{2(m+m_c^{'})}\right).$$

Equality holds if $G = (K_n)_c$. □

4. Color energy of derived graphs

Theorem 4.1. *For $n \geq 3$, the characteristic polynomial of a derived color graph $S_n^{\dagger}$ of a star graph S_n is $(\lambda+1)^{n-3}[\lambda^3 - (n-3)\lambda^2 - (n-1)\lambda + (n-3)]$.*

Proof. Let $S_n^{\dagger}$ be derived color graph of star graph S_n. Since $\chi(S_n^{\dagger}) = n-1$, we have

$$A_c\left(S_n^{\dagger}\right) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \ldots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \end{array}$$

Consider $\det\left(\lambda I - A_c\left(S_n^{\dagger}\right)\right)$.

Step 1: Replace R_i by $R_i^{'} = R_i - R_{i-1}$, for $i = v_4, v_5, \ldots, v_{n-1}, v_n$. Then, $\det(\lambda I - A(S_n^{\dagger})) = (\lambda+1)^{n-3}\det(C)$.

Step 2: In $\det(C)$, replace C_i by $C_i^{'} = C_i + C_{i+1}$, for $i = v_{n-1}, v_{n-2}, \ldots, v_3$. Then it reduces to a new determinant,

$$\det(D) = \begin{vmatrix} \lambda & 1 & 2-n \\ 1 & \lambda & 0 \\ -1 & 0 & \lambda - n + 3 \end{vmatrix}$$

Hence, $\det(D) = \lambda^3 - (n-3)\lambda^2 - (n-1)\lambda + (n-3)$.

Substituting $\det(D)$ in step 1, we get $\det(\lambda I - A(S_n^{\dagger})) = (\lambda+1)^{n-3}[\lambda^3 - (n-3)\lambda^2 - (n-1)\lambda + (n-3)]$. □

Theorem 4.2. *For $n \geq 3$, the energy of derived color graph ${S_{2n}^0}^{\dagger}$ of crown graph S_{2n}^0 is $4(n-1)$.*

Proof. Let ${S_{2n}^0}^{\dagger}$ be derived graph of crown graph. Since $\chi({S_{2n}^0}^{\dagger})$ is n, we have

$$A_c\left({S_{2n}^0}^{\dagger}\right) = \left[\begin{array}{c|c} (J-I)_n & -I_n \\ \hline -I_n & (J-I)_n \end{array}\right]_{2n\times 2n}$$

where J denotes matrix with all entries equal to unity, I is an identity matrix.
The result is proved by showing $AZ = \lambda Z$ for certain vector Z and by making use of the fact that geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_c\left({S_{2n}^0}^{\dagger}\right)$ is real and symmetric.
Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $A_c\left({S_{2n}^0}^{\dagger}\right)$.
Note that

$$\left(A_c\left({S_{2n}^0}^{\dagger}\right) - \lambda I_{2n}\right)\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (J-(\lambda+1)I)X - YI \\ -IX + (J-(\lambda+1)I)Y \end{bmatrix} \tag{6}$$

Case 1: let $X = \mathbf{1}_n$ and $Y = (n-(\lambda+1))\mathbf{1}_n$, where λ is any root of equation

$$\lambda^2 + (2-2n)\lambda + n^2 - 2n = 0. \tag{7}$$

From Equation (6),

$$(J-(\lambda+1)I)\mathbf{1}_n - I(n-(\lambda+1))\mathbf{1}_n = (n-\lambda-1)\mathbf{1}_n - (n-\lambda-1)\mathbf{1}_n = 0$$

and

$$\begin{aligned} I\mathbf{1}_n + (J-(\lambda+1)I)(n-(\lambda+1))\mathbf{1}_n &= [1+(n-\lambda-1)^2]\mathbf{1}_n \\ &= [\lambda^2 + (2-2n)\lambda + n^2 - 2n]\mathbf{1}_n \\ &= 0, \quad \text{follows from equation (7).} \end{aligned}$$

Hence, n and $n-2$ are eigenvalues of $A_c\left({S_{2n}^0}^{\dagger}\right)$, each with multiplicity at least one.

Case 2: Let $X = X_i$ be an eigenvector vector with first element 1 and i^{th} element -1, for $i = 2, 3, \ldots, n$ and remaining elements zero. Now $Y_i = -(\lambda+1)X_i$, where λ is any root of $\lambda^2 + 2\lambda = 0$.
Noting $JX_i = 0$ and from Equation (6),

$$(J-\lambda-1)X_i + I(\lambda+1)X_i = -(\lambda+1)X_i + (\lambda+1)X_i = 0$$

and

$$-I_nX_i + [J-(\lambda+1)I_n](\lambda+1)X_i = (\lambda^2+2\lambda)X_i.$$

From Equation (7), $\lambda^2+2\lambda=0$. Thus $\lambda=0$ and $\lambda=-2$ are eigenvalues, each with multiplicity at least $(n-1)$, as there are $(n-1)$ independent vectors of the form X_i.
Since order of the graph is $2n$, spectrum of ${S_{2n}^0}^\dagger$ is
$\left\{\begin{matrix} 0 & -2 & n-2 & n \\ n-1 & n-1 & 1 & 1 \end{matrix}\right\}$. Hence, $E_c({S_{2n}^0}^\dagger)=4(n-1)$. □

Theorem 4.3. *For $n\geq 3$, energy of a derived color graph $K_{n\times 2}^\dagger$ of a cocktail party graph $K_{n\times 2}$ is $4(n-1)$.*

Proof. Let $K_{n\times 2}^\dagger$ be derived color graph of cocktail party graph of order $2n$. Since $\chi(K_{n\times 2}^\dagger)$ is 2, we have

$$A_c\left(K_{n\times 2}^\dagger\right)=\left[\begin{array}{c|c}(I-J)_n & I_n\\ \hline I_n & (I-J)_n\end{array}\right]_{2n\times 2n}$$

The result is proved by showing $AZ=\lambda Z$ for certain vector Z and by making use of the fact that geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $A_c\left(K_{n\times 2}^\dagger\right)$ is real and symmetric.
Let $Z=\begin{bmatrix}X\\Y\end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $A_c\left(K_{n\times 2}^\dagger\right)$.
Note that

$$\left(A_c\left(K_{n\times 2}^\dagger\right)-\lambda I_{2n}\right)\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}-(J+(\lambda-1)I)X-IY\\ IX-(J+(\lambda-1)I)Y\end{bmatrix} \tag{8}$$

Case 1: let $X=\mathbf{1}_n$ and $Y=(n+(\lambda-1))\mathbf{1}_n$, where λ is any root of equation

$$\lambda^2-(2-2n)\lambda-n(2-n)=0. \tag{9}$$

From Equation (8),

$$-(J+(\lambda-1)I)\mathbf{1}_n+I(n+(\lambda-1))\mathbf{1}_n=-(n+\lambda-1)\mathbf{1}_n+(n+\lambda-1)\mathbf{1}_n=0$$

and

$$\begin{aligned}I\mathbf{1}_n-(J+(\lambda-1)I)(n+(\lambda-1))\mathbf{1}_n&=[1-(n+\lambda-1)^2]\mathbf{1}_n\\&=[\lambda^2-(2-2n)\lambda+n^2-2n]\mathbf{1}_n\\&=0.\quad\text{follows from equation (9).}\end{aligned}$$

Thus we conclude that $-n$ and $2-n$ are eigenvalues of $A_c\left(K_{n\times 2}^\dagger\right)$, each with multiplicity at least one.

Case 2: Let $X=X_i$ be the vector with first element 1 and i^{th} element -1, for $i=2,3,\ldots,n$ and remaining zero. Now $Y_i=(\lambda-1)X_i$, where λ is any root of $\lambda^2-2\lambda=0$.
Noting $JX_i=0$ and from Equation (8),

$$-(J+\lambda-1)X_i+I(\lambda-1)X_i=-(\lambda-1)X_i+(\lambda-1)X_i=0$$

and

$$I_nX_i-[J+(\lambda-1)I_n](\lambda-1)X_i=(\lambda^2-2\lambda)X_i.$$

From Equation (9), $\lambda^2 - 2\lambda = 0$. Thus $\lambda = 0$ and $\lambda = 2$ are eigenvalues, each with multiplicity at least $(n-1)$, as there are $(n-1)$ independent vectors of the form X_i.
Since order of graph is $2n$, spectrum of $K^{\dagger}_{n\times 2}$ is
$\left\{\begin{matrix} 0 & 2 & 2-n & -n \\ n-1 & n-1 & 1 & 1 \end{matrix}\right\}$. Hence, $E_c(K^{\dagger}_{n\times 2}) = 4(n-1)$. □

Remark 4.4. *Color energy of ${S^0_{2n}}^{\dagger}$ and $K^{\dagger}_{n\times 2}$ are same, but color spectrum of these graphs are different. Hence ${S^0_{2n}}^{\dagger}$ and $K^{\dagger}_{n\times 2}$ are non co spectral color equi-energetic graphs.*

Theorem 4.5. *For $m \geq n$, the characteristic polynomial of a derived color graph $S^{\dagger}_{m,n}$ of double star graph $S_{m,n}$ is $\lambda^{n-1}(\lambda+1)^{m-n-1}(\lambda+2)^{n-1}[\lambda^3 - (m+n-3)\lambda^2 + ((n-2)m - (2n-2))\lambda + n(m-2)]$.*

Proof. Let $S^{\dagger}_{m,n}$ be the derived color graph of double star graph. Since $\chi(S^{\dagger}_{m,n})$ is m, we have

$$A_c\left(S^{\dagger}_{m,n}\right) = \left[\begin{array}{c|c|c} (J-I)_{n\times n} & J_{n\times(m-n)} & -I_{n\times n} \\ \hline J_{(m-n)\times n} & (J-I)_{(m-n)\times(m-n)} & 0_{(m-n)\times n} \\ \hline -I_{n\times n} & 0_{n\times(m-n)} & (J-I)_{n\times n} \end{array}\right]_{(m+n)\times(m+n)}$$

Consider $\det\left(\lambda I - A_c\left(S^{\dagger}_{m,n}\right)\right)$.

Step 1: Replace R_i by $R'_i = R_i - R_{i+1}$, for $i = 1, 2, \ldots, n-1, n+1, \ldots, m-1, m+1, \ldots, m+n-1$. Then, $\det\left(\lambda I - A_c\left(S^{\dagger}_{m,n}\right)\right)$ will reduces to new determinant, say $\det(C)$.
Step 2: In $\det(C)$, replacing C_i by $C'_i = C_i + C_{i+1} + \cdots + C_{m+n}$, for $i = 1, 2, \ldots, m+n-1$, a new determinant, $\det(D)$ is obtained.
Step 3: In $\det(D)$, replacing R_i by $R'_i = R_i - (\lambda+1)R_{n+i+1}$, for $i = 1, 2, \ldots, n-1$, we get $\det(E)$.
Step 4: On expanding the $\det(E)$ along the rows R_i, for $i = 1, 2, \ldots, n-1, n+1, \ldots, m-1, m+1, \ldots, m+n-1$, we obtain

$$\det(E) = \lambda^n(\lambda+1))^{m-n-1}(\lambda+2)^{n-1}\begin{vmatrix} \lambda-m+2 & -m+n+1 & 1 \\ \lambda-m+1 & \lambda-m+n+1 & 0 \\ \lambda-n+2 & \lambda-n+1 & \lambda-n+1 \end{vmatrix}$$

$= \lambda^n(\lambda+1)^{m-n-1}(\lambda+2)^{n-1}[\lambda^3 - (m+n-3)\lambda^2 + ((n-2)m - (2n-2))\lambda + n(m-2)]$.
Thus, $\det\left(\lambda I - A_c\left(S^{\dagger}_{m,n}\right)\right) = \lambda^n(\lambda+1)^{m-n-1}(\lambda+2)^{n-1}[\lambda^3 - (m+n-3)\lambda^2 + ((n-2)m - (2n-2))\lambda + n(m-2)]$. □

Theorem 4.6. *If $K^{m\dagger}_n$ is the derived color graph of one point union of complete graph of order $m(n-1)+1$, then characteristic polynomial of $K^{m\dagger}_n$ is $(\lambda+2n-3)^{m-2}\ (\lambda-1)^{(n-2)m}[\lambda^3 + (m(1-n) + 4n - 6)\lambda^2 + (2n^2 - 5n + 3)m - (4n^2 - 13n + 10))\lambda + ((n^2 - 2n + 1)m - (3n^2 - 7n + 4)]$.*

Proof. Let $K^{m\dagger}_n$ be the derived color graph of one point union of complete graph of order $m(n-1)+1$. Since $\chi(K^{m\dagger}_n)$ is m, we have

$$A_c\left(K_n^{m\dagger}\right)=\left[\begin{array}{c|c|c|c|c|c} 0_{1\times 1} & -1_{1\times(n-1)} & 0_{1\times(n-1)} & \cdots & 0_{1\times(n-1)} & 0_{1\times(n-1)} \\ \hline -1_{(n-1)\times 1} & (I-J)_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} & \cdots & J_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} \\ \hline 0_{(n-1)\times 1} & J_{(n-1)\times(n-1)} & (I-J)_{(n-1)\times(n-1)} & \cdots & J_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0_{(n-1)\times 1} & J_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} & \cdots & (I-J)_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} \\ \hline 0_{(n-1)\times 1} & J_{(n-1)\times(n-1)} & J_{(n-1)\times(n-1)} & \cdots & J_{(n-1)\times(n-1)} & (I-J)_{(n-1)\times(n-1)} \end{array}\right]$$

Consider $\det\left(\lambda I - A_c\left(K_n^{m\dagger}\right)\right)$.

Step 1: Replace R_i by $R_i' = R_i - R_{i-1}$, for $i = m(n-1)+1, \ldots, (m-1)(n-1)+3, (m-1)(n-1)+1, \ldots, (m-2)(n-1)+3, \ldots, 2(n-1)+1, \ldots, (n-1)+3, (n-1)+1, \ldots, 3$. Then,
$\det\left(\lambda I - A(K_n^{m\dagger})\right) = (\lambda-1)^{(n-2)m}\det(C)$.

Step 2: In $\det(C)$, replacing C_i by $C_i' = C_i + C_{i+1}$, for $i = (n-1), \ldots, 1, 2(n-1), \ldots, (n-1)+2, 3(n-1), \ldots, 2(n-1)+2, \ldots, m(n-1), \ldots, (m-1)(n-1)+2$, we get a new determinant, let it be $\det(D)$.

Step 3: Expanding $\det(D)$ over last row, we get

$$\det(E) = \begin{vmatrix} \lambda+n-1 & n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda-1 & 1-n & 1-n & \cdots & 1-n & 1-n \\ 1-n & 1-n & \lambda+n-2 & 1-n & \cdots & 1-n & 1-n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1-n & 1-n & 1-n & 1-n & \cdots & \lambda+n-2 & 1-n \\ 1-n & 1-n & 1-n & 1-n & \cdots & 1-n & \lambda+n-2 \end{vmatrix}_{(m+1)\times(m+1)}$$

Step 4: In $\det(E)$, replacing R_i by $R_i' = R_i - R_{i-1}$, for $i = m+1, m, m-1, \ldots, 1$ and C_i by $C_i' = C_i + C_{i+1} + \cdots + C_m$, for $i = 3, 4, \ldots, m-1$ and simplifying we get

$$\det(E) = (\lambda+2n-3)^{m-2}\begin{vmatrix} \lambda+n-1 & n-1 & 0 \\ 0 & \lambda-1 & -(m-1)(n-1) \\ 1-n & 1-n & \lambda+n(3-m)+m-4 \end{vmatrix}$$

$$= (\lambda+2n-3)^{m-2}[\lambda^3 + (m(1-n)+4n-6)\lambda^2 + (2n^2-5n+3)m - (4n^2-13n+10))\lambda + ((n^2-2n+1)m - (3n^2-7n+4))].$$

Substituting $\det(E)$ in step 2, we get $\det\left(\lambda I - A_c\left(K_n^{m\dagger}\right)\right) = (\lambda+2n-3)^{m-2}\,(\lambda-1)^{(n-2)m}[\lambda^3 + (m(1-n)+4n-6)\lambda^2 + (2n^2-5n+3)m - (4n^2-13n+10))\lambda + ((n^2-2n+1)m - (3n^2-7n+4)]$.

□

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