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RELATIONS ON BERNOULLI AND EULER POLYNOMIALS RELATED TO TRIGONOMETRIC FUNCTIONS

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Abstract. The cosine-Euler polynomials, the sine-Euler polynomials, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials have been recently studied by Kim and Ryoo [4] and also Masjed-Jamei et al. [5]-[6] and Srivastava et al. [11]. The purpose of this paper is to derive some new identities and relations for these polynomials with aid of generating functions and trigonometric functions.

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1. Introduction

Mathematicians and other scientists have studied on trigonometric functions, special numbers and polynomials and their applications because these functions have various mathematical easily usages which are derivative, integrals and other algebraic properties. By using these functions with their functional equations and derivative equations, various properties of these special numbers and polynomials have been investigated (*cf.* [1]-[11]).

By using these functions involving trigonometric function, we not only study some special families of polynomials and numbers including the Bernoulli polynomials and Euler polynomials, but also derive some identities and relations for these polynomials and numbers.

We use the following notations and definitions:

Let $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers.

Now we introduce generating functions for very useful special numbers and polynomials.

The Bernoulli polynomials $B_n(x)$ are defined by means of the following generating function:

$$F_B(t, x) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1}$$

where $|t| < 2\pi$ (*cf.* [1] -[11]); and the references therein).

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We observe that

$$B_n(0) = B_n$$

which denote the Bernoulli numbers (*cf.* [1]-[11]; and the references therein).

The Euler polynomials $E_n(x)$ are defined by means of the following generating function:

$$F_E(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{2}$$

where $|t| < \pi$ (*cf.* [1]-[11]; and the references therein).

We observe that

$$E_n(0) = E_n$$

which denote the Euler numbers (*cf.* [1]-[11]; and the references therein).

Masjed-Jamei et al. [5]-[6], Kim and Ryoo [4], and Srivastava et al. [11] have studied on the cosine-Bernoulli polynomials and the sine-Bernoulli polynomials, and also the cosine-Euler polynomials and the sine-Euler polynomials with their generating functions. Therefore, in work of Kim and Ryoo [4], the cosine-Euler polynomials and the sine-Euler polynomials were defined by means of the following generating functions, respectively,

$$F_{EC}(t,x,y) = \frac{2}{e^t+1}e^{xt}\cos(yt) = \sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!}, \tag{3}$$

$$F_{ES}(t,x,y) = \frac{2}{e^t+1}e^{xt}\sin(yt) = \sum_{n=0}^{\infty} E_n^{(S)}(x,y)\frac{t^n}{n!} \tag{4}$$

(*cf.* [4], [6]).

Similarly, in [4], the cosine-Bernoulli polynomials and the sine-Bernoulli polynomials were defined by means of the following generating functions, respectively,

$$F_{BC}(t,x,y) = \frac{t}{e^t-1}e^{xt}\cos(yt) = \sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{t^n}{n!}, \tag{5}$$

$$F_{BS}(t,x,y) = \frac{t}{e^t-1}e^{xt}\sin(yt) = \sum_{n=0}^{\infty} B_n^{(S)}(x,y)\frac{t^n}{n!} \tag{6}$$

(*cf.* [4], [6]).

In [11], Srivastava et al. studied and investigated various properties of Apostol type cosine-Bernoulli polynomials, Apostol type cosine-Euler polynomials, Apostol type sine-Bernoulli polynomials and Apostol type sine-Euler polynomials. For the names of these polynomials, they gave the names two parametric kinds of Apostol-Bernoulli, Apostol-Euler polynomials.

The other new families of polynomials are given by the following generating functions:

$$F_C(t,x,y) = e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{7}$$

$$F_S(t,x,y)=e^{xt}\sin(yt)=\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!} \tag{8}$$

(*cf.* [4], [5], [6], [11]). By using above equations, we get

$$C_n(x,y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k} \tag{9}$$

and

$$S_n(x,y)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}x^{n-2k-1}y^{2k+1} \tag{10}$$

(*cf.* [4], [5], [6], [11]).

2. Identities and relations related to special numbers and polynomials

In this section, by using generating functions and their functional equations, we obtain some identities and relations including the Bernoulli numbers and polynomials, the Euler polynomials, the cosine-Euler polynomials, the sine-Euler polynomials, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the polynomials $C_n(x,y)$ and $S_n(x,y)$.

Theorem 2.1. *Let $n\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}E_n^{(S)}(2x,2y) &= \sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}E_j^{(S)}(x,y)E_{v-j}^{(C)}(x,y)\\ &\quad+\sum_{j=0}^{n}\binom{n}{j}E_j^{(S)}(x,y)E_{n-j}^{(C)}(x,y).\end{aligned}$$

Proof. By using (3) and (4), we derive the following functional equation:

$$F_{ES}(t,2x,2y)=\left(e^t+1\right)F_{ES}(t,x,y)\,F_{EC}(t,x,y).$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty}E_n^{(S)}(2x,2y)\frac{t^n}{n!}=\left(e^t+1\right)\sum_{n=0}^{\infty}E_n^{(S)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}E_n^{(C)}(x,y)\frac{t^n}{n!}.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty}E_n^{(S)}(2x,2y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}E_j^{(S)}(x,y)E_{v-j}^{(C)}(x,y)\frac{t^n}{n!}\\ &\quad+\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E_j^{(S)}(x,y)E_{n-j}^{(C)}(x,y)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.2. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} B_{n-1}^{(S)}(2x,2y) &= \frac{2}{n}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}B_{j}^{(S)}(x,y)B_{v-j}^{(C)}(x,y) \\ &\quad -\frac{2}{n}\sum_{j=0}^{n}\binom{n}{j}B_{j}^{(S)}(x,y)B_{n-j}^{(C)}(x,y). \end{aligned}$$

Proof. By using (5) and (6), we derive the following functional equation:

$$\frac{t}{2}F_{BS}\left(t,2x,2y\right) = \left(e^{t}-1\right)F_{BS}\left(t,x,y\right)F_{BC}\left(t,x,y\right).$$

From the above equation, we get

$$\sum_{n=0}^{\infty}B_{n}^{(S)}(2x,2y)\frac{t^{n+1}}{n!} = 2\left(e^{t}-1\right)\sum_{n=0}^{\infty}B_{n}^{(S)}(x,y)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}B_{n}^{(C)}(x,y)\frac{t^{n}}{n!}.$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty}nB_{n-1}^{(S)}(2x,2y)\frac{t^{n}}{n!} &= 2\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}B_{j}^{(S)}(x,y)B_{v-j}^{(C)}(x,y)\frac{t^{n}}{n!} \\ &\quad -2\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}B_{j}^{(S)}(x,y)B_{n-j}^{(C)}(x,y)\frac{t^{n}}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_{n}(x) = 2^{-n}\left(\sum_{j=0}^{n}\binom{n}{j}\left(E_{j}^{(C)}(x,y)B_{n-j}^{(C)}(x,y)+E_{j}^{(S)}(x,y)B_{n-j}^{(S)}(x,y)\right)\right).$$

Proof. By using (1), (3), (4), (5) and (6), we derive the following functional equation:

$$F_{B}\left(2t,x\right) = F_{EC}\left(t,x,y\right)F_{BC}\left(t,x,y\right)+F_{ES}\left(t,x,y\right)F_{BS}\left(t,x,y\right).$$

From the above equation, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}B_{n}(x)\frac{(2t)^{n}}{n!} &= \sum_{n=0}^{\infty}E_{n}^{(C)}(x,y)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}B_{n}^{(C)}(x,y)\frac{t^{n}}{n!} \\ &\quad +\sum_{n=0}^{\infty}E_{n}^{(S)}(x,y)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}B_{n}^{(S)}(x,y)\frac{t^{n}}{n!}. \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}2^{n}B_{n}(x)\frac{t^{n}}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\left(E_{j}^{(C)}(x,y)B_{n-j}^{(C)}(x,y)+E_{j}^{(S)}(x,y)B_{n-j}^{(S)}(x,y)\right)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(C)}(x,y) B_{n-j}^{(S)}(x,y) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} B_j(x) B_{n-j}^{(S)}(x,2y).$$

Proof. By using (1), (5) and (6), we derive the following functional equation:

$$F_{BC}(t,x,y)\, F_{BS}(t,x,y) = \frac{1}{2} F_B(t,x)\, F_{BS}(t,x,2y).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} B_n^{(C)}(x,y) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n^{(S)}(x,y) \frac{t^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n^{(S)}(x,2y) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} B_j^{(C)}(x,y) B_{n-j}^{(S)}(x,y) \frac{t^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} B_j(x) B_{n-j}^{(S)}(x,2y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} E_j^{(C)}(x,y) E_{n-j}^{(S)}(x,y) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} E_j(x) E_{n-j}^{(S)}(x,2y).$$

Proof. By using (2), (3) and (4), we derive the following functional equation:

$$F_{EC}(t,x,y)\, F_{ES}(t,x,y) = \frac{1}{2} F_E(t,x)\, F_{ES}(t,x,2y).$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty} E_n^{(C)}(x,y) \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{(S)}(x,y) \frac{t^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{(S)}(x,2y) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} E_j^{(C)}(x,y) E_{n-j}^{(S)}(x,y) \frac{t^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} E_j(x) E_{n-j}^{(S)}(x,2y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.6. *Let $n \in \mathbb{N}$. Then we have*

$$B_{n-1}^{(C)}(x,y) = \frac{1}{n} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} 4^j B_{2j} y^{2j-1} B_{n-2j}^{(S)}(x,y).$$

Proof. By using (5) and (6), we derive the following functional equation:

$$F_{BC}(t,x,y) = F_{BS}(t,x,y) \cot(yt).$$

By combining the above functional equation, with the following well-known relation

$$t \cot(t) = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2t)^{2n}}{(2n)!} \tag{11}$$

(*cf.* [1], [3]), we get

$$\sum_{n=0}^{\infty} ynB_{n-1}^{(C)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(S)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2yt)^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} nB_{n-1}^{(C)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}4^j B_{2j}y^{2j-1}B_{n-2j}^{(S)}(x,y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.7. *Let $n \in \mathbb{N}$. Then we have*

$$E_{n-1}^{(C)}(x,y) = \frac{1}{n}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}4^j B_{2j}y^{2j-1}E_{n-2j}^{(S)}(x,y).$$

Proof. By using (5) and (6), we derive the following functional equation:

$$F_{EC}(t,x,y) = F_{ES}(t,x,y)\cot(yt).$$

Combining (11) with the above equation, we obtain

$$\sum_{n=0}^{\infty} ynE_{n-1}^{(C)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(S)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2yt)^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} nE_{n-1}^{(C)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}4^j B_{2j}y^{2j-1}E_{n-2j}^{(S)}(x,y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_n(2x,2y) = 2\sum_{j=0}^{n}\binom{n}{j}S_j(x,y)\,C_{n-j}(x,y).$$

Proof. By using (7) and (8), we derive the following functional equation:

$$F_S(t,2x,2y) = 2F_S(t,x,y)\,F_C(t,x,y).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} S_n(2x,2y)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} S_n(2x,2y)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}S_j(x,y)\,C_{n-j}(x,y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation we arrive at the desired result. □

Theorem 2.9. *Let $n \in \mathbb{N}$. Then we have*

$$S_n(x,y) = 2^{1-n} \sum_{j=1}^{n} \binom{n}{j} S_j(x,y) C_{n-j}(x,y).$$

Proof. By using (7) and (8), we derive the following functional equation:

$$F_S(2t,x,y) = 2F_S(t,x,y) F_C(t,x,y).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} S_n(x,y) \frac{(2t)^n}{n!} = 2\sum_{n=0}^{\infty} S_n(x,y) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n(x,y) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2^{n-1} S_n(x,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S_j(x,y) C_{n-j}(x,y) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, and since $S_0(x,y) = 0$, we arrive at the desired result. $\square$

Theorem 2.10. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$C_n(mx,my) = m^n C_n(x,y).$$

Proof. By using (7), we derive the following functional equation:

$$F_C(t,mx,my) = F_C(mt,x,y).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} C_n(mx,my) \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n(x,y) m^n \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Theorem 2.11. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$S_n(mx,my) = m^n S_n(x,y).$$

Proof. By using (8), we derive

$$F_S(t,mx,my) = F_S(mt,x,y).$$

From the above equation, thus we have

$$\sum_{n=0}^{\infty} S_n(mx,my) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S_n(x,y) m^n \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

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References

[1] M. Abramowitz and I. A. Stegun,(Eds): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, *Washington, D.C.: U.S. Dept. of Commerce, National Bureau of Standards,* 1972.

[2] E. Y. Deeba and D. M. Rodriguez, *Bernoulli numbers and trigonometric functions*, Int. J. Math. Educ. Sci. Technol. **21**(2) (1990), 275-282.

[3] D. S. Kim and T. Kim, *Some p-adic integrals on $\mathbb{Z}_p$ Associated with trigonometric functions*, Russ. J. Math. Phys. **25**(3) (2018), 300–308.

[4] T. Kim and C. S. Ryoo, *Some identities for Euler and Bernoulli polynomials and their zeros*, Axioms **7**(3) 56, (2018), 1–19.

[5] M. Masjed-Jamei and W. Koepf, *Symbolic computation of some power-trigonometric series*, J. Symb. Comput. (2016), http://dx.doi.org/10.1016/j.jsc.2016.03.004

[6] M. Masjed-Jamei, M. R. Beyki and W. Koepf, *A new type of Euler polynomials and numbers*, Mediterr. J. Math. **15**(138) (2018), 1–17.

[7] Y. Simsek, *Special functions related to Dedekind-type DC-sums and their applications*, Russ. J. Math. Phys. **17**(4) (2010), 495–508.

[8] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12** (2018), 1–35.

[9] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inform. Sci. **5** (2011), 390–444.

[10] H. M. Srivastava, T. Kim and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12** (2005), 241–268.

[11] H. M. Srivastava, M. Masjed-Jamei and M. R. Beyki, *A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Appl. Math. Inf. Sci. **12**(5) (2018), 907-916.

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