Advanced Studies in Contemporary Mathematics
29 (2019), No. 1, pp. 147 - 159
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.1.147

A NOTE ON TYPE 2 DEGENERATE EULER AND BERNOULLI POLYNOMIALS

GWAN-WOO JANG AND TAEKYUN KIM

Abstract. Recently, several authors have studied degenerate Bernoulli and Euler polynomials in connection with many special numbers and polynomials. In this paper, we study the type 2 degenerate Bernoulli and Euler polynomials and give some properties for these polynomials. In particular, these polynomials are closely related to central factorial numbers of the second kind. In addition, we give some identities of these polynomials associated with the central factorial numbers of the second kind.

1. Introduction

It is well known that the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!},\quad (see[1-15]). \tag{1.1}$$

When $x=0$, $E_n^*=E_n^*(0)$ are called the Euler numbers.

The type 2 Euler polynomials are also defined by the generating function to be

$$e^{xt}\sec h\frac{t}{2}=\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (see[6]). \tag{1.2}$$

When $x=0$, $E_n=E_n(0)$ are called the type 2 Euler numbers.

The stirling number of the first kind is defined by

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\quad (see[14,15]), \tag{1.3}$$

where $(x)_0=1$, $(x)_n=x(x-1)\cdots(x-n+1)$, $(n\geq 1)$.

In the point of view of inverse of (1.2), the stirling number of the second kind is defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \quad (see[14]). \tag{1.4}$$

The Bernoulli polynomials are given by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (see[1-15]). \tag{1.5}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

Recently, the type 2 Bernoulli polynomials are introduced by

$$\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad (see[6,7]). \tag{1.6}$$

When $x = 0$, $b_n = b_n(0)$ are called the type 2 Bernoulli numbers.

For $\lambda \in \mathbb{R}$, L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad (see[2]). \tag{1.7}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

He also introduced the degenerate Euler polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \varepsilon_{n,\lambda}(x) \frac{t^n}{n!}, \quad (see[2]). \tag{1.8}$$

In the special case, $x = 0$, $\varepsilon_{n,\lambda} = \varepsilon_{n,\lambda}(0)$, $(n \geq 0)$ are called degenerate Euler numbers.

For $n \geq 0$, the central factorial is defined as

$$x^{[0]} = 1, \quad x^{[n]} = x\big(x + \frac{n}{2} - 1\big)\big(x + \frac{n}{2} - 2\big) \cdots \big(x - \frac{n}{2} + 1\big), \quad (n \geq 1).$$

The central factorial numbers of the second kind is defined by

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, \quad (n \geq 0), \quad (see[8]). \tag{1.9}$$

From (1.9), we can derive the generating function of the central factorial numbers of the second kind as follows:

$$\frac{1}{k!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!},\quad (k\geq 0). \tag{1.10}$$

It is known that the degenerate Changhee polynomials of the second kind is defined by the generating function to be

$$\frac{2}{1+\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^n}{n!},\quad (see[5]). \tag{1.11}$$

When $x=0$, $Ch_{n,\lambda}=Ch_{n,\lambda}(0)$ are called the degenerate Changhee numbers of the second kind $(see[5])$.

The degenerate Daehee polynomials of the second kind is defined by the generating function to be

$$\frac{\log(1+t)}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}-1}\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.12}$$

When $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the degenerate Daehee numbers of the second kind, $(see[13])$.

In [7], the degenerate central factorial polynomials of the second kind are given by the generating function to be

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=k}^{\infty}T_{2,\lambda}(n,k|x)\frac{t^n}{n!}, \tag{1.13}$$

where k is non-negative integer.
When $x=0$, $T_{2,\lambda}(n,k)=T_{2,\lambda}(n,k|0)$, $(n,k\geq 0)$, are called the degenerate central factorial numbers of the second kind.
Note that $\lim_{\lambda\longrightarrow 0}T_{2,\lambda}(n,k)=T(n,k)$, $(n,k\geq 0)$.

In this paper, we consider the type 2 degenerate Bernoulli and Euler polynomials and investigate some properties for these polynomials. In addition, we give some identities for the type 2 degenerate Bernoulli and Euler polynomials associated with special polynomials.

2. Type 2 degenerate Bernoulli and Euler polynomials.

For $\lambda \in \mathbb{R}$, we define type 2 degenerate Euler polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the type 2 degenerate Euler numbers. Now, we define $\cosh_\lambda(t)$ which are given by

$$\cosh_\lambda(t)=\frac{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{-\frac{1}{\lambda}}}{2}, \quad (\lambda \in \mathbb{R}). \tag{2.2}$$

By (2.2), we easily get

$$\sec h_\lambda(t)=\frac{1}{\cosh_\lambda(t)}, \quad \cosh_\lambda(-t)=\cosh_{-\lambda}(t). \tag{2.3}$$

From (2.1) and (2.3), we have

$$\sec h_{2\lambda}\big(\frac{t}{2}\big)=\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}=\sum_{n=0}^{\infty}E_{n,\lambda}\frac{t^n}{n!}. \tag{2.4}$$

In the viewpoint of (1.2), we define the type 2 degenerate Changhee polynomials which are given by the generating function to be

$$\frac{2}{\big(1+\lambda\log(1+t)\big)^{\frac{1}{2\lambda}}+\big(1+\lambda\log(1+t)\big)^{-\frac{1}{2\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}C_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.5}$$

When $x=0$, $C_{n,\lambda}=C_{n,\lambda}(0)$ are called the type 2 degenerate Changhee numbers. By replacing t by $\log(1+t)$ in (2.1), we get

$$\begin{aligned}&\frac{2}{\big(1+\lambda\log(1+t)\big)^{\frac{1}{2\lambda}}+\big(1+\lambda\log(1+t)\big)^{-\frac{1}{2\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}\\&=\sum_{k=0}^{\infty}E_{k,\lambda}(x)\frac{1}{k!}\big(\log(1+t)\big)^k=\sum_{k=0}^{\infty}E_{k,\lambda}(x)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\big(\sum_{k=0}^{n}E_{k,\lambda}(x)S_1(n,k)\big)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem

Theorem 2.1. *For $n \geq 0$, we have*

$$C_{n,\lambda}(x)=\sum_{k=0}^{n}E_{k,\lambda}(x)S_1(n,k).$$

In (2.5), by replacing t by e^t-1, we get

$$\begin{aligned}&\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{k=0}^{\infty}C_{k,\lambda}(x)\frac{1}{k!}(e^t-1)^k\\&=\sum_{k=0}^{\infty}C_{k,\lambda}(x)\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}C_{k,\lambda}(x)S_2(n,k)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

Therefore, by (2.1) and (2.7), we obtain the following theorem

Theorem 2.2. *For $n\geq 0$, we have*

$$E_{n,\lambda}(x)=\sum_{k=0}^{n}C_{k,\lambda}(x)S_2(n,k).$$

From (1.2) and (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}&=\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{tx}=\sum_{k=0}^{\infty}E_{k,\lambda}(x)\frac{1}{k!}\Big(\frac{1}{\lambda}(e^{\lambda t}-1)\Big)^k\\&=\sum_{k=0}^{\infty}\lambda^{-k}E_{k,\lambda}(x)\sum_{n=k}^{\infty}\lambda^nS_2(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\lambda^{n-k}S_2(n,k)E_{k,\lambda}(x)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.8}$$

Therefore, by comparing the coefficients on the both sides of (2.8), we obtain the following theorem

Theorem 2.3. *For $n\geq 0$, we have*

$$E_n(x)=\sum_{k=0}^{n}\lambda^{n-k}S_2(n,k)E_{k,\lambda}(x),$$

and

$$E_n=\sum_{k=0}^{n}\lambda^{n-k}S_2(n,k)E_{k,\lambda}.$$

Let us take $t=-2t$, $\lambda=2$, and $x=-\frac{1}{2}$ in (2.1). Then we have

$$\begin{aligned}\frac{2}{(1-4t)^{\frac{1}{4}}+(1-4t)^{-\frac{1}{4}}}(1-4t)^{-\frac{1}{4}}&=\frac{2}{1+\sqrt{1-4t}}\\&=\sum_{n=0}^{\infty}(-2)^nE_{n,2}(-\frac{1}{2})\frac{t^n}{n!}.\end{aligned}\tag{2.9}$$

It is well known that the Catalan numbers are given by the generating function to be

$$\frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}, \quad (see[4,9]). \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem

Theorem 2.4. *For $n \geq 0$, we have*

$$C_n = (-2)^n E_{n,2}(-\frac{1}{2}).$$

That is,

$$\frac{1}{n+1}\binom{2n}{n} = (-2)^n E_{n,2}(-\frac{1}{2}).$$

The λ-analogue of falling factorial sequence, $(x)_{n,\lambda}$, are defined by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big), \quad (n \geq 1).$$

Note that $\lim_{\lambda \longrightarrow 1}(x)_{n,\lambda} = (x)_n$, $(n \geq 0)$.
By (2.1), we get

$$\begin{aligned} 2 &= \Big(\sum_{l=0}^{\infty} E_{l,\lambda}\frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty}(\frac{1}{2})_{m,\lambda}\frac{t^m}{m!} + \sum_{m=0}^{\infty}\big(\frac{1}{2}+(m-1)\lambda\big)_{m,\lambda}(-1)^m\frac{t^m}{m!}\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}(\frac{1}{2})_{n-l,\lambda}E_{l,\lambda} + \sum_{l=0}^{n}\binom{n}{l}E_{l,\lambda}\big(\frac{1}{2}+(n-l-1)\lambda\big)_{n-l,\lambda} \\ &\times (-1)^{n-l}\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

Thus, by (2.11), we have

$$\begin{aligned} &\sum_{l=0}^{n}\binom{n}{l}(\frac{1}{2})_{n-l,\lambda}E_{l,\lambda} + \sum_{l=0}^{n}\binom{n}{l}E_{l,\lambda}\big(\frac{1}{2}+(n-l-1)\lambda\big)_{n-l,\lambda}(-1)^{n-l} \\ &\quad = 2\delta_{0,n}, \quad (n \geq 0), \end{aligned} \tag{2.12}$$

where $\delta_{n,k}$ is the kronecker's symbol.
We observe that

$$\begin{aligned}
&\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}\left(x+\frac{1}{2}\right)}\\
&=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}-1+2}(1+\lambda t)^{\frac{1}{\lambda}\left(x+\frac{1}{2}\right)}\\
&=\sum_{m=0}^{\infty}(-1)^m\Big(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{2}\Big)^m(1+\lambda t)^{\frac{1}{\lambda}\left(x+\frac{1}{2}\right)}\\
&=\sum_{m=0}^{\infty}(-1)^m 2^{-m}m!\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\big)^m(1+\lambda t)^{\frac{1}{\lambda}\left(x+\frac{m+1}{2}\right)} \qquad (2.13)\\
&=\sum_{m=0}^{\infty}(-1)^m 2^{-m}m!\sum_{n=m}^{\infty}T_{2,\lambda}\big(n,m|x+\frac{m+1}{2}\big)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}(-1)^m 2^{-m}m!T_{2,\lambda}\big(n,m|x+\frac{m+1}{2}\big)\Big)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.1) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$E_{n,\lambda}(x)=\sum_{m=0}^{n}(-1)^m 2^{-m}m!T_{2,\lambda}\big(n,m|x+\frac{m+1}{2}\big).$$

For $r\in\mathbb{N}$, the higher-order type 2 degenerate Euler polynomials are defined by the generating function to be

$$\Big(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}\Big)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \qquad (2.14)$$

When $x=0$, $E_{n,\lambda}^{(r)}=E_{n,\lambda}^{(r)}(0)$ are called the higher-order type 2 degenerate Euler numbers.

First, we observe that

$$
\begin{aligned}
&\Big(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}\Big)^{r} = (1+\lambda t)^{\frac{r}{2\lambda}}\Big(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{2}+1\Big)^{-r}\\
&= (1+\lambda t)^{\frac{r}{2\lambda}}\sum_{l=0}^{\infty}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}\big((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\big)^{l}\\
&\times (1+\lambda t)^{\frac{l}{2\lambda}}\\
&= \sum_{l=0}^{\infty}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}(1+\lambda t)^{\frac{r+l}{2\lambda}}l!\frac{1}{l!}\big((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\big)^{l}\\
&= \sum_{l=0}^{\infty}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}l!\sum_{n=l}^{\infty}T_{2,\lambda}\big(n,l|\frac{r+l}{2}\big)\frac{t^{n}}{n!}\\
&= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}l!T_{2,\lambda}\big(n,l|\frac{r+l}{2}\big)\Big)\frac{t^{n}}{n!}.
\end{aligned}
\tag{2.15}
$$

Therefore, by (2.14) and (2.15), we obtain the following theorem

Theorem 2.6. *For $n \geq 0$, we have*

$$E_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}l!T_{2,\lambda}\big(n,l|\frac{r+l}{2}+x\big).$$

In particular,

$$E_{n,\lambda}^{(r)} = \sum_{l=0}^{n}\binom{r+l-1}{l}\big(-\frac{1}{2}\big)^{l}l!T_{2,\lambda}\big(n,l|\frac{r+l}{2}\big).$$

First, we define the degenerate of $\sinh t$ which are given by

$$\sinh_{\lambda}(t) = \frac{1}{2}\big((1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}\big). \tag{2.16}$$

Note that $\lim_{\lambda\longrightarrow 0}\sinh_{\lambda}(t) = \sinh t = \frac{e^{t}-e^{-t}}{2}$.
Now, we define the type 2 degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{2.17}$$

When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the type 2 degenerate Bernoulli numbers. Form (2.16), we note that

$$\frac{t}{2}\csc h_{2\lambda}\big(\frac{t}{2}\big)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{2.18}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.17), we get

$$\begin{aligned}\Big(\frac{e^{\lambda t}-1}{\lambda t}\Big)\Big(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\Big)e^{xt} &= \sum_{k=0}^{\infty} B_{k,\lambda}(x)\lambda^{-k}\frac{1}{k!}(e^{\lambda t}-1)^k \\ &= \sum_{k=0}^{\infty} B_{k,\lambda}(x)\lambda^{-k}\sum_{n=k}^{\infty} S_2(n,k)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\big(\sum_{k=0}^{n}\lambda^{n-k}B_{k,\lambda}(x)S_2(n,k)\big)\frac{t^n}{n!}.\end{aligned} \tag{2.19}$$

Also, by (1.6) and (2.19), we have

$$\begin{aligned}\Big(\frac{e^{\lambda t}-1}{\lambda t}\Big)\Big(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\Big)e^{xt} &= \frac{1}{\lambda t}\Big(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\Big)(e^{(\lambda+x)t}-e^{xt}) \\ &= \frac{1}{\lambda t}\sum_{n=0}^{\infty}\big(b_n(\lambda+x)-b_n(x)\big)\frac{t^n}{n!} \\ &= \frac{1}{\lambda t}\sum_{n=1}^{\infty}\big(b_n(\lambda+x)-b_n(x)\big)\frac{t^n}{n!} \\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\Big(\frac{b_{n+1}(\lambda+x)-b_{n+1}(x)}{n+1}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$\frac{b_{n+1}(\lambda+x)-b_{n+1}(x)}{n+1} = \sum_{k=0}^{n}\lambda^{n-k}B_{k,\lambda}(x)S_2(n,k).$$

In the viewpoint of (1.2), we define the type 2 degenerate Daehee polynomials which are given by the generating function to be

$$\begin{aligned}&\frac{\log(1+t)}{\big(1+\lambda\log(1+t)\big)^{\frac{1}{2\lambda}}-\big(1+\lambda\log(1+t)\big)^{-\frac{1}{2\lambda}}}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.21}$$

When $x=0$, $d_{n,\lambda}=d_{n,\lambda}(0)$ are called the type 2 degenerate Daehee numbers. From (2.21), we can derive the following equation.

$$\begin{aligned}\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}&=\sum_{k=0}^{\infty}d_{k,\lambda}(x)\frac{1}{k!}(e^t-1)^k\\=\sum_{k=0}^{\infty}d_{k,\lambda}(x)\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}d_{k,\lambda}(x)S_2(n,k)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.22}$$

Therefore, by (2.17) and (2.22), we obtain the following theorem.

Theorem 2.8. *For $n\geq 0$, we have*

$$B_{n,\lambda}(x)=\sum_{k=0}^{n}d_{k,\lambda}(x)S_2(n,k).$$

By (2.17), we get

$$\begin{aligned}&\frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\&=\sum_{k=0}^{\infty}B_{k,\lambda}(x)\frac{1}{k!}\big(\log(1+t)\big)^k=\sum_{k=0}^{\infty}B_{k,\lambda}(x)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}B_{k,\lambda}(x)S_1(n,k)\Big)\frac{t^n}{n!}.\end{aligned}\tag{2.23}$$

Therefore, by (2.21) and (2.23), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$, we have*

$$d_{n,\lambda}(x)=\sum_{k=0}^{n}B_{k,\lambda}(x)S_1(n,k).$$

For $\alpha\in\mathbb{R}$, let us define the type 2 degenerate Daehee polynomials of order α which are given by the generating function to be

$$\begin{aligned}\Big(\frac{\log(1+t)}{\big(1+\lambda\log(1+t)\big)^{\frac{1}{2\lambda}}-\big(1+\lambda\log(1+t)\big)^{-\frac{1}{2\lambda}}}\Big)^{\alpha}\big(1+\lambda\log(1+t)\big)^{\frac{x}{\lambda}}\\=\sum_{n=0}^{\infty}d_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.24}$$

When $x=0$, $d_{n,\lambda}^{(\alpha)}=d_{n,\lambda}^{(\alpha)}(0)$ are called the type 2 degenerate Daehee numbers.

For $k \in \mathbb{N}$, let $\alpha = -k$, $x = 0$ in (2.24). Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda}^{(-k)} \frac{t^n}{n!} &= \Big(\frac{\big(1+\lambda\log(1+t)\big)^{\frac{1}{2\lambda}} - \big(1+\lambda\log(1+t)\big)^{-\frac{1}{2\lambda}}}{\log(1+t)} \Big)^k \\ &= \Big(\frac{t}{\log(1+t)}\Big)^k \frac{k!}{t^k} \sum_{m=k}^{\infty} T_{2,\lambda}(m,k) \frac{1}{m!} \big(\log(1+t)\big)^m \\ &= \Big(\sum_{l=0}^{\infty} B_l^{(l-k+1)} \frac{t^l}{l!} \Big) \Big(\frac{k!}{t^k} \sum_{m=k}^{\infty} T_{2,\lambda}(m,k) \sum_{i=m}^{\infty} S_1(i,m) \frac{t^i}{i!} \Big) \\ &= \Big(\sum_{l=0}^{\infty} B_l^{(l-k+1)} \frac{t^l}{l!} \Big) \Big(\frac{k!}{t^k} \sum_{i=k}^{\infty} \big(\sum_{m=k}^{i} T_{2,\lambda}(m,k) S_1(i,m) \big) \frac{t^i}{i!} \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} \sum_{m=k}^{i+k} \Big(\frac{\binom{n}{i}}{\binom{i+k}{i}} T_{2,\lambda}(m,k) S_1(i+k,m) \\ &\quad \times B_{n-i}^{(n-i-k+1)}(1) \Big) \frac{t^n}{n!},\end{aligned} \tag{2.25}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α which are given by

$$\Big(\frac{t}{e^t-1}\Big)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on the both sides of (2.25), we obtain the following theorem.

Theorem 2.10. *For $m, n, k \in \mathbb{N} \cup \{0\}$, we have*

$$d_{n,\lambda}^{(-k)} = \sum_{i=0}^{n} \sum_{m=k}^{i+k} \frac{\binom{n}{i}}{\binom{i+k}{i}} T_{2,\lambda}(m,k) S_1(i+k,m) B_{n-i}^{(n-i-k+1)}(1).$$

Now, we define the type 2 degenerate Bernoulli polynomials of order $\alpha (\in \mathbb{R})$ which are given by the generating function to be

$$\Big(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}} \Big)^{\alpha} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(\alpha)}(x) \frac{t^n}{n!}. \tag{2.26}$$

When $x = 0$, $B_{n,\lambda}^{(\alpha)} = B_{n,\lambda}^{(\alpha)}(0)$ are called the type 2 degenerate Bernoulli numbers of order α.

In particular, $\alpha = k \in \mathbb{N}$, we have

$$\Big(\frac{t}{2}\Big)^k \underbrace{\operatorname{csc} h_{2\lambda}\Big(\frac{t}{2}\Big) \times \operatorname{csc} h_{2\lambda}\Big(\frac{t}{2}\Big) \times \cdots \times \operatorname{csc} h_{2\lambda}\Big(\frac{t}{2}\Big)}_{k-times} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)} \frac{t^n}{n!},$$

where $\csc h_\lambda(t) = \frac{1}{\sinh_\lambda(t)}$.
From (2.24), we note that

$$\begin{aligned}\Big(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}\Big)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{k=0}^{\infty} d_{k,\lambda}^{(\alpha)}(x)\frac{1}{k!}(e^t-1)^k \\ = \sum_{k=0}^{\infty} d_{k,\lambda}^{(\alpha)}(x)\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} S_2(n,k)d_{k,\lambda}^{(\alpha)}(x)\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(\alpha)}(x) = \sum_{k=0}^{n} S_2(n,k)d_{k,\lambda}^{(\alpha)}(x),$$

and

$$d_{n,\lambda}^{(\alpha)}(x) = \sum_{k=0}^{n} S_1(n,k)B_{k,\lambda}^{(\alpha)}(x).$$

For $k \in \mathbb{N}$, let $\alpha = -k$ and $x = 0$ in (2.26). Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{(-k)}\frac{t^n}{n!} &= \frac{1}{t^k}\big((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\big)^k \\ &= \frac{k!}{t^k}\sum_{n=k}^{\infty} T_{2,\lambda}(n,k)\frac{t^n}{n!} = k!\sum_{n=0}^{\infty}\frac{T_{2,\lambda}(n+k,k)n!}{(n+k)!}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{\binom{n+k}{k}}T_{2,\lambda}(n+k,k)\frac{t^n}{n!}.\end{aligned} \tag{2.28}$$

Therefore, by comparing the coefficients on the both sides of (2.28), we obtain the following theorem.

Theorem 2.12. *For $n, k \geq 0$, we have*

$$\binom{n+k}{k}B_{n,\lambda}^{(-k)} = T_{2,\lambda}(n+k,k).$$

References

1. P. L. Butzer, M. Schmidt, E. L. Stark, L. Vogt, *Central factorial numbers; their main properties and some applications.* Numer. Funct. Anal. Optim. 10 (1989), no. 5-6, 419–488
2. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. 15 (1979), 51-88.
3. L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions.* Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974. xi+343 pp. ISBN: 90-277-0441-4

4. D. V. Dolgy, G.-W. Jang, D. S. Kim, T. Kim, *Explicit expressions for Catalan-Daehee numbers,* Proc. Jangjeon Math. Soc. 20 (2017), no. 1, 1-9.
5. G.-W. Jang, D. S. Kim, T. Kim, *Degenerate Changhee numbers and polynomials of the second kind,* Adv. Stud. Contemp. Math. 27(2017), no. 4, 609-624
6. D. S. Kim, T. Kim, *A note on type 2 Changhee and Daehee polynomials,* arXiv:1809.05217 (preprint)
7. T. Kim, D. S. Kim, *Degenerate central factorial numbers of the second kind* (communicated)
8. T. Kim, *A note on central factorial numbers,* Proc. Jangjeon Math. Soc.2192019), no. 4, 575–588.
9. T. Kim, *A note on Catalan numbers associated with p-adic integral on* $\mathbb{Z}_p$*,* Proc. Jangjeon Math. Soc. 19 (2016), no. 3, 493501.
10. M. Merca, *Connections between central factorial numbers and Bernoulli polynomials.* Period. Math. Hungar. 73 (2016), no. 2, 259–264.
11. I. Mezo, J. L. Ramirez, *some identities of the r-Whitney numbers,* Aequationes Math.90(2016), no. 2, 393–406.
12. S. Nadarajah, I. E. Okorie, *Factorial, raw and central moments.* Math. Sci. 41 (2016), no. 1, 65–71.
13. S.-S. Pyo, T. Kim, S.=H. Rim, *Degenerate Daehee numbers of the third kind,* Mathematics 2018, 6(11), 239; https://doi.org/10.3390/math6110239
14. R. Riordan, *Combinatorial identities.* John Wiley & Sons, Inc., New York-London-Sydney 1968 xiii+256 pp.
15. S. Roman, *The umbral calculus. Pure and Applied Mathematics,* 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. x+193 pp. ISBN: 0-12-594380-6

Department of Mathematics, Kwnagwoon University, Seoul 139-701, Republic of Korea

E-mail address: `gwjang@kw.ac.kr`

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

E-mail address: `tkkim@kw.ac.kr`