Advanced Studies in Contemporary Mathematics
29 (2019), No. 1, pp. 99 - 108
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.1.99

THE RIGIDITY OF RECTANGULAR FRAMEWORKS

KEUNBAE CHOI

ABSTRACT. In general, the stability problem of rectangular frameworks consisting of rectangular array of girder beams and riveted joints is determined by the connectivity of the bipartite graph. In this paper, we study how to solve the stability problem using the rank of the matrix induced by the rectangular framework.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65F30, 65F99.

KEYWORDS AND PHRASES. Rectangular Framework, bipartite graph, rank

1. INTRODUCTION

The following are excerpts from Wilson & Watkins' book ([6], pp.59-61). Many buildings are supported by steel frameworks consisting of rectangular array of girder beams and welded or riveted joints. In particular, this is the case if high-rise buildings is designed by steel rectangular arrays. However, for many reasons, these structures are treated as planar (rather than spatial) structures with pin-joints rather than rigid welds when joining the beams together. The simplest type is a rectangle consisting of four beams and four pin-joints as Figure 1.

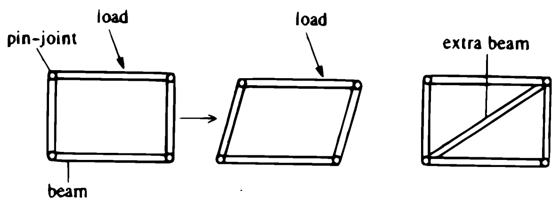

FIGURE 1. Simplest form of rectangular framework([6])

This structure is unstable because it can be easily deformed under sufficiently high loads into parallelogram. For the sake of stability of the structure, we have to reinforce the structure with extra beam as Figure 1.

This research was supported by the 2017 scientific promotion program funded by Jeju National University.

In the case of a larger structure (see, Figure 2) containing many rectangular cells, it is possible to ensure the rigidity by attaching support rods (extra beams) to all the rectangular cells, but it is costly in terms of economy. This natually leads to the following two mathematical problems.

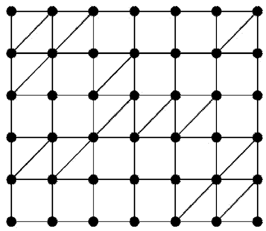

FIGURE 2. Rectangular framework

- The rigidity problem: whether the rectangular framework with bracings is rigid or not.
- Optimization problem: what is the minimum number of braces when the braced rectangular framework is rigid?

It is well known that the above problem can be solved with the connectivity of the bipartite graph induced by a braced rectangular framework (see, [1], [2], [3], [4], [5], [6]). Figure 3 shows a rigid rectangular framework and a corresponding connected bipartite graph. Figure 4, on the other hand, shows non-rigid structure and a disconnected bipartite graph that corresponds to it.

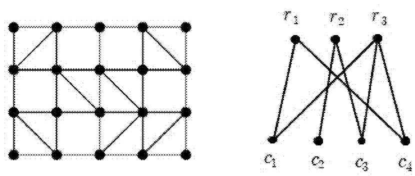

FIGURE 3. Rigid rectangular framework and connected bipartite graph

In particular, in [3], the rigidity of a rectangular framework is verified in more detail by using the relationship between the connectivity of the bipartite graph induced by given rectangular framework and the variation of angles of parallelograms constituting the rectangular framework.

In this paper, we study how to solve the stability problem using the rank of the matrix induced by the braced rectangular framework. This study can be seen as a sequential study of [3].

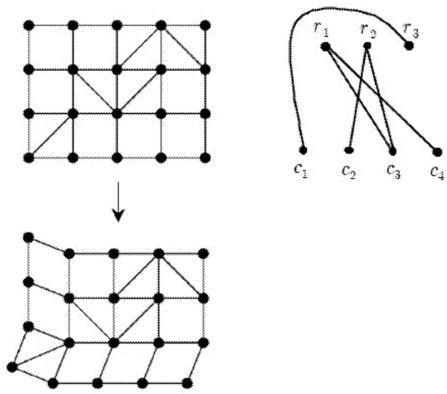

FIGURE 4. Non-rigid rectangular framework and disconnected bipartite graph

2. On the Bracing Rectangular Frameworks

Let R_{mn} be a $m \times n$ rectangular framework with braces. For each $i, j (1 \leq i \leq m, 1 \leq j \leq n)$, we let θ_{ij} be the angle of the upper left corner of the parallelogram that lies in the i-th row and j-th column of R_{mn}. For instance,

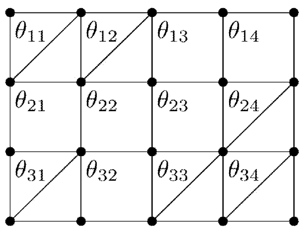

FIGURE 5. The upper left corner angles in a rectangular framework

If $\theta_{ij} = 90°$ for any $i, j (1 \leq i \leq m, 1 \leq j \leq n)$, then the rectangular framework R_{mn} is rigid. Notice that the angle of the upper left corner of a parallelogram with bracing is 90°. In Figure 5, since $\theta_{11} = \theta_{12} = \theta_{24} = \theta_{31} = \theta_{33} = \theta_{34} = 90°$, to determine the rigidity of the framework, we have to examine whether the remaining corner angles are also right angles or not.

The following lemma plays an important role in our approaches.

Lemma 2.1. *Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework. For each $i, j (1 \leq i < m, 1 \leq j < n)$, we have*

$$\theta_{ij} + \theta_{i+1j+1} = \theta_{ij+1} + \theta_{i+1j}$$

Proof. This is because the torsional motion of a rectangular framework is eventually depend on the torsional motion of the parallelograms that make up each cell in rectangular framework. □

Remark 2.2. From Lemma 2.1, using the parallelogram chasing (see, figure 7) with the basic form (rotating form of it if we need) as shown in the figure 6, it is possible to determine whether the rectangular framework is rigid even if it takes time. That is to say, if all rectangular cells without support rods can be chasing with basic forms, the rectangular framework is rigid.

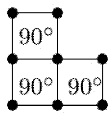

FIGURE 6. A basic form to determine the rigidity of rectangular framework

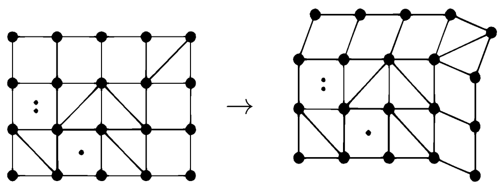

FIGURE 7. A parallelogram chasing with the basic form

For a rectangular framework R_{mn} $(m, n \geq 2)$, for each $i = 1, 2, \ldots, m-1$, we have the following equations;

$$\begin{cases} \theta_{i1} + \theta_{i+1\,2} = \theta_{i2} + \theta_{i+1\,1} \\ \theta_{i2} + \theta_{i+1\,3} = \theta_{i3} + \theta_{i+1\,2} \\ \cdots \\ \theta_{i\,n-1} + \theta_{i+1\,n} = \theta_{in} + \theta_{i+1\;n-1} \end{cases}$$

Equivalently, we have a matrix equation of the form

$$FY = O, \tag{2.1}$$

where

$$F = \begin{bmatrix} A & -A & O & O & \cdots & O & O & O \\ O & A & -A & O & \cdots & O & O & O \\ & & & & \ddots & & & \\ O & O & O & O & \cdots & O & A & -A \end{bmatrix}_{\{(m-1)\times(n-1)\}\times(m\times n)}$$

with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}_{(n-1)\times n},$$

a zero matrix $O = [0]_{(n-1)\times n}$, and

$$Y = [\theta_{11}\theta_{12}\cdots\theta_{1n}\theta_{21}\theta_{22}\cdots\theta_{2n}\cdots\theta_{m1}\theta_{m2}\cdots\theta_{mn}]^T.$$

Now, if we consider the bracings of given rectangular framework R_{mn}, then we have the following system of linear equations from (2.1),

$$F_R X = B, \tag{2.2}$$

where

- F_R is the matrix obtained by deleting θ_{ij} columns in F corresponding to braced parallelograms in R_{mn}.
- X is the matrix obtained by deleting θ_{ij} columns in Y corresponding to braced parallelograms in R_{mn}. That is to say, X is determined by non-braced parallelograms in R_{mn}.
- B is the matrix obtained by deleted θ_{ij} columns in F corresponding to braced parallelograms in R_{mn}.

For example, in Figure 5, we have the following equations

$$\begin{cases} \theta_{11} + \theta_{22} = \theta_{12} + \theta_{21} \\ \theta_{12} + \theta_{23} = \theta_{13} + \theta_{22} \\ \theta_{13} + \theta_{24} = \theta_{14} + \theta_{23} \end{cases}$$

and

$$\begin{cases} \theta_{21} + \theta_{32} = \theta_{22} + \theta_{31} \\ \theta_{22} + \theta_{33} = \theta_{23} + \theta_{32} \\ \theta_{23} + \theta_{34} = \theta_{24} + \theta_{33} \end{cases}$$

Equivalently,

$$FY = O, \tag{2.3}$$

where O is a zero matrix,

$$F = \begin{bmatrix} A & -A & O \\ O & A & -A \end{bmatrix}$$

with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

and

$$Y = [\theta_{11}\,\theta_{12}\,\theta_{13}\,\theta_{14}\,\theta_{21}\,\theta_{22}\,\theta_{23}\,\theta_{24}\,\theta_{31}\,\theta_{32}\,\theta_{33}\,\theta_{34}]^T.$$

Now, if we consider bracings in Figure 5, then we have

$$F_R X = B \tag{2.4}$$

where

$$F_R = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$X = [\theta_{13}\,\theta_{14}\,\theta_{21}\,\theta_{22}\,\theta_{23}\,\theta_{32}]^T,$$

and

$$B = -90^\circ \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0^\circ \\ -90^\circ \\ -90^\circ \\ 90^\circ \\ -90^\circ \\ 90^\circ \end{bmatrix}$$

Clearly, the system (2.4) of linear equations is consistent. In particular, for all $i, j (1 \leq i \leq 3,\ 1 \leq j \leq 4)$, $\theta_{ij} = 90^\circ$ is a solution of the system.

Notice that the rectangular framework in Figure 5 is rigid if and only if the rank of the matrix F_R is 6. Since $\text{rank}(F_R) = 6$, the rectangular framework in Figure 5 is rigid.

Theorem 2.3. *Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework. Then R_{mn} is rigid if and only if the rank of F_R is the number of columns of F_R.*

Proof. Clearly, the system $F_R X = B$ of (2.2) is consistent. Thus we have the rank of F_R is equal to the number of columns of F_R if and only if the system $F_R X = B$ has only one solution, say $X = [90^\circ\,90^\circ\,\cdots 90^\circ]^T$. $\square$

Corollary 2.4. *Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework. Then we have*

(1) *The nullity of F, the dimension of the solution space of $FX = O$, is $m - n + 1$.*
(2) *The quantity $m + n - 1$ is the minimum number of supports for R_{mn} to be rigid.*

Proof. Notice that the size of the matrix F is

$$(m-1)(n-1) \times mn$$

and

$$\begin{aligned} mn - (m+n-1) &= (m-1)(n-1) \\ &= \operatorname{rank}(F) \geq \operatorname{rank}(F_R). \end{aligned}$$

This completes the proof. $\square$

Remark 2.5. The rectangular framework R_{mn} $(m, n \geq 2)$ of Figure 8 is rigid with minimum bracings. Because of the framework does not change the rank of F. In fact,

$$\operatorname{rank}(F) = (m-1)(n-1) = \operatorname{rank}(F_R)$$

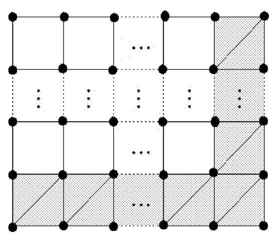

FIGURE 8. The rectangular framework with bracings in m-th row and n-th column

For the future development of the theory, we consider the example of Figure 9, in this case we have

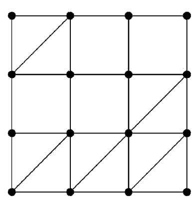

FIGURE 9. R_{33} with braces.

$$F = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

and

$$F_R = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework and let

$$\overline{F}_R = F\,E,$$

where

$$E = \begin{bmatrix} C_1 & & & & \\ & \ddots & & \mathbf{O} & \\ & & C_i & & \\ & \mathbf{O} & & \ddots & \\ & & & & C_m \end{bmatrix} \tag{2.5}$$

The size of the matrix C_i is $n \times p$, where p is the number of non-braced parallelograms in i-th row in R_{mn}. In the case of all parallelograms in i-th row in R_{mn} are bacings, for convenience, we consider the size of C_i to $n \times 1$.

The entries of C_i $(i = 1, \ldots, m)$ are constructed by the informations of R_{mn}; first, if we give the i-th row of R_{mn} to the number 0 if a parallelogram has a bracing, otherwise give 1, then i-th row of R_{mn} is represented by the sum of the rows having only one component of 1, say 'd-row'. And then the transpose matrix of the matrix consisting of d-rows in i-row of R_{mn} is C_i. If the components of i-th row are all 0 (all parallelograms in i-th row in R_{mn} are bacings), the entries of C_i are all zeros (see, the 3rd row of the rectangular framework in Figure 9). For instance, in Figure 5,

- 1st row: $[0\ 0\ 1\ 1] = [0\ 0\ 1\ 0] + [0\ 0\ 0\ 1]$, the sum of d-rows

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T$$

- 2nd row: $[1\ 1\ 1\ 0] = [1\ 0\ 0\ 0] + [0\ 1\ 0\ 0] + [0\ 0\ 1\ 0]$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T$$

- 3rd row: $[0\ 1\ 0\ 0]$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

from C_i $(i = 1, 2, 3)$, we have

$$E = \begin{bmatrix} C_1 & O & O \\ O & C_2 & O \\ O & O & C_3 \end{bmatrix}$$

and $F_R = \overline{F}_R$. In the case of Figure 9,

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T, \ C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T, \ C_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

and

$$\overline{F}_R = FE = \begin{bmatrix} -1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be easily proved that $\overline{F}_R$ is just a matrix with a few zero columns added to matrix F_R. Thus the rank of $\overline{F}_R$ is equal to the rank of F_R.

Corollary 2.6. *Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework. Then R_{mn} is rigid if and only if rank(F_R) = rank(E).*

Proof. We note that

$$\begin{aligned} \operatorname{rank}(F_R) &= \operatorname{rank}(\overline{F}_R) = \operatorname{rank}(FE) \\ &\leq \operatorname{rank}(E) = \operatorname{rank}(C_1) + \cdots + \operatorname{rank}(C_m) \\ &= \text{The number of columns of } F_R \end{aligned}$$

Thus we have that the rectangular framework R_{mn} is rigid if and only if rank(F_R) = rank(E). □

Corollary 2.7. *Let R be a rectangular framework obtained by permute the rows (or columns) of any rigid rectangular framework. Then R is also rigid.*

Proof. Notice that two matrices that are row(or column) equivalent have the same rank. □

3. Algorithm

Let R_{mn} be a $m \times n$ $(m, n \geq 2)$ rectangular framework with bracings.

Step 1: If the number of bracings is less than $m + n - 1$, the rectangular framework R_{mn} is NOT rigid. Otherwise, go to Step 2.

Step 2: Construct a Toeplitz type matrix A :

$$A = \begin{bmatrix} \underline{1 \quad -1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \underline{1 \quad -1} & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & 0 \cdots & 0 & \underline{1 \quad -1} \end{bmatrix}_{(n-1)\times n}$$

Step 3: Construct a block Toeplitz type matrix F from A :

$$F = \begin{bmatrix} \underline{A \quad -A} & O & O & \cdots & O & O & O \\ O & \underline{A \quad -A} & O & \cdots & O & O & O \\ & & & \ddots & & & \\ O & O & O & O \cdots & O & \underline{A \quad -A} \end{bmatrix}_{\{(m-1)\times(n-1)\}\times(m\times n)}$$

Step 4: Construct the matrix $\overline{F}_R$ from the block diagonal matrix E: $\overline{F}_R = F\,E$

$$E = \begin{bmatrix} C_1 & & & & \\ & \ddots & & \mathbf{O} & \\ & & C_i & & \\ & \mathbf{O} & & \ddots & \\ & & & & C_m \end{bmatrix} \qquad \text{(see, (2.5))}$$

Step 5: Find rank$(\overline{F}_R)$ and rank(E) : if rank$(\overline{F}_R)$ = rank(E), then rectangular framework R_{mn} is rigid. Otherwise, R_{mn} is non-rigid.

References

[1] Bolker, E. D. and Henry Crapo., *Bracing Rectangular Frameworks I*, SIAM J. Appl. Math. **36** (1997), June, 473-490.

[2] Graver, J. E., Counting on frameworks: Mathematics to Aid the design of Rigid Structures, the Mathematical Association of America, United States of America. 2001.

[3] Lee, J., Kwon, Y. S and Choi, K., *A Study on the Bracing Rectangular Frameworks*, J. Korea Soc. Math. Ed. Ser. E: Communications of Mathematical Education, **30** (2) (2016), 251-262.

[4] Laine, S. T., The Grid Bracing Problem and a Generalization, Master's Thesis of Worcester Polytechnic Institute, United States of America. 2006.

[5] Servatius, Brigitte., *Graphs, Digraphs, and the Rigidity of Grids*, The UMAP Journal, **16** (1995), 37-63.

[6] Wilson, R. J. and Wakins, J. J., Graphs; An introductory Approach, John Wiley and Sons, Inc. 1990.

Department of Mathematics, Teachers College, Jeju National University, Jeju 63294, Korea

E-mail address: kbchoe@jejunu.ac.kr