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ON THE TOPOLOGY OF n-NORMED SPACES WITH RESPECT TO NORMS OF ITS QUOTIENT SPACES

HARMANUS BATKUNDE AND HENDRA GUNAWAN

Abstract. In this paper, we study some topological characteristics of n-normed spaces. We construct quotient spaces of an n-normed space and define a norm in each quotient space. Using these norms, we observe convergence sequences, closed sets, and bounded sets in n-normed spaces. These norms will be a new viewpoint in observing the characteristics of n-normed spaces. We also review the completeness of n-normed spaces.

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1. Introduction

In the 1960's, the concept of n-normed spaces for $n \geq 2$ was introduced by Gähler [6, 7, 8, 9]. Some other researchers then studied and developed this concept further [1, 2, 3, 5, 10]. They also studied some characteristics of n-normed spaces. Ekariani *et al.* [4] provided a contractive mapping theorem for the n-normed space of p-summable sequence. They proved the theorem using a linearly independent set consisting of n vectors. Moreover, Gunawan *et al.* [12] proved fixed point theorems on bounded sets in an n-normed space also using a linearly independent set of n vectors. Meitei and Singh [13] studied bounded n-linear operators in n-normed spaces using a similar approach. As we can see, most researchers usually investigate n-normed spaces using a set of n linear independent vectors.

Moreover, we will consider some quotient spaces of an n-normed space. We will construct these quotient spaces with respect to a linearly independent set. We also define a norm in each quotient space. We use these norms to observe some characteristics of n-normed spaces. Before we present our main results, here are some basic concepts of n-normed spaces.

Let n be a nonnegative integer and X is a real vector space with $dim(X) \geq n$. An n-**norm** on X is a function $\|\cdot,\ldots,\cdot\| : X^n \to \mathbb{R}$ which satisfies the following conditions:

i. $\|x_1,\cdots,x_n\| \geq 0$;
   $\|x_1,\cdots,x_n\| = 0$ if and only if $x_1,\cdots,x_n$ linearly dependent.
ii. $\|x_1,\cdots,x_n\|$ is invariant under permutation.
iii. $\|\alpha x_1,\ldots,x_n\| = |\alpha|\ \|x_1,\ldots,x_n\|$ for any $\alpha \in \mathbb{R}$.
iv. $\|x_1 + x_1', x_2,\ldots,x_n\| \leq \|x_1,x_2,\ldots,x_n\| + \|x_1',x_2,\ldots,x_n\|$.

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The pair $(X, \|\cdot, \dots, \cdot\|)$ is called **an n-normed space** [10].

For example, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, we can define the standard n-norm on X by

$$\|x_1, \dots, x_n\|^S := \begin{vmatrix} \langle x_1, x_1\rangle & \cdots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}^{1/2}. \tag{1}$$

The determinant on the right hand side of (1) is known as Gram's determinant. Its value is always nonnegative. Geometrically, the value of $\|x_1, \dots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$. Moreover, if $(X, \|\cdot, \dots, \cdot\|)$ is an n-normed space, then

$$\|x_1, \dots, x_n\| = \|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \dots, x_n\|,$$

for all $x_1, \dots, x_n \in X$ and $\alpha_2, \dots, \alpha_n \in \mathbb{R}$ [10].

Now, we construct the quotient spaces of an n-normed space with respect to a linearly independent set. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $Y = \{y_1, \dots, y_n\}$ be a linearly independent set in X. For a $j \in \{1, \dots, n\}$ we consider $Y \setminus \{y_j\}$. We define a subspace of X, namely

$$Y_j^0 := \operatorname{span} Y \setminus \{y_j\} := \left\{ \sum_{i \neq j} \alpha_i y_i : \quad \alpha_i \in \mathbb{R}. \right\}.$$

For any $u \in X$, the corresponding coset in X is

$$\overline{u} := \left\{ u + \sum_{i \neq j} \alpha_i y_i : \quad \alpha_i \in \mathbb{R} \right\}.$$

Then we have $\overline{0} = \operatorname{span} Y \setminus \{y_j\} = Y_j^0$ and if $\overline{u} = \overline{v}$ then $u - v \in \operatorname{span} Y \setminus \{y_j\}$. We define the quotient space

$$X_j^* := X/Y_j^0 := \{\overline{u} : u \in X\}.$$

The addition and scalar multiplication apply in this space. Moreover, we define a norm of X_j^* defined by

$$\|\overline{u}\|_j^* := \|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\|. \tag{2}$$

Using the above construction, we can get n quotient spaces. These quotient spaces have same structure but different elements. The set that contains all quotient spaces constructed above is called class-1 collection. [1]

In the above construction we 'eliminate' one vector from Y. Now we construct quotient spaces by 'eliminating' m vectors from Y with $m \in \{1, \dots, n\}$. For $i_1, \dots, i_m \in \{1, \dots, n\}$, observe $Y \setminus \{y_{i_1}, \dots, y_{i_m}\}$. We define a subspace of X, namely

$$Y_{i_1,\dots,i_m}^0 := \operatorname{span} Y \setminus \{y_{i_1}, \dots, y_{i_m}\} := \left\{ \sum_{i \neq i_1,\dots,i_m} \alpha_i y_i \ ; \ \alpha_i \in \mathbb{R}. \right\}.$$

For any $u \in X$, the corresponding coset in X is

$$\overline{u} := \left\{ u + \sum_{i \neq i_1,\ldots,i_m} \alpha_i y_i : \alpha_i \in \mathbb{R} \right\}.$$

Then we have $\overline{0} = \operatorname{span} Y \setminus \{y_{i_1}, \ldots, y_{i_m}\} = Y^0_{i_1,\ldots,i_m}$ and, if $\overline{u} = \overline{v}$, then $u - v \in \operatorname{span} Y \setminus \{y_{i_1}, \ldots, y_{i_m}\}$. We define the quotient space

$$X^*_{i_1,\ldots,i_m} := X/Y^0_{i_1,\ldots,i_m} := \{\overline{u} : u \in X\}.$$

The addition and scalar multiplication apply in this space. Moreover, we define a norm of $X^*_{i_1,\ldots,i_m}$ defined by

$$\begin{aligned} \|\overline{u}\|^*_{i_1,\ldots,i_m} :=& \|u, y_1, \ldots, y_{i_1-1}, y_{i_1+1}, \ldots, y_n\| + \ldots \\ &+ \|u, y_1, \ldots, y_{i_m-1}, y_{i_m+1}, \ldots, y_n\|. \end{aligned} \tag{3}$$

Using the above construction, we can get $\binom{n}{m}$ quotient spaces. For an $m \in \{1, \ldots, n\}$ the set that contains all quotient spaces constructed above is called class-m collection. [1]

One can see that the right hand side of equation (3) is a summation of norms of class-1 collection (see (2)). Then equation (3) can be rewritten as

$$\|\overline{u}\|^*_{i_1,\ldots,i_m} = \|\overline{u}\|^*_{i_1} + \cdots + \|\overline{u}\|^*_{i_m}.$$

Moreover, by saying 'norms of class-m collection' we mean 'all the norms of each quotient space in a class-m collection'. Actually, some previous researchers who worked on n-normed spaces have used this approach. They used the norm of class-n collection to investigate some characteristics of the n-normed spaces. If we compare our approach to their works, ours provides more than one viewpoint to observe characteristics of the n-normed spaces. One can see that the summation of all norms of class-1 collection is the norm of class-n collection. Furthermore, for any $m \in \{1, \ldots, n\}$ we will use the norms of the class-m collection to study some topological structures of the n-normed spaces.

2. Results and Discussion

As mentioned before, the construction of quotient spaces depends on a set of n linearly independent vectors we choose. The choice of the set would not matter. We can choose any n linearly independent vectors in the n-normed spaces to form the set. So from here on, we will not mention the linearly independent set explicitly, unless it is necessary.

Here we introduce some topological characteristics we observe by using the norms of class-m collection with $m \in \{1, \ldots, n\}$.

Definition 2.1. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. We say a sequence $\{x_k\} \subset X$* ***converges with respect to the norms of class-m collection to*** *x if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $k \geq N$ we have*

$$\|\overline{x_k - x}\|^*_{i_1,\ldots,i_m} < \epsilon,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. In this case we also say

$$\lim_{n\to\infty} \|\overline{x_k - x}\|^*_{i_1,\ldots,i_m} = 0.$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. *If* $\{x_k\}$ *does not converge, we say it diverges.*

Note that this definition also says that for an $m \in \{1, \dots, n\}$, a sequence converges with respect to the norms of a class-m collection to x if and only if the sequence converges in each quotient space of the class-m collection.

We want to know if there is a relation between convergence with respect to the norms of class-m_1 collection and class-m_2 collection for any $m_1, m_2 \in \{1, \dots, n\}$. First, we examine the following example.

Example 2.2. *Let* $(\mathbb{R}^d, \|\cdot,\cdot,\cdot\|)$ *be a* 3*-normed space and for* $m = 1, 2, 3$ *consider all norms in each class-m collection. A sequence* $\{x_k\}$ *converges with respect to the norms of class-*2 *collection to* x *if and only if, for any* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that for any* $k \geq N$ *we have*

$$\begin{aligned} \|\overline{x_k - x}\|^*_{1,2} &= \|\overline{x_k - x}\|^*_1 + \|\overline{x_k - x}\|^*_2 < \frac{2}{3}\epsilon, \\ \|\overline{x_k - x}\|^*_{1,3} &= \|\overline{x_k - x}\|^*_1 + \|\overline{x_k - x}\|^*_3 < \frac{2}{3}\epsilon, \\ \|\overline{x_k - x}\|^*_{2,3} &= \|\overline{x_k - x}\|^*_2 + \|\overline{x_k - x}\|^*_3 < \frac{2}{3}\epsilon. \end{aligned} \tag{4}$$

Equation (4) implies

$$\|\overline{x_k - x}\|^*_{1,2,3} = \|\overline{x_k - x}\|^*_1 + \|\overline{x_k - x}\|^*_2 + \|\overline{x_k - x}\|^*_3 < \epsilon. \tag{5}$$

*The left hand side of equation (5) is nothing but the norm of class-*3 *collection. It means at the same time* $\{x_k\}$ *converges with respect to the norms of class-*3 *collection to* x. *Equation (4) also implies*

$$\|\overline{x_k - x}\|^*_1 < \epsilon\,; \quad \|\overline{x_k - x}\|^*_2 < \epsilon\,; \quad \|\overline{x_k - x}\|^*_3 < \epsilon. \tag{6}$$

*The left hand sides of three equations in (6) are the norms of class-*1 *collection. By definition, this means the sequence* $\{x_k\}$ *converges with respect to the norms of class-*1 *collection to* x. *Here we have if* $\{x_k\}$ *converges with respect to the norms of class-*2 *collection to* x, *then it converges with respect to the norms of class-*1 *collection to* x. *Also it converges with respect to the norms of class-*3 *collection to* x. *One can see that the converses are also true.*

Example 2.2 indicates that all types of convergence are equivalent. This is true as we state in the following theorem.

Theorem 2.3. *Let* $(X, \|\cdot, \dots, \cdot\|)$ *be an n-normed space and* $m \in \{1, \dots, n\}$. *A sequence* $\{x_k\} \subset X$ *is convergent with respect to the norms of class-*1 *collection if and only if it is convergent with respect to the norms of class-m collection.*

Proof. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $m \in \{1, \dots, n\}$. Suppose that $\{x_k\} \subset X$ converges with respect to the norms of class-1 collection to x. Then for any $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that for $n \geq N_0$ we have

$$\|\overline{x_k - x}\|_1^* < \frac{1}{m}\epsilon,$$
$$\vdots$$
$$\|\overline{x_k - x}\|_n^* < \frac{1}{m}\epsilon,$$

for any $m \in \{1, \ldots, n\}$. Therefore, we have

$$\|\overline{x_k - x}\|_{i_1}^* + \cdots + \|\overline{x_k - x}\|_{i_m}^* < \epsilon,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. By the definition, this means $\{x_k\}$ converges with respect to the norms of class-m collection to x.

Conversely, suppose that $\{x_k\}$ converges with respect to the norms of class-m collection to x. Then for any $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that for $n \geq N_1$ we have

$$\|\overline{x_k - x}\|_{i_1}^* + \cdots + \|\overline{x_k - x}\|_{i_m}^* < \epsilon,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. Then we have

$$\|\overline{x_k - x}\|_j^* < \epsilon,$$

for any $j \in \{1, \ldots, n\}$. By the definition $\{x_k\}$ also converges with respect to the norms of class-1 collection to x. $\square$

Theorem (2.3) states that for any $m_1, m_2 \in \{1, \ldots, n\}$, the types of convergence with respect to class-m_1 and class-m_2 collection are equivalent. So unless we need to specify the class explicitly, we may simply use the word 'converges' instead of 'converges with respect to the norms of class-m collection'. We also denote $\{x_k\}$ converges to x by $x_k \longrightarrow x$. Moreover we present teh definition with respect to the normss of class-m collection for an $m \in \{1, \ldots, n\}$ closed set in the n-normed space.

Definition 2.4. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space, $K \subseteq X$, and $m \in \{1, \ldots, n\}$. The set K is called* ***closed with respect to the norms of class-m collection*** *if for any sequence $\{x_k\}$ in K that converges with respect to the norms of class-m collection in X, its limit belongs to K.*

Since all types of convergence are equivalent, all types of closedness of a set also euivalent. Moreover, one can see that for observing the convergence of a sequence we have to use all norms of a class-m collection. For simplicity, we need to reduce the number of the norms we used. So, is it possible to reduce the number of the norms? First, we consider the following example.

Example 2.5. *Let $(\mathbb{R}^d, \|\cdot, \cdot, \cdot\|)$ be a 3-normed space. We define class-1,2,3 collections of $\mathbb{R}^d$. Suppose that a sequence $\{x_k\} \subset \mathbb{R}^d$ converges with respect to the norms of class-2 collection to x. For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for $k \geq N$ we have*

$$\|\overline{x_k - x}\|_{1,2}^* < \epsilon, \tag{7}$$

$$\|\overline{x_k - x}\|_{1,3}^* < \epsilon, \tag{8}$$

$$\|\overline{x_k - x}\|_{2,3}^* < \epsilon. \tag{9}$$

Now, let us just consider (7) and (8). The two equations imply

$$\|\overline{x_k - x}\|_1^* < \epsilon, \quad \|\overline{x_k - x}\|_2^* < \epsilon, \quad \|\overline{x_k - x}\|_3^* < \epsilon.$$

By the definition, we have $\{x_k\}$ converges with respect to the norms of class-1 collection to x. Since for all $m \in \{1, \dots, n\}$ the types of convergence are equivalent collection, we can check the convergence of a sequence just by using two norms of class-2 collection, namely $\|\cdot\|_{1,2}$ and $\|\cdot\|_{1,3}$.

Remark. Generally, for an $m \in \{1, \dots, n\}$ the convergence (of a sequence) with respect to the norms of class-m collection can be observed by using some norms $\|\cdot\|^*_{i_1,\dots,i_m}$ we choose such that

$$\bigcup \{i_1, \dots, i_m\} \supseteq \{1, \dots, n\}. \tag{10}$$

Moreover, the least number of norms that can be used to observe the convergence with respect to the norms of class-m collection is $\lceil \frac{n}{m} \rceil$ norms. For some class-m collections, we do not need to use all norms to check the convergence. One can see that for $m = 1$ or $m = n$ we have to use all norms of the class-m collection.

For example, let $(X, \cdot, \cdot, \cdot, \cdot, \cdot|\|)$ be a 5-normed space and consider the norms of class-2 collection. The least number of norms that we can use is $\lceil \frac{5}{2} \rceil = 3$, for instance $\|\cdot\|^*_{1,2}$, $\|\cdot\|^*_{1,3}$, $\|\cdot\|^*_{4,5}$. If it is less than three norms, then it will not meet the definition requirements. The following example shows what happened if we choose less than $\lceil \frac{n}{m} \rceil$ norms.

Example 2.6. *Let $(\mathbb{R}^5, \langle \cdot, \cdot \rangle)$ be an inner product space and Y is a set of standard basis vectors of $\mathbb{R}^5$. We define a standard 5-norm as*

$$\|x_1, \dots, x_5\|^S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_5 \rangle \\ \vdots & \ddots & \vdots \\ \langle x_5, x_1 \rangle & \cdots & \langle x_5, x_5 \rangle \end{vmatrix}^{1/2}. \tag{11}$$

We consider sequence $x_k = (0, 0, 0, 0, k)$ in $\mathbb{R}^5$. By the definition, one can see that this sequence $\{x_k\}$ does not converge with respect to the norms of class-m collection, for any $m = 1, \dots, 5$.

Let us consider class-2 collection of $\mathbb{R}^5$ and observe the convergence. Note that three is the least number of norms we propose to observe the convergence with respect to the norms of class-2 collection. Instead of using three norms, here we try to use two norms of class-2 collection. Let $x = (0, 0, 0, 0, 0)$, choose two norms of class-2 collection, namely $\|\cdot\|^*_{1,2}, \|\cdot\|^*_{3,4}$. For any $k \in \mathbb{N}$ we have

$$\|\overline{x_k - x}\|^*_{1,2} = \|\overline{x_k}\|^*_{1,2} = \|x_k, y_2, y_3, y_4, y_5\|^S + \|x_k, y_1, y_3, y_4, y_5\|^S = 0,$$

and

$$\|\overline{x_k - x}\|^*_{3,4} = \|\overline{x_k}\|^*_{1,2} = \|x_k, y_1, y_2, y_4, y_5\|^S + \|x_k, y_1, y_2, y_3, y_5\|^S = 0.$$

the value of these norms is 0 because $x_k = k\, y_5$, for every $k = 1, 2, \dots,$ which means x_k and y_5 are linearly dependent. If we just take these two norms to investigate the convergence of $\{x_k\}$, then we have $x_k \longrightarrow x$. This is a false *conclusion. But if we add another norm of class-2, for example $\|\overline{x_k - x}\|^*_{1,5}$, then we have*

$$\begin{aligned}\|\overline{x_k - x}\|^*_{1,5} &= \|x_k, y_2, y_3, y_4, y_5\|^S + \|x_k, y_1, y_2, y_3, y_4\|^S \\ &= 0 + \begin{vmatrix} \langle x_k, x_k\rangle & \langle x_k, x_1\rangle & \langle x_k, x_2\rangle & \langle x_k, x_3\rangle & \langle x_k, y_4\rangle \\ \langle x_1, x_k\rangle & \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \langle x_1, x_3\rangle & \langle x_1, y_4\rangle \\ \langle x_2, x_k\rangle & \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \langle x_2, x_3\rangle & \langle x_2, y_4\rangle \\ \langle x_3, x_k\rangle & \langle x_3, x_1\rangle & \langle x_3, x_2\rangle & \langle x_3, x_3\rangle & \langle x_3, y_4\rangle \\ \langle y_4, x_k\rangle & \langle x_4, x_1\rangle & \langle x_4, x_2\rangle & \langle x_4, x_3\rangle & \langle y_4, y_4\rangle \end{vmatrix}^{1/2} \\ &= n.\end{aligned}$$

Then, for $n \to \infty$, $\|\overline{x_k - x}\|^*_{1,5} \to \infty$, which means that $\{x_k\}$ diverges.

The above example shows that in this case, we can not take less than $\lceil \frac{5}{2} \rceil$ norms to examine the convergence of a sequence. Also, the norms we choose must satisfy condition (10). Furthermore, we will study bounded sets with respect to the norms of quotient sets of the n-normed spaces.

Definition 2.7. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, $K \subseteq X$ be a nonempty set, and $m \in \{1, \dots, n\}$. The set K is called **bounded with respect to the norms of class-m collection** if and only if there exists an $M > 0$ for any $x \in K$ such that*

$$\|\overline{x}\|^*_{i_1,\dots,i_m} \le M,$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$.

Moreover, we will show that all types of the boundedness of a set with respect to the norms of class-m collection for all $m \in \{1, \dots, n\}$ are equivalent. Now, let us consider the following example.

Example 2.8. *Let $(\mathbb{R}^d, \|\cdot, \cdot, \cdot\|)$ be a 3-normed space, $K \subset \mathbb{R}$ and define class-1, 2, 3 collections in $\mathbb{R}^d$. Suppose that $K \subseteq \mathbb{R}^d$ is bounded with respect to the norms of class-2 collection. Then for any $x \in K$, there is an $M > 0$ such that*

$$\begin{aligned}\|\overline{x}\|^*_{2,3} &= \|\overline{x}\|^*_2 + \|\overline{x}\|^*_3 \le M, \\ \|\overline{x}\|^*_{1,3} &= \|\overline{x}\|^*_1 + \|\overline{x}\|^*_3 \le M, \\ \|\overline{x}\|^*_{1,2} &= \|\overline{x}\|^*_1 + \|\overline{x}\|^*_2 \le M.\end{aligned} \tag{12}$$

These imply

$$2\left(\|\overline{x}\|^*_1 + \|\overline{x}\|^*_2 + \|\overline{x}\|^*_3\right) = \|\overline{x}\|^*_{2,3} + \|\overline{x}\|^*_{1,3} + \|\overline{x}\|^*_{1,2} \le 3M.$$

or

$$\|\overline{x}\|^*_{1,2,3} = \|\overline{x}\|^*_1 + \|\overline{x}\|^*_2 + \|\overline{x}\|^*_3 \le \frac{3}{2}M = C, \tag{13}$$

with $C > 0$. One can see that the left hand side of equation (13) is the norm of class-3 collection. This means that K is bounded with respect to the norm in class-3 collection. Equation (13) implies

$$\|\overline{x}\|^*_1 \le C, \quad \|\overline{x}\|^*_2 \le C, \quad \|\overline{x}\|^*_3 \le C. \tag{14}$$

The left hand sides of the three equations in (14) are nothing but the norms of class-1 collection. This means K is bounded with respect to the norms of class-1 collection.

In the above example, if K is bounded with respect to the norms of class-2 collection, then K is bounded with respect to the norms of class-3 collection. It also implies K is bounded with respect to the norms of class-1 collection. One can see that the converses are also true. The example indicates that all types of boundedness are equivalent. Then we have the following theorem.

Theorem 2.9. *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, $K \subset X$ nonempty and $m \in \{1, \dots, n\}$. The set K is bounded with respect to the norms of class-1 collection if and only if it is bounded with respect to class-m collection.*

Proof. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, $K \subset X$ nonempty and $m \in \{1, \dots, n\}$. Suppose that the set K is bounded with respect to the norms of class-1 collection. Then there is an $M > 0$ for any $x \in K$, such that

$$\|\overline{x}\|_1^* \leq M, \quad \|\overline{x}\|_2^* \leq M, \quad \cdots, \quad \|\overline{x}\|_n^* \leq M.$$

Hence we have

$$\|\overline{x}\|_{i_1,\dots,i_m}^* = \|\overline{x}\|_{i_1}^* + \cdots + \|\overline{x}\|_{i_m}^* \leq m \cdot M, \tag{15}$$

for any $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. The left hand side of equation (15) represents all norms of class-m collection. By the definition, K is bounded with respect to the norms of class-m collection.

Conversely, supposed that K is bounded with respect to the norms of class-m collection. Then there exists an $M > 0$ for any $x \in K$ such that

$$\|\overline{x}\|_{i_1}^* + \cdots + \|\overline{x}\|_{i_m}^* = \|\overline{x}\|_{i_1,\dots,i_m}^* \leq M.$$

for every $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. It implies that

$$\|\overline{x}\|_j^* \leq M,$$

for every $j \in \{1, \dots, n\}$. Consider the norms of class-1 collection. Then by the definition this means K is bounded with respect to the norms of class-1 collection. □

Since all types of boundedness are equivalent, we will not mention the type of boundedness explicitly. We may simply use the word 'bounded' instead of 'bounded with respect to the norms of class-m collection'.

We will give an example to show that we do not need to choose all norms of class-m collection for some $m \in \{1, \dots, n\}$ to examine the boundedness.

Example 2.10. *We observe $(\mathbb{R}^d, \|\cdot, \cdot, \cdot\|)$ as a 3-normed space and consider all norms of class-2 collection. We have three norms in class-2 collection, namely $\|\cdot\|_{1,2}^*, \|\cdot\|_{1,3}^*, \|\cdot\|_{2,3}^*$. It is sufficient to examine the boundedness of a set $K \subseteq \mathbb{R}^d$ just by using two norms of class-2, namely $\|\overline{x}\|_{1,2}^*, \|\overline{x}\|_{1,3}^*$. If, for every $x \in K$,*

$$\|\overline{x}\|_{1,2}^* \leq M, \text{ and } \|\overline{x}\|_{1,3}^* \leq M; \ M > 0,$$

then we have

$$2\|\overline{x}\|_1^* + \|\overline{x}\|_2^* + \|\overline{x}\|_3^* \leq 2M = C; \ C > 0.$$

This implies

$$\|\overline{x}\|_{1,2,3}^* \leq 2\|\overline{x}\|_1^* + \|\overline{x}\|_2^* + \|\overline{x}\|_3^* \leq C. \tag{16}$$

The left hand side of equation (16) is the norm of class-3 collection. By the definition, this means K is bounded with respect to the norms of class-3 collection. Since all types of boundedness are equivalent, K is bounded.

Generally, the definition of bounded set of class-m can be observed by just using some norms $\|\cdot\|^*_{i_1,\ldots,i_m}$ such that condition (10) applies. Moreover, the sufficient number of norms that we use to define bounded with respect to the norms of class-m collection is $\lceil \frac{n}{m} \rceil$ norms. One can see that these conditions match with the conditions used to investigate convergent sequences.

Next, we will present a concept of completeness in the n-normed spaces with respect to the norms of class-m collection, with $m \in \{1, \ldots, n\}$. Here we give some basic definitions related to completeness.

Definition 2.11. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. A sequence $\{x_k\} \subset X$ is called a* ***Cauchy sequence with respect to the norms of class-m collection*** *if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for every $k, l \geq N$, we have*

$$\|\overline{x_k - x_l}\|^*_{i_1,\ldots,i_m} < \epsilon,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. In other words

$$\lim_{k,l\to\infty} \|x_k - x_l\|^*_{i_1,\ldots,i_m} = 0,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$.

Theorem 2.12. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. If $\{x_k\}$ is convergent with respect to the norms of class-m collection, then $\{x_k\}$ is a Cauchy sequence with respect to the norms of class-m collection.*

Proof. For an $m \in \{1, \ldots, n\}$, let $\{x_k\}$ converge with respect to class-m collection to x. Then we have for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for every $k, l \geq N$ we have

$$\|\overline{x_k - x_l}\|^*_{i_1,\ldots,i_m} \leq \|\overline{x_k - x}\|^*_{i_1,\ldots,i_m} + \|\overline{x_l - x}\|^*_{i_1,\ldots,i_m} < \epsilon,$$

for every $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. This means $\{x_k\}$ is a Cauchy sequence. □

Moreover, all types of Cauchy sequence with respect to the norms of class-m collection for all $m \in \{1, \ldots, n\}$ are equivalent. We state it in the following theorem.

Theorem 2.13. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $m \in \{1, \ldots, n\}$. A sequence $\{x_k\} \subset X$ is Cauchy with respect to the norms of class-1 collection if and only if $\{x_k\}$ is Cauchy with respect to the norms of class-m collection.*

Proof. The proof is analogous with the proof of Theorem (2.3). □

Hereafter, we may simply use the word 'Cauchy' instead of 'Cauchy with respect to the norms of class-m collection'. We will mention the type if we need to specify it explicitly. Moreover, if every Cauchy sequence in X is convergent, then X is **complete**.

3. Remark and Conslusion

We have observed some topological characteristics of the n-normed spaces. We studied closed sets, bounded sets, convergence and Cauchy sequences. We also studied the completeness of the n-normed spaces. We reviewed these characteristics using norms of the quotient spaces we constructed from the n-normed spaces. Some researchers have used this approach. They used the norms of class-n collection to observe some characteristics of the n-normed spaces. Here, we provide more than one class collection that can be used to see the characteristics of the n-normed spaces. This means we use more general viewpoints. This allows us to present more general definitions, properties and theorems as part of the characteristics of the n-normed spaces. Using this viewpoint, we can investigate not only the topological characteristics, but also geometry and other aspects of the n-normed spaces.

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Analysis and Geometry Group, Faculty of Mathematics and Natural Sciences Bandung Institute of Technology. Jl. Ganesha 10, Bandung 40132
E-mail address: `h.batkunde@fmipa.unpatti.ac.id`

Analysis and Geometry Group, Faculty of Mathematics and Natural Sciences Bandung Institute of Technology. Jl. Ganesha 10 Bandung 40132
E-mail address: `hgunawan@math.itb.ac.id`