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BACKWARD BIFURCATION OF AN SIR-SI MODEL WITH VACCINATION AND TREATMENT

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Abstract. In the presence of treatment, most epidemic models demonstrate behavior of backward bifurcation. This is important in epidemiology because it provides significant information for disease control. However, most models consider only one single population. In this paper, an extended model of two populations in the form SIR-SI involving vaccination and treatment is analyzed. The analysis of local and global stability of equilibria is discussed. By using the center manifold theorem, this model has backward bifurcation behavior when the number of infected people exceeds the treatment capacity. Vaccination decreases the basic reproduction number, but does not affect the backward bifurcation behavior. This study also showed that under vaccination and treatment, an endemic equilibrium always occurs when $R_0 > 1$.

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1. Introduction

Infectious diseases are a major challenge to survival and threaten the health of hundreds of millions of people around the world. They killed more than 17 million people a year, including about 9 million deaths in young children [16]. Today, another serious problem is the development of resistance of some microbes and bacteria to antibiotics. This leads to longer duration of the disease, more testing and more costly medicine use [15].

Mathematical models are important tools for understanding the transmission mechanism of infectious diseases. This approach can provide advice for prevention and control strategies to reduce disease. The SIR compartment model of epidemic disease was first proposed by McKermack and McKendrick [20]. Subsequently, other researchers have developed the model by involving various factors (age, seasonality, treatment, vaccination), see for example [2, 3, 7, 12, 17, 19, 24, 25, 26, 28], and applied it to study the transmission of various infectious diseases, see for example [4, 8, 10, 14, 18, 21].

The development of the treatment function in the SIR model is done as follows. One of the early treatment functions was proposed in 1992 and is formulated as follows

$$T(I) = kI \tag{1}$$

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where k is a positive constant and I is the number of the infected population [1]. In [23], the treatment function is expressed by

$$T(I) = \begin{cases} 0, & I = 0 \\ k, & I > 0. \end{cases} \tag{2}$$

In [17, 24], the treatment function is proportional to the number of infected individuals when it is below the treatment capacity and constant when the number of infected individuals reaches the treatment capacity. The following treatment function was used

$$T(I) = \begin{cases} \gamma I, & 0 \leq I \leq I_0 \\ \gamma I_0, & I > I_0. \end{cases} \tag{3}$$

The results showed that the existence of a bistable endemic equilibrium and backward bifurcation occurs when I_0 is small [24]. A backward bifurcation SIR model by involving the saturated incidence rate, $\lambda SI/(1 + kI)$, and the saturated treatment function

$$T(I) = \frac{rI}{1 + \alpha I} \tag{4}$$

are discussed in [27]. The model showed that delays in treatment affect global dynamics behavior. Backward bifurcation occurs when the delay is long. The model from [27] was extended by adding an expose compartment [27]. It suggests that a diseases can be controlled by increasing the efficiency and the capacity of its treatment. In [7], the authors introduced quadratic treatment function

$$T(I) = \{rI - gI^2, 0\}. \tag{5}$$

This describes a case where health resources have limitations such that when the number of infected people increases and after treatment reaches its maximum, health resources continue to decline. In the model, treatment parameter r plays an important role in determining the dynamic behavior of the system. At certain r values, the system has four equilibrium points and a limit cycle.

Vaccination can strengthen the body's immune system, which is very important for preventing the spread of infectious diseases. It prevents the deaths of more than 1.5 million children every year [15]. In the literature, many studies on infectious disease epidemics and vaccination can be found, see for example [2, 12, 19, 26].

An SEIR epidemic model that takes vaccination and treatment into account is presented in [11]. The results showed the existence of backward bifurcation using the center manifold theorem and global stability of the endemic equilibrium point using a geometric approach. This model does not consider infectious diseases involving vectors. In reality, however, many infectious diseases involve vectors, such as dengue, chikungunya, zika, chagas and etc. To extend the SIR-SI model with vaccination and treatment in [11], we propose the addition of a vector as the carrier of the disease. Here, SIR represents the host dynamics and SI represents the vector dynamics. The treatment function was adopted from [17, 24]. The vaccine is given to healthy people and is considered perfect so that people who are vaccinated cannot get infected with the disease.

This paper is organized as follows. In the second section, we review SIR-SI model with vaccination and treatment. Then, we discuss the formulation of the basic reproduction number of this model. The fourth section discusses finding the equilibria and their local stability. The next section discusses the global stability of the disease free and endemic equilibria. The existence of backward bifurcation in this model is discussed in the sixth section. Furthermore, we present numerical results to support the analysis. The last section discusses all results and concludes this paper.

2. Model

Host-vector models have been applied in various studies on the transmission of infectious diseases, for example such as dengue, chikungunya, zika, and chagas [6, 8, 10, 14]. We adopted a basic host vector model in previous work [10], which studied dengue transmission. Here, we do not consider alternative hosts as a source of food for the vector. We will first explain this model briefly. Based on their health status, the host population is divided into three compartments, namely susceptible (S), infected (I) and recovery (R). Meanwhile, the vector is divided into two compartments, namely susceptible (S) and infectious (I). The susceptible compartment contains healthy individuals and the infected compartment contains sick individuals. The infection process occurs through the interaction between individuals in the infected compartment of the vector and individuals in the susceptible compartment of the host, or vice versa. Some susceptible individual will get perfect vaccination so they become immune to the disease. Consequently, the vaccination makes them move directly to the recovery compartment. The model was also complemented with treatment given to individuals in the infected compartment. Transmission of the disease in this model can be seen in Figure 1. The diagram can be written as the following differential equations

$$
\begin{aligned}
\dot{S}_h &= \mu_h N_h - \frac{\beta_h I_v S_h}{N_h} - (\mu_h + u) S_h \\
\dot{I}_h &= \frac{\beta_h I_v S_h}{N_h} - (\mu_h + \alpha_h) I_h - T(I_h) \\
\dot{R}_h &= \alpha_h I_h + T(I_h) - \mu_h R_h + u S_h \\
\dot{S}_v &= \mu_v N_v - \frac{\beta_v I_h S_v}{N_h} - \mu_v S_v \\
\dot{I}_v &= \frac{\beta_v I_h S_v}{N_h} - \mu_v I_v
\end{aligned}
\tag{6}
$$

Indeces h and v in the this system are related to the host and the vector. Let $S_h + I_h + R_h := N_h$ and $S_v + I_v := N_v$. By adding all the above equations, it is easy to verify that N_h and N_v are constant. Thus, System (6) has the following biologically feasible region

$$\Omega = \{(S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}^5_+ \mid S_h + I_h + R_h = N_h, S_v + I_v = N_v\}$$

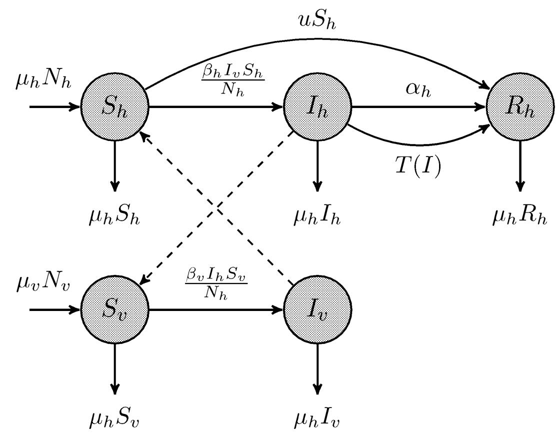

FIGURE 1. Disease transmission diagram

and parameters $N_h, N_v, \beta_h, \beta_v, \mu_h, \mu_v, \alpha_h, \gamma_h, u > 0$ with the following descriptions: N_h is the host population size, N_v is the vector population size, β_h is the human infection rate, β_v is the vector infection rate, μ_h is the recruitment/natural death rate of host per unit time, μ_v is the natural death rate of vector per unit time, α_h is the recovery rate per unit time, u represents the susceptible fraction that gets a vaccination per unit time, γ_h is the rate of treatment per unit of time. The function $T(I_h)$ is formulated as in (3). Thus, the term $\gamma_h I_0$ indicates the capacity of infected persons receiving treatment. This function also indicates that the treatment is proportional to the infected population. Since health resources are limited, if the infected population exceeds the treatment capacity, the treatment will be constant. In the next sections, we will analyze this system.

3. Basic Reproduction Number

It is well known that the basic reproduction number plays an important role in epidemiology. It describes whether the spread of a disease will continue or if the disease become extinct in the population. This parameter is defined as the number of secondary infections when an infected individual enters a virgin population (all individuals are healthy) during the period of the infection. When all individuals are healthy or there is no infection, this is represented by the disease-free equilibrium. Substituting $I_h = 0$ and $I_v = 0$ into System (6) will easily give this equilibrium as follows

$$E_0 = \left\{ \frac{\mu_h N_h}{\mu_h + u}, 0, \frac{u N_h}{(\mu_h + u)}, N_v, 0 \right\}.$$

Furthermore, we derive the formulation of this parameter using the next generation matrix; for more details see [22]. Now, we consider the case of $0 < I_h < I_0$. If $x = (I_h, I_v, S_h, S_v, R_h)$, System (6) can be rewritten in the following form

$$\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x) \tag{7}$$

where

$$\mathcal{F} = \begin{bmatrix} \frac{\beta_h I_v S_h}{N_h} \\ \frac{\beta_v I_h S_v}{N_h} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} (\mu_h + \alpha_h + \gamma_h) I_h \\ \mu_v I_v \\ -\mu_h N_h + \frac{\beta_h I_v S_h}{N_h} + (\mu_h + u) S_h \\ -\mu_v N_v + \frac{\beta_v I_h S_v}{N_h} + \mu_v S_v \\ (\alpha_h + \gamma_h) I_h + \mu_h R_h - u S_h \end{bmatrix}$$

Due to there are only two infected compartments (I_h and I_v), thus $m = 2$. Hence, evaluation of the Jacobian matrix $\mathcal{F}$ and $\mathcal{V}$ at E_0 are given by

$$F = \begin{bmatrix} 0 & \frac{\beta_h \mu_h}{(\mu_h + u)} \\ \frac{\beta_v N_v}{N_h} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} (\mu_h + \alpha_h + \gamma_h) & 0 \\ 0 & \mu_v \end{bmatrix}.$$

It follows

$$V^{-1} = \begin{bmatrix} \frac{1}{\mu_v} & 0 \\ 0 & \frac{1}{(\mu_h + \alpha_h + \gamma_h)} \end{bmatrix}, \quad FV^{-1} = \begin{bmatrix} 0 & \frac{\beta_h \mu_h}{(\mu_h + u)(\mu_h + \alpha_h + \gamma_h)} \\ \frac{\beta_v N_v}{N_h \mu_v} & 0 \end{bmatrix}.$$

Here, the basic reproduction number is denoted by $\bar{\mathcal{R}}_0$ and is formulated as the spectral radius of the next generation matrix. Hence, we have

$$\bar{\mathcal{R}}_0 = \rho(FV^{-1}) = \sqrt{\frac{\mu_h N_v \beta_h \beta_v}{\mu_v N_h (\mu_h + u)(\mu_h + \alpha_h + \gamma_h)}}. \tag{8}$$

Similarly, in the case of $I_h > I_0$, we will have the same results as (8), but $\gamma_h = 0$. It is clear, in this case, that the presence of vaccination can reduce the spread of the disease while treatment cannot. For the model analysis in the next section, we introduce a new parameter $R_0 := \bar{\mathcal{R}}_0^2$. This is equivalent to the basic reproduction number.

4. Analysis of Equilibria

In the case of $0 \leq I_h^* \leq I_0$, the endemic equilibria satisfy the following equations

$$\begin{cases} \mu_h N_h - \dfrac{\beta_h I_v^* S_h^*}{N_h} - (\mu_h + u) S_h^* = 0 \\ \dfrac{\beta_h I_v^* S_h^*}{N_h} - (\mu_h + \alpha_h + \gamma_h) I_h^* = 0 \\ (\alpha_h + \gamma_h) I_h^* - \mu_h R_h^* + u S_h^* = 0 \\ \mu_v N_v - \dfrac{\beta_v I_h^* S_v^*}{N_h} - \mu_v S_v^* = 0 \\ \dfrac{\beta_v I_h^* S_v^*}{N_h} - \mu_v I_v^* = 0. \end{cases} \tag{9}$$

By solving the first and last two equations in (9), we obtain

$$S_h^* = \frac{\mu_h N_h^2}{I_v^* \beta_h + N_h (u + \mu_h)} \text{ and } I_v^* = \frac{I_h^* N_v \beta_v}{\left(I_h^* \beta_v + \mu_v N_h\right)}. \tag{10}$$

Substituting (10) in the second equation of (7) yields

$$(11)\quad I_h^*(-\beta_v(\mu_h+\alpha_h+\gamma_h)(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h)I_h^* + \beta_h\beta_v\mu_h N_h\mu_v N_v-{\mu_v}^2N_h^2(\mu_h+u)(\mu_h+\alpha_h+\gamma_h))=0.$$

The following is the non-trivial solution of (11)

$$(12)\quad I_h^*=\frac{\mu_v^2N_h^2(\mu_h+u)(R_0-1)}{\beta_v(\mu_v N_v\beta_h+u\mu_h N_h+\mu_h\mu_v N_h)}.$$

This gives the endemic equilibrium: $E_1=\{S_h^*,I_h^*,R_h^*,S_v^*,I_v^*\}$, where

$$\begin{aligned}
I_v^* &= \frac{\mu_v N_h(\mu_h+u)(\mu_h+\alpha_h+\gamma_h)(R_0-1)}{\beta_h(\mu_h\beta_v+\mu_v(\alpha_h+\gamma_h+\mu_h))}\\
S_h^* &= \frac{\mu_v N_h^2(\mu_v(\alpha_h+N_h\gamma_h+N_h\mu_h)+\mu_h\beta_v)}{\beta_v(uN_h\mu_v+N_h\mu_h\mu_v+\beta_h\mu_v N_v)}\\
S_v^* &= \frac{(\mu_h+\alpha_h+\gamma_h)(N_h\mu_v(u+\mu_h)+\beta_h\mu_v N_v)}{\beta_h(\mu_v(\alpha_h+\gamma_h+\mu_h)+\beta_v\mu_h)}\\
R_h^* &= \frac{u\mu_v^2N_h^2((\mu_h+\alpha_h+\gamma_h)+\beta_v\mu_h N_h)+(\alpha_h+\gamma_h)\mu_v N_h(\mu_h+u)(R_0-1)}{\mu_h(N_h(\mu_h+u)\mu_v+\mu_v N_v\beta_h)\beta_v}.
\end{aligned}$$

From (12) and $I_h^*\le I_0$, the existence condition for E_1 is

$$(13)\quad 1<R_0<P_1=1+\frac{\beta_v(\mu_v N_v\beta_h+\mu_v N_h(u+\mu_h))I_0}{\mu_v^2N_h^2(\mu_h+u)}.$$

In the case of $I_h^*>I_0$, the endemic equilibria satisfy the following equations

$$(14)\quad \begin{cases}
\mu_h N_h-\dfrac{\beta_h I_v^*S_h^*}{N_h}-(\mu_h+u)S_h^*=0\\
\dfrac{\beta_h I_v^*S_h^*}{N_h}-(\mu_h+\alpha_h)I_h^*-\gamma_h I_0=0\\
\alpha_h I_h^*+\gamma_h I_0-\mu_h R_h^*+uS_h^*=0\\
\mu_v N_v-\dfrac{\beta_v I_h^*S_v^*}{N_h}-\mu_v S_v^*=0\\
\dfrac{\beta_v I_h^*S_v^*}{N_h}-\mu_v I_v^*=0.
\end{cases}$$

Some algebraic manipulations to (14) yield

$$(15)\quad a(I_h^*)^2+bI_h^*+c=0$$

where the coefficients are given by

$$\begin{aligned}
a &=\beta_v(\mu_h+\alpha_h)(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h)\\
b &=\beta_v\gamma_h(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h)I_0-{\mu_v}^2N_h^2(\mu_h+u)(\mu_h+\alpha_h)(R_0-1)\\
c &=I_0\gamma_h{\mu_v}^2N_h^2(\mu_h+u).
\end{aligned}$$

In the case of $I_0=0$, we obtain the explicit solution

$$I_h^*=\frac{(\mu_h+u){\mu_v}^2(R_0-1)}{\beta_v((\mu_h+u)\mu_v+\mu_v N_v\beta_h)}.$$

Equation (15) will has a real positive result if $b < 0$ and $\Delta = b^2 - 4ac > 0$. Note that $b < 0$ is equivalent to

(16) $$R_0 > P_2 = 1 + \frac{\beta_v \gamma_h \left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right) I_0}{{\mu_v}^2 N_h^2 \left(\mu_h + u\right)\left(\mu_h + \alpha_h\right)}$$

and $\Delta > 0$ is equivalent to

(17)
$$R_0 > P_3 = P_2 + 2\,\frac{\sqrt{I_0 \beta_v \gamma_h (\mu_h + u)\left(\mu_h + \alpha_h\right)\left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right)}}{\mu_v N_h \left(\mu_h + u\right)\left(\mu_h + \alpha_h\right)}.$$

If (17) is satisfied, then we have the two real solutions (15) below

$$I_{h1} = \frac{-b + \sqrt{\Delta}}{2\beta_v \left(\mu_h + \alpha_h\right)\left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right)}$$

and

$$I_{h2} = \frac{-b - \sqrt{\Delta}}{2\beta_v \left(\mu_h + \alpha_h\right)\left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right)}.$$

It follows that there are two endemic equilibria

$$E_{2i} = \{S_{hi}, I_{hi}, R_{hi}, S_{vi}, I_{vi}\}$$

where

$$S_{vi} = \frac{\mu_v N_v N_h}{I_{hi}\beta_v + \mu_v N_h}$$

$$I_{vi} = \frac{\mu_v N_v I_{hi} \beta_v}{\mu_v \left(I_{hi}\beta_v + \mu_v N_h\right)}$$

$$S_{hi} = \frac{\mu_h \mu_v N_h^2 \left(I_{hi}\beta_v + \mu_v\right)}{\mu_v N_v I_{hi} \beta_h \beta_v + \mu_v N_h \left(\mu_h + u\right)\left(I_{hi}\beta_v + \mu_v N_h\right)}$$

$$R_{hi} = \frac{u\mu_v N_h N_h \left(I_{hi}\beta_v + \mu_v\right)}{\left(\mu v N_h I_{hi} \beta_h \beta_v + \mu_v N_h \left(\mu_h + u\right)\left(I_{hi}\beta_v + \mu_v N_h\right)\right)} + \frac{\gamma_h I_0 + \alpha_h I_h}{\mu_h}$$

and $i = 1, 2$.

Since $I_h^* > I_0$, we will have

(18) $$b_1 + 2\beta_v \left(\mu_h + \alpha_h\right)\left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right) I_0 < 0.$$

This implies

(19) $$R_0 > P_4 = P_2 + \frac{2\beta_v \left(\mu_v N_v \beta_h + u\mu_v N_h + \mu_h \mu_v N_h\right) I_0}{{\mu_v}^2 N_h^2 \left(\mu_h + u\right)}.$$

These results are collected in the theorem below

Theorem 4.1. *Let*

$$
\begin{aligned}
P_1 &= 1+\frac{\beta_v\left(\mu_v N_v\beta_h+\mu_v N_h(u+\mu_h)\right)I_0}{\mu_v^2 N_h^2\left(\mu_h+u\right)+\mu_v N_v\beta_h}\\
P_2 &= 1+\frac{\beta_v\gamma_h\left(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h\right)I_0}{{\mu_v}^2 N_h^2\left(\mu_h+u\right)\left(\mu_h+\alpha_h\right)}\\
P_3 &= P_2+2\,\frac{\sqrt{I_0\beta_v\gamma_h(\mu_h+u)\left(\mu_h+\alpha_h\right)\left(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h\right)}}{\mu_v N_h\left(\mu_h+u\right)\left(\mu_h+\alpha_h\right)}\\
P_4 &= P_2+\frac{2\beta_v\left(\mu_v N_v\beta_h+u\mu_v N_h+\mu_h\mu_v N_h\right)I_0}{{\mu_v}^2 N_h^2\left(\mu_h+u\right)}
\end{aligned}
$$

then

(1) *System* (6) *always has disease-free equilibrium* E_0.
(2) *The endemic equilibrium* $E_1=\{S_h^*,I_h^*,R_h^*,S_v^*,I_v^*\}$ *exist if and only if* $1<R_0<P_1$.
(3) *The two endemic equilibria* $E_{2i}=\{S_{hi},I_{hi},R_{hi},S_{vi},I_{vi}\}$, *where* $i=1,2$, *exist if and only if* $R_0>P_3$ *and* $R_0>P_4$.

Now, we will use the theorem to derive the region of existence of endemic point E_{2i} in parameter space u and I_0 for the following special case

$$
\frac{\gamma_h\beta_h\left(\mu_h+u\right){\mu_v}^2}{I_0\left(\mu_h+\alpha_h\right)\left(u\mu_v N_h+N_h\mu_h\mu_v+\mu_v N_v\beta_h\right){\beta_v}^2}<1. \tag{20}
$$

This case implies $P_4>P_3$. If Theorem 4.1 for point (3) is satisfied when $u=0$, then $R_0>P_4>P_3$. For $u>0$, we obtain

$$
\frac{dR_0}{du}<0,\ \frac{dP_4}{du}<0,\ \lim_{u\to\infty}R_0=0,\ \lim_{u\to\infty}P_4=\frac{\beta_v I_0}{\mu_v N_h}\left(\frac{\gamma_h}{\mu_h+\alpha_h}+\frac{2}{\mu_h}\right).
$$

Hence, R_0 intersects P_4 at $u=u^*>0$. By solving $R_0=P_4$ for u, we obtain

$$
\begin{aligned}
u^* = {} & \frac{A\mu_v N_v\beta_h\beta_v-In_0\beta_v\left(\mu_v N_v\beta_h+\mu_h\mu_v\right){N_h}^2\gamma_h}{\mu_v N_h^2(\beta_v\left(2\,\alpha_h+\gamma_h+2\,\mu_h\right)I_0+\mu_v\left(\mu_h+\alpha_h\right))}\\
&+\frac{-N_h\left(\mu_h+\alpha_h\right)\left(2\,\beta_v\left(N_h\mu_h\mu_v+\mu_v N_v\beta_h\right)In_0+N_h\mu_h{\mu_v}^2\right)}{\mu_v N_h^2(\beta_v\left(2\,\alpha_h+\gamma_h+2\,\mu_h\right)I_0+\mu_v\left(\mu_h+\alpha_h\right))}.
\end{aligned} \tag{21}
$$

Furthermore, the signs of the first and second derivative (21) to I_0 are

$$
\frac{du^*}{dI_0}<0 \quad\text{and}\quad \frac{d^2u^*}{dI_0^2}>0.
$$

Hence, the curve u^* decreases and is concave up. Figure 2 illustrates this curve, where endemic equilibria E_{2i} exist in Region I.

In Figure 2, $\bar{u}$ and $\bar{I}_0$, respectively, are formulated as follows

$$
\begin{aligned}
\bar{u} &= \frac{\mu_h\mu_v N_v\beta_v\beta_h-\mu_h{\mu_v}^2 N_h\left(\mu_h+\alpha_h\right)}{\mu_v N_h\left(\alpha_h\mu_v+\mu_h\mu_v\right)}\\
\bar{I}_0 &= \frac{\mu_h\mu_v N_h N_v\beta_v\beta_h-\mu_h{\mu_v}^2 N_h^2\left(\mu_h+\alpha_h\right)}{\gamma_h\beta_v\left(\mu_v N_v\beta_h N_h+\mu_h\mu_v\right)+2\,N_h\beta_v\left(\mu_h+\alpha_h\right)\left(N_h\mu_h\mu_v+\mu_v N_v\beta_h\right)}.
\end{aligned}
$$

Now, we will discuss the stability of the equilibrium E_0 in the following theorem.

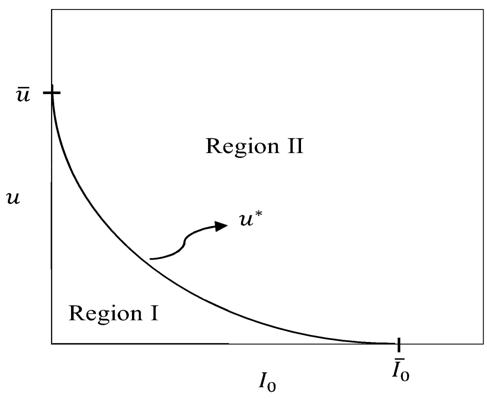

FIGURE 2. The existence of endemic equilibria E_{2i} in parameter space u and I_0

Theorem 4.2. *If $R_0 < 1$, then E_0 is locally asymptotically stable and if $R_0 > 1$, then E_0 is unstable and E_1 is locally asymptotically stable.*

Proof. Calculating the Jacobian matrix at E_0 gives

$$J(E_0) = \begin{bmatrix} -u-\mu_h & 0 & 0 & 0 & -\frac{\beta_h\mu_h}{(\mu_h)+u} \\ 0 & -\mu_h-\alpha_h-\gamma_h & 0 & 0 & \frac{\beta_h\mu_h}{(\mu_h+u)} \\ u & \alpha_h+\gamma_h & -\mu_h & 0 & 0 \\ 0 & -\frac{N_v\beta_v}{N_h} & 0 & -\mu_v & 0 \\ 0 & \frac{N_v\beta_v}{N_h} & 0 & 0 & -\mu_v \end{bmatrix} \tag{22}$$

where the characteristic of the polynomial (22) is given by

$$P_0(\lambda) = (\lambda+\mu_h+u)(\lambda+\mu_v)(\lambda+\mu_h) \\ (\lambda^2 + (\mu_h+\alpha_h+\gamma_h+\mu_v)\,\lambda \\ \mu_v\,(\mu_h+\alpha_h+\gamma_h)\,(1-R_0)) \tag{23}$$

and the eigenvalues of $J(E_0)$ are $-\mu_h, -\mu_v, -\mu_h - u$, and the roots of the quadratic equation. Clearly, if $R_0 < 1$, then the real part of the roots is negative, which implies that E_0 is locally asymptotically stable.

In order to prove the stability of endemic E_1, we compute the Jacobian matrix at the endemic point as follows

$$J(E_1) = \begin{bmatrix} -I_v^*\beta_h-u-\mu_h & 0 & 0 & 0 & -\beta_h S_h^* \\ I_v^*\beta_h & -\mu_h-\alpha_h-\gamma_h & 0 & 0 & \beta_h S_h^* \\ u & \alpha_h+\gamma_h & -\mu_h & 0 & 0 \\ 0 & -S_v^*\beta_v & 0 & -I_h^*\beta_v-\mu_v & 0 \\ 0 & S_v^*\beta_v & 0 & I_h^*\beta_v & -\mu_v \end{bmatrix} \tag{24}$$

which implies the characteristic polynomial below

$$P_1(\lambda) = (\lambda+\mu_h)(\lambda+\mu_v)(a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0). \tag{25}$$

Clearly, the eigenvalues are $-\mu_h, -\mu_v$, and the roots of the following cubic polynomial

$$(a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0) = 0$$

where

$$\begin{aligned}
a_3 =& 1\\
a_2 =& \frac{N_h\mu_v\left(\mu_h+u\right)\left(R_0-1\right)}{uN_h+N_h\mu_h+N_v\beta_h} + \frac{\mu_v\left(\mu_h+u\right)\left(\mu_h+\alpha_h+\gamma_h\right)\left(R_0-1\right)}{\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)}\\
&+u+\alpha_h+\gamma_h+2\,\mu_h+\mu_v\\
a_1 =& \frac{N_h{\mu_v}^2\left(\mu_h+u\right)^2\left(\mu_h+\alpha_h+\gamma_h\right)\left(R_0-1\right)^2}{\left(\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)\right)\left(uN_h+N_h\mu_h+N_v\beta_h\right)}\\
&+\frac{\mu_vN_h\left(u+\alpha_h+\gamma_h+2\,\mu_h\right)\left(R_0-1\right)\left(\mu_h+u\right)}{uN_h+N_h\mu_h+N_v\beta_h}\\
&+\frac{\mu_v\left(\mu_h+\alpha_h+\gamma_h+\mu_v\right)\left(\mu_h+u\right)\left(\mu_h+\alpha_h+\gamma_h\right)\left(R_0-1\right)}{\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)}\\
&+\left(\mu_h+u\right)\left(\mu_h+\alpha_h+\gamma_h+\mu_v\right)\\
a_0 =& \frac{\left(\mu_h+\alpha_h+\gamma_h\right)^2N_h{\mu_v}^2\left(\mu_h+u\right)^2\left(R_0-1\right)^2}{\left(\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)\right)\left(uN_h+N_h\mu_h+N_v\beta_h\right)}\\
&+\frac{{\mu_v}^2\left(\mu_h+\alpha_h+\gamma_h\right)^2\left(\mu_h+u\right)\left(R_0-1\right)}{\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)}\\
&+\frac{\mu_vN_h\left(\mu_h+u\right)^2\left(\mu_h+\alpha_h+\gamma_h\right)\left(R_0-1\right)}{uN_h+N_h\mu_h+N_v\beta_h}.
\end{aligned}$$

If $R_0 > 1$, then $a_2, a_1, a_0 > 0$. After some algebraic manipulations, we obtain

$$\begin{aligned}
a_2a_1 >& \frac{{\mu_v}^2N_h\left(2\mu_h+\alpha_h+\gamma_h+\mu_v+u\right)\left(\mu_h+\alpha_h+\gamma_h\right)\left(\mu_h+u\right)^2\left(R_0-1\right)^2}{\left(\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)\right)\left(uN_h+N_h\mu_h+N_v\beta_h\right)}\\
&+\frac{\mu_v(\mu_v+2\mu_h+\alpha_h+\gamma_h+u)\left(\mu_h+\alpha_h+\gamma_h\right)^2\left(\mu_h+u\right)\left(R_0-1\right)}{\mu_h\beta_v+\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)}\\
&+\frac{\left(\mu_v\mu_h+u\right)^2\left(\mu_h+\alpha_h+\gamma_h+\mu_v\right)\left(R_0-1\right)}{u+\mu_h+\beta_h}\\
>& \frac{\left(\mu_h+\alpha_h+\gamma_h\right)^2{N_h}^2{\mu_v}^3\left(\mu_h+u\right)^2\left(R_0-1\right)^2}{\left(\mu_hN_h\beta_v+N_h\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)\right)\left(uN_h\mu_v+N_h\mu_h\mu_v+\mu_vN_v\beta_h\right)}\\
&+\frac{N_h{\mu_v}^2\left(\mu_h+\alpha_h+\gamma_h\right)^2\left(\mu_h+u\right)\left(R_0-1\right)}{\mu_hN_h\beta_v+N_h\mu_v\left(\mu_h+\alpha_h+\gamma_h\right)}\\
&+\frac{\mu_vN_h\left(\mu_h+u\right)^2\left(\mu_h+\alpha_h+\gamma_h\right)\left(R_0-1\right)}{uN_h+N_h\mu_h+N_v\beta_h}\\
=& a_0.
\end{aligned}$$

According to the Routh-Hurwitz criterion, we conclude that the real part of the roots is negative. Thus, all eigenvalues are negative. This implies that if $R_0 > 1$, then E_1 is locally asymptotically stable. □

5. Global Stability

In order to analyze the global stability of E_0, we use the proposed theorem by Castillo-Chavez *et al.*, 2002 [4]. Consider the epidemic compartment model below

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) \tag{26}$$

where $\mathbf{x} \in \mathbf{R}^{m+n}$. It can be written in the following form

$$\begin{aligned} \frac{d\mathbf{y}}{dt} &= F(\mathbf{y}, \mathbf{I}) \\ \frac{d\mathbf{I}}{dt} &= G(\mathbf{y}, \mathbf{I}), \quad G(\mathbf{I}, 0) = 0 \end{aligned} \tag{27}$$

where $\mathbf{y} \in \mathbf{R}^m$ represents the non-infected compartments, $\mathbf{I} \in \mathbf{R}^n$ represents the infected compartments. Let $\mathbf{U}_0 = \{\mathbf{y}^*, 0\}$ is the disease-free equilibrium. If $\mathbf{U}_0$ is locally asymptotically stable and and System (26) meets the following conditions

(H1) For $\frac{d\mathbf{y}}{dt} = F(\mathbf{y}^*, 0)$, $\mathbf{y}^*$ is globally asymptotically stable

(H2) $G(\mathbf{y}, \mathbf{I}) = \mathbf{AI} - \hat{G}(\mathbf{y}, \mathbf{I})$, $\hat{G}(\mathbf{y}, \mathbf{I}) \geq 0$ for $(\mathbf{y}, \mathbf{I}) \in \Omega$

where Ω is the region where the model form the biological region, then the following theorem applies.

Theorem 5.1. *The fixed point* $\mathbf{U}_0(\mathbf{x}^*, 0)$ *is a globally asymptotically stable equilibrium of System* (26) *provided that* $R_0 < 1$ *and that assumptions (H1) and (H2) are satisfied.*

In the case of System (6), we have $\mathbf{y} = (S_h, R_h, S_v), \mathbf{I} = (I_h, I_v), \mathbf{U}_0 = E_0$ and

$$F(\mathbf{y}, 0) = \begin{pmatrix} \mu_h N_h - \mu_h S_h \\ -\mu_h R_h + u S_h \\ \mu_v N_v - \mu_v S_v \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -(\mu_h + \alpha_h + \gamma_h) & \beta_h \\ \beta_v & -\mu_v \end{pmatrix},$$

and

$$G(\mathbf{y}, \mathbf{I}) = \begin{pmatrix} \beta_h I_v \left(1 - \frac{S_h}{N_h}\right) \\ \beta_v I_h \left(1 - \frac{S_v}{N_h}\right) \end{pmatrix}.$$

In the case of $N_h > N_v$, inequality $0 \leq \frac{S_h}{N_h}, \frac{S_v}{N_h} \leq 1$ is satisfied, we conclude that $G(\mathbf{y}, \mathbf{I}) \geq 0$. Furthermore, it is easy to show that

$$\mathbf{y}^* = \{\frac{\mu_h N_h}{\mu_h + u}, \frac{u N_h}{(\mu_h + u)}, N_v\}$$

is globally asymptotically stable of $\frac{dF}{dt} = F(\mathbf{y}, 0)$. Then, according to theorem 5.1, thus if $R_0 < 1$, then $\mathbf{U}_0$ is globally asymptotically stable and if $R_0 > 1$, then $\mathbf{U}_0$ is unstable. We summarize the above results in the following theorem.

Theorem 5.2. *If* $R_0 < 1$ *and* $N_h > N_v$, *then* E_0 *is globally asymptotically stable.*

In order to determine the global stability of E_1, we define the following Lyapunov function

$$
\begin{aligned}
(28)\quad V = c_1 &\left(S - S^* - S^* \ln \frac{S}{S^*}\right) \\
&+ c_1 \left(I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*}\right) + \left(R_h - R_h^* - R_h^* \ln \frac{R_h}{R_h^*}\right) \\
&\quad + c_2 \left(S_v - S_v^* - S_v^* \ln \frac{S_v}{S_v^*}\right) + c_2 \left(I_v - I_v^* - I_v^* \ln \frac{I_v}{I_v^*}\right)
\end{aligned}
$$

where $c_1 = \dfrac{\beta_v I_h^* S_v^*}{N_h}$ and $c_2 = \dfrac{\beta_h I_h^* S_h^*}{N_h}$.

Derivative V with respect to time is a long solution of System (6)

$$
\begin{aligned}
\dot{V} \;=\; & \frac{\beta_v I_h^* S_v^*}{N_h}\left(1 - \frac{S_h^*}{S_h}\right)\left(\mu_h N_h - \frac{\beta_h I_v S_h}{N_h} - (\mu_h + u)\, S_h\right) \\
& + \frac{\beta_v I_h^* S_v^*}{N_h}\left(1 - \frac{I_h^*}{I_h}\right)\left(\frac{\beta_h I_v S_h}{N_h} - (\mu_h + \alpha_h + \gamma_h)\, I_h\right) \\
& + \left(1 - \frac{R_h^*}{R_h}\right)\left((\alpha_h + \gamma_h)\, I_h - \mu_h R_h\right) \\
& + \frac{\beta_h I_v^* S_h^*}{N_h}\left(1 - \frac{S_v^*}{S_v}\right)\left(\mu_v N_v - \frac{\beta_v I_h S_v}{N_h} - \mu_v S_v\right) \\
& + \frac{\beta_h I_v^* S_h^*}{N_h}\left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v I_h S_v}{N_h} - \mu_v I_v\right).
\end{aligned}
$$

By using the following equilibrium condition

$$
\begin{aligned}
&\mu_h N_h = \frac{\beta_h I_v^* S_h^*}{N_h} + (\mu_h + u)\, S_h^*, \quad \frac{\beta_h I_v^* S_h^*}{N_h} = (\mu_h + \alpha_h + \gamma_h)\, I_h^*, \\
&(\alpha_h + \gamma_h) I_h^* = \mu_h R_h^*,\ \mu_v N_v = \frac{\beta_v I_h^* S_v^*}{N_h} + \mu_v S_v^*,\ \frac{\beta_v I_h^* S_v^*}{N_h} = \mu_v I_v^*,
\end{aligned}
$$

we will have

$$
\begin{aligned}
\dot{V} \;=\; & -\frac{(\mu_h + u)\beta_v I_h^* S_v^* (S_h - S_h^*)^2}{S_h N_h} - \frac{\mu_h}{R_h}(R_h - R_h^*)^2 \\
& - \frac{\mu_v \beta_h I_v^* S_h^*}{S_v N_h}(S_v - S_v^*)^2 + \frac{\beta_h \beta_v I_h^* I_v^* S_h^* S_v^*}{N_h^2}\left(1 - \frac{S_h^*}{S_h}\right) \\
& + \frac{\beta_h \beta_v I_h^* I_v^* S_h^* S_v^*}{N_h^2}\left(1 - \frac{I_h}{I_h^*}\right) + \frac{\beta_h \beta_v I_h^* I_v^* S_h^* S_v^*}{N_h^2}\left(1 - \frac{S_v^*}{S_v}\right) \\
& - \frac{\beta_h \beta_v I_h^* I_v S_h S_v^*}{N_h^2}\frac{I_h^*}{I_h} - \frac{\beta_h \beta_v I_h I_v^* S_h^* S_v}{N_h^2}\frac{I_v^*}{I_v} + \frac{\beta_h \beta_v I_h^* I_v^* S_h^* S_v^*}{N_h^2} \\
=\; & \frac{-(\mu_h + u)\beta_v I_h^* S_v^* (S_h - S_h^*)^2}{S_h N_h} - \frac{\mu_h}{R}(R - R^*)^2 - \frac{\mu_v \beta_h I_v^* S_h^*}{S_v N_h}(S_v - S_v^*)^2 \\
& + \frac{\beta_h \beta_v I_h^* I_v^* S_h^* S_v^*}{N_h^2}\left(5 - \frac{S_h^*}{S_h} - \frac{I_h^*}{I_h} - \frac{I_v S_h I_h^*}{I_v^* S_h^* I_h} - \frac{I_h S_v I_v^*}{I_h^* S_v^* I_v} - \frac{S_h^*}{S_h}\right).
\end{aligned}
$$

By introducing function $g(x) = 1 - x + \ln x$, we obtain

$$5 - \frac{S_h^*}{S_h} - \frac{I_h^*}{I_h} - \frac{I_v S_h I_h^*}{I_v^* S_h^* I_h} - \frac{I_h S_v I_v^*}{I_h^* S_v^* I_v} - \frac{S_v^*}{S_v} = g\left(\frac{S_h^*}{S_h}\right) + g\left(\frac{I_h^*}{I_h}\right) + g\left(\frac{I_v S_h I_h^*}{I_v^* S_h^* I_h}\right) + g\left(\frac{I_h S_v I_v^*}{I_h^* S_v^* I_v}\right) + g\left(\frac{S_v^*}{S_v}\right). \tag{29}$$

Because $g(x)$ is monotone decreasing for all $x > 0$, then $g(x) \leq 0$ for all $x > 0$, hence we have $\dot{V} \leq 0$. $\dot{V} = 0$, if only if $S_h = S_h^*, I_h = I_h^*, I_v = I_v^*$. We conclude that the largest compact invariant set in $\{(S_h, I_h, I_v) \in \Omega \mid \dot{V} = 0\}$ is the singleton E_1. The LaSalle-Lyapunov theorem [13] implies that E_1 is globally asymptotically stable in Ω. The results of this analysis are stated in the following theorem.

Theorem 5.3. *If $R_0 > 1$, then E_1 is globally asymptotically stable.*

6. Backward Bifurcation

Before we show the existence of backward bifurcation, we will briefly review the application of the center manifold theorem developed by Castillo-Chavez and Song, 2004 [5]. Consider the following general system of ordinary differential equations with a parameter θ

$$\frac{dx(t)}{dt} = f(x, \theta) \tag{30}$$

where function $f(x, \theta) : R^n \times R \to R^n$ with $f \in C^2$. Assume that $x = 0$ is an equilibrium of System (6), that is, $f(0, \theta) = 0$ for all θ. Let

$$Q = \left(\frac{\partial f_i}{\partial x_j}(0, 0)\right)$$

be the Jacobian matrix of $f(x, \theta)$ at $(0, 0)$.

Lemma 6.1. *Assume that*
(H3) Zero is a simple eigenvalue of Q and all other eigenvalues have negative real part;
(H4) Q has (non-negative) right vector $\mathbf{w} = (\omega_1, \omega_2, ..., \omega_n)^T$ and left eigenvector $\mathbf{v} = (v_1, v_2, ..., v_n)^T$ corresponding to zero eigenvalue.
Let $f_k(x, \theta)$ denote the $k-th$ component of $f(x, \theta)$ and

$$\mathbf{a} = \sum_{k,i,j=1}^{n} v_k \omega_i \omega_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0) \quad \mathbf{b} = \sum_{k,i}^{n} v_k \omega_i \frac{\partial^2 f_k}{\partial x_i \partial \theta}(0, 0)$$

then, the local dynamics of system (6) around $x = 0$ are totally determined by the signs of $\mathbf{a}$ and $\mathbf{b}$.
(1) If $\mathbf{a} > 0$ and $\mathbf{b} > 0$, then when $\theta < 0$ with $|\theta| \ll 1$, $x = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium, and when $0 < \theta \ll 1$, $x = 0$ is unstable and there exists a negative unstable equilibrium
(2) If $\mathbf{a} < 0$ and $\mathbf{b} > 0$, then when θ changes from negative to positive, $x = 0$ changes its stability from stable to unstable.Correspondingly, a negative unstable equilibrium become positive and locally asymptotically.
Particularly, if $\mathbf{a} < 0$ and $\mathbf{b} > 0$, then a forward bifurcation occurs $\theta = 0$, and if $\mathbf{a} > 0$ and $\mathbf{b} > 0$, a backward bifurcation occurs at $\theta = 0$.

Now, we will examine the existence of the backward bifurcation in System (6) with $T(I_h) = \gamma_h I_0$. Let $\theta = \beta_v$ is the bifurcation parameter and $x = (x_1, x_2, x_3, x_4, x_5)$, where $x_1 = S_h, x_2 = I_h, x_3 = R_h, x_4 = S_v, x_5 = I_v$. We can rewrite System (6) in the following form

$$\begin{aligned}
\dot{x}_1 =& \mu_h N_h - \frac{\beta_h x_5 x_1}{N_h} - (\mu_h + u)\, x_1 := f_1 \\
\dot{x}_2 =& \frac{\beta_h x_5 x_1}{N_h} - (\mu_h + \alpha_h)\, x_2 - T(x_2) := f_2 \\
\dot{x}_3 =& \alpha_h x_2 + T(x_2) - \mu_h x_3 + u x_1 := f_3 \\
\dot{x}_4 =& \mu_v N_v - \frac{\beta_v x_2 x_4}{N_h} - \mu_v x_4 := f_4 \\
\dot{x}_5 =& \frac{\beta_v x_2 x_4}{N_h} - \mu_v x_5 := f_5.
\end{aligned} \tag{31}$$

$R_0 = 1$ is equivalent to

$$\beta_v^* = \frac{\mu_v^2(\mu_h + \alpha_h)(\mu_h + u)}{\mu_h N_h B \beta_h}.$$

The Jacobian matrix at E_0 when $R_0 = 1$ is given by

$$J(E_0) = \begin{bmatrix} -u - \mu_h & 0 & 0 & 0 & -\frac{\beta_h \mu_h}{(\mu_h + u)} \\ 0 & -\mu_h - \alpha_h & 0 & 0 & \frac{\beta_h \mu_h}{(\mu_h + u)} \\ u & \alpha_h & -\mu_h & 0 & 0 \\ 0 & -\frac{N_v \beta_v^*}{N_h} & 0 & -\mu_v & 0 \\ 0 & \frac{N_v \beta_v^*}{N_h} & 0 & 0 & -\mu_v \end{bmatrix} \tag{32}$$

with the characteristic polynomial

$$P_0(\lambda) = \lambda(\lambda + \mu_h + u)(\lambda + \mu_v)(\lambda + \mu_h)(\lambda + (\mu_h + \alpha_h + \mu_v)).$$

The eigenvalues are $0, -\mu_h, -\mu_v, -\mu_h - u, -\mu_h - \alpha_h - \gamma_h - \mu_v$. One of the eigenvalues is zero and the others are negative. So, the condition *H3* in the lemma above is satisfied. In order to satisfy the condition *H4*, we will determine the right eigenvector (w) and the left eigenvector (v) corresponding to zero eigenvalue. The right eigenvector $w = (w_1, w_2, w_3, w_4, w_5)^T$ satisfies

$$\begin{aligned}
-(\mu_h + u) w_1 - \frac{\beta_h \mu_h}{(\mu_h + u)} w_5 &= 0 \\
-(\mu_h + \alpha_h) w_2 + \frac{\beta_h \mu_h}{(\mu_h + u)} w_5 &= 0 \\
u w_1 + \alpha_h w_2 - \mu_h w_3 &= 0 \\
\frac{-N_v \beta_v^*}{N_h} w_2 - \mu_v w_4 &= 0 \\
\frac{N_v \beta_v^*}{N_h} w_2 - \mu_v w_5 &= 0
\end{aligned}$$

is

$$w = \begin{bmatrix} \mu_h + \alpha_h \\ -u - \mu_h \\ u - \alpha_h \\ \dfrac{\left(u\alpha_h + u\mu_h + \alpha_h\mu_h + {\mu_h}^2\right)(\mu_h + u)}{\mu_h\beta_h} \\ -\dfrac{\left(u\alpha_h + u\mu_h + \alpha_h\mu_h + {\mu_h}^2\right)(\mu_h + u)}{\mu_h\beta_h} \end{bmatrix}. \tag{33}$$

Next, we find the left eigenvector $v = (v_1, v_2, v_3, v_4, v_5)^T$, which is satisfies

$$\begin{aligned} (-u - \mu_h)\, v_1 + u v_3 &= 0 \\ v_2\, (-\mu_h - \alpha_h) + v_3\, \alpha_h - v_4 N_v \beta_v + v_5 N_v \beta_v &= 0 \\ -\mu_v\, v_3 &= 0 \\ -\mu_v\, v_4 &= 0 \\ -\frac{\beta_h \mu_h N_h v_1}{\mu_h + u} + \frac{\beta_h \mu_h N_h v_2}{\mu_h + u} - \mu_v v_5 &= 0 \end{aligned} \tag{34}$$

and $v{\cdot}w = 1$. We obtain $v = (0, v_2, 0, 0, v_5)^T$, where

$$v_2 = -\frac{\mu_v}{(\mu_h + u)(\alpha_h + \mu_h + \mu_v)}, \qquad v_5 = -\frac{\mu_h\beta_h}{(\mu_h + u)^2(\alpha_h + \mu_h + \mu_v)}.$$

Calculating the partial derivative at E_0 gives

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x_1 \partial x_5} &= \frac{\partial^2 f_1}{\partial x_5 \partial x_1} = -\frac{\beta_h}{N_h}, & \frac{\partial^2 f_2}{\partial x_1 \partial x_5} &= \frac{\partial^2 f_2}{\partial x_5 \partial x_1} = \frac{\beta_h}{N_h} \\ \frac{\partial^2 f_4}{\partial x_2 \partial x_4} &= \frac{\partial^2 f_4}{\partial x_4 \partial x_2} = -\frac{\beta_v}{N_h}, & \frac{\partial^2 f_5}{\partial x_2 \partial x_4} &= \frac{\partial^2 f_5}{\partial x_4 \partial x_2} = \frac{\beta_v}{N_h} \\ \frac{\partial^2 f_4}{\partial x_2 \partial \beta_v} &= \frac{\partial^2 f_4}{\partial \beta_v \partial x_2} = -\frac{N_v}{N_h}, & \frac{\partial^2 f_5}{\partial x_2 \partial \beta_v} &= \frac{\partial^2 f_5}{\partial \beta_v \partial x_2} = \frac{N_v}{N_h} \end{aligned}$$

and all the other second-order partial derivatives are equal to zero. The formulas for coefficients **a** and **b** in the above lemma are

$$\mathbf{a} = \sum_{k,i,j=1}^{n} v_k \omega_i \omega_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_0, \beta_v^*), \quad \mathbf{b} = \sum_{k,i}^{n} v_k \omega_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_v}(E_0, \beta_v^*).$$

For System (31), we obtain

$$\begin{aligned} \mathbf{a} &= 2v_1 w_1 w_5 \frac{\partial^2 f_1}{\partial x_1 \partial x_5} + 2v_2 w_1 w_5 \frac{\partial^2 f_2}{\partial x_1 \partial x_5} + 2v_4 w_2 w_4 \frac{\partial^2 f_4}{\partial x_2 \partial x_4} + 2v_5 w_2 w_4 \frac{\partial^2 f_5}{\partial x_2 \partial x_4} \\ &= \frac{2(\mu_h + u)(\mu_h + \alpha_h)(\mu_h\beta_v N_h + \alpha_h\mu_v + \mu_h\mu_v)}{\mu_h(\alpha_h + \mu_h + \mu_v)} \\ \mathbf{b} &= 2v_4 w_2 \frac{\partial^2 f_4}{\partial x_2 \partial \beta_v} + 2v_5 w_2 \frac{\partial^2 f_5}{\partial x_2 \partial \beta_v} \\ &= \frac{2\mu_h\mu_v N_v \beta_h}{\mu_v(\mu_h + u)(\mu_h + \alpha_h + \mu_v)}. \end{aligned}$$

Obviously, for any arbitrary parameter value, we have $\mathbf{a}, \mathbf{b} > 0$. It follows that $\beta_v = \beta_v^*$ is the backward bifurcation point. In the same way as above for System (6) with $T(I_h) = \gamma_h I_h$, we get the coefficients as follows

$$\mathbf{a} = -\frac{2\left(\mu_h + u\right)\left(\mu_h + \alpha_h + \gamma_h\right)\left(\mu_h \beta_v N_h + \mu_v \alpha_h + \mu_v \gamma_h + \mu_h \mu_v\right)}{\mu_h\left(\alpha_h + \gamma_h + \mu_h + \mu_v\right)}$$

$$\mathbf{b} = \frac{2\mu_h \mu_v N_v \beta_h}{\mu_v\left(\mu_h + u\right)\left(\mu_h + \alpha_h + \mu_v + \gamma_h\right)}.$$

Since $\mathbf{a} < 0$ and $\mathbf{b} > 0$, then forward bifurcation occurs when $R_0 = 1$. The following theorem summarizes the results of the bifurcation analysis.

Theorem 6.2. *If $T(I_h) = \gamma_h I_0$, then the direction of bifurcation system* (6) *is backward at $R_0 = 1$, and if $T(I_h) = \gamma_h I_h$, then the direction of bifurcation system* (6) *is forward at $R_0 = 1$.*

This theorem shows that the system has backward bifurcation behavior under the presence of treatment. The behavior does not depend on vaccination parameter u.

Now, we present some numerical simulation performances to provide an illustration and support the results of the above analysis, as shown in Figure 3. The solid line indicates that the equilibrium is stable, while the dashed line indicates that the equilibrium is unstable. Forward bifurcation is demonstrated in Figure 3 (a) and (b) where the system is under treatment $T(I_h) = \gamma_h I_h$. Figure 3 (c) and (d) show backward bifurcation where the system is under treatment $T(I_h) = \gamma_h I_0$. In addition, we also present numerical simulations to investigate the effect of vaccination and treatment on the dynamic of the infected population in equilibrium condition. Here, we varied, the values of u and I_0. The results can be seen in Figure 4. It shows that an increase in vaccination caused a decrease in the infected population, see Figure 4 (a). Furthermore, when the treatment parameter I_0 increased, the infected population also decreased, but it was accompanied by a greater basic reproduction number, as we can see in Figure 4 (b). It follows that more treatment leads to a larger basic reproduction number for existence of an endemic equilibrium.

7. Discussion and Conclusion

In this work, we have analyzed an SIR-SI model with vaccination and treatment to study the transmission of an infectious disease in the host and the vector population. The treatment in this model reflects that there are limited resources for treatment of the infected population. Preventive measures in the form of vaccinations are also considered in this model. This vaccination allows people to become immune to the disease. For $T(I_h) = \gamma_h I_h$, we got two equilibria, respectively, representing the disease-free and the endemic equilibrium. Their stability depends on the basic reproduction number. If $R_0 < 1$, then the disease free is only equilibrium in System (6) and it is locally asymptotically stable. If $R_0 > 1$, then the endemic equilibrium appears and we showed by using the Lyapunov function that it is globally asymptotically stable. Furthermore, we showed analytically using the center manifold theorem that the forward bifurcation always occurs at $R_0 = 1$. For

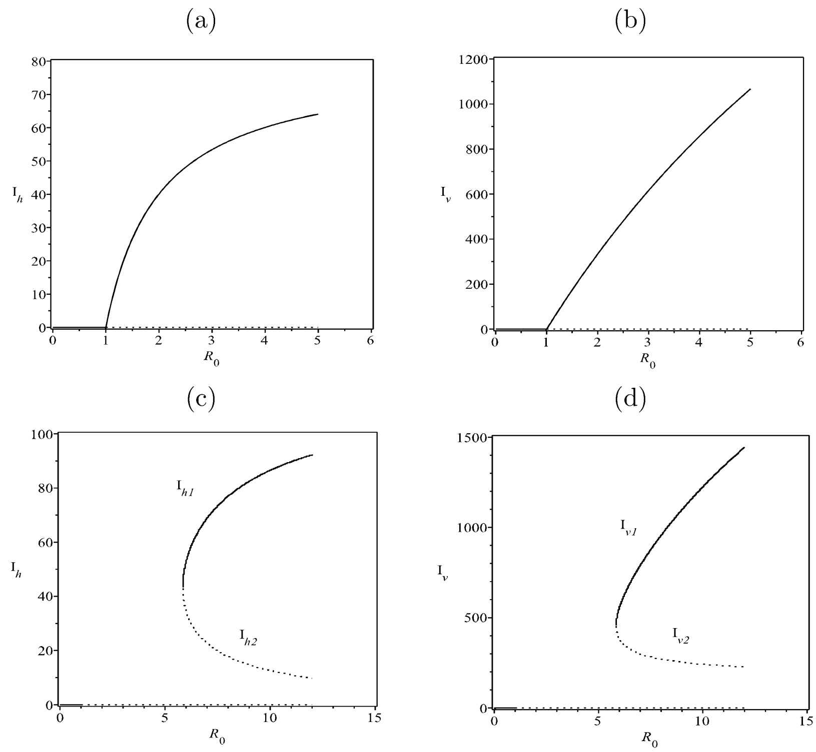

FIGURE 3. Forward and Backward bifurcation diagram. In (a) and (b), forward bifurcation diagram for $T(I_h) = \gamma_h I_h$ is described in plane I_h vs R_0 and I_v vs R_0. In (c) and (d), backward bifurcation diagram for $T(I_h) = \gamma_h I_0$ is described in plane I_h vs R_0 and I_v vs R_0. The parameters used are $\alpha_h = 0.1, \gamma_h = 0.125, \mu_h = 0.002, \mu_v^{-1} = 20, I_0 = 50, \beta_h = 0.3, N_h = 10000, N_v = 4000$.

$T(I) = \gamma_h I_0$, System (6) has two endemic equilibria. Analytically, this system always shows backward bifurcation at $R_0 = 1$. We showed numerically that endemic equilibria become more difficult to exist because of treatment. This study demonstrated that vaccination and adequate treatment lead to better disease control.

In the future, we will develop an extended model by taking into account seasonality. This is an important factor in epidemiology since in the reality, some epidemics of infectious diseases are strongly related to the weather, for example vector born diseases with mosquito as vector, such as dengue, chikungunya, zika, etc.

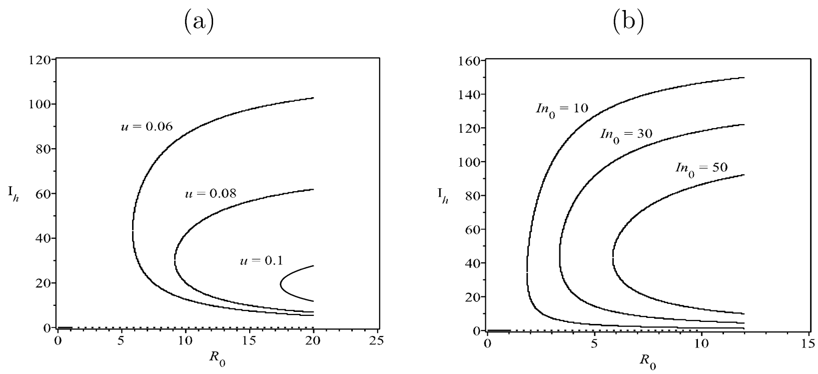

FIGURE 4. Backward bifurcation diagram when u and I_0 vary. In (a), the vaccination parameter u varies at 0.06, 0.08, and 0.1. In (b), the treatment parameter I_0 varies at a value of 10, 30, and 50. The other parameters are the same in the previous simulation

REFERENCES

[1] R. M. Anderson and R. M. May, *Infectious diseases of humans: dynamics and control*, Oxford University Press, 1992.

[2] J. Arino, C. C. McCluskey, and P. van den Driessche, "*Global results for an epidemic model with vaccination that exhibits backward bifurcation*", SIAM J. Appl. Math. 64 (2003), 260-276.

[3] Z. Bai, Y. Zhou, and T. Zhang, "*Existence of multiple periodic solutions for an SIR model with seasonality*", Nonlinear Anal.-Theor. 74(11) (2011), 3548-3555.

[4] C. Castillo-Chavez, Z. Feng, and W. Huang. *On the computation of ro and its role on global stability*, In: Castillo-Chavez C, van den Driessche P, Kirschner D, Yakubu A.-A, editors, Mathematical approaches for emerging and reemerging infection diseases: an introduction, Springer: New York, 2002, 229-250.

[5] C. Castillo-Chavez and B. Song,"*Dynamical models of Tuberculosis and their applications*", Math. Biosci. Eng. 1 (2004), 361-404.

[6] Y. Dumont, F. Chiroleu, and C. Domerg, "*On a temporal model for the chikungunya disease: modeling, theory and numerics*", Math. Biosci. 213(1)(2008), 80-91.

[7] J.C. Eckalbar and W.L. Eckalbar, "*Dynamics of epidemic model with quadratic treatment*", Nonlinear Anal. Real World Appl. 12(1) (2011), 320-332.

[8] E. S. Nugraha, J. Naiborhu, N. Nuraini, and Ahmadin. "*Mathematical modeling for control zika transmission*", AIP Conf. Proc. 1905 (2017), 030026.

[9] S. Elkahar and A. Kaddar, "*Stability Anaysis of an SEIR Model with Treatment*", Res. Appl. Math. 1(2017), 1-16.

[10] L. Esteva and C. Vargas, "*Analysis of a dengue disease transmission model*", Math. Biosci. 150(2) (1998), 131-151.

[11] X. Feng, Z. Teng, K. Wang, and F. Zhang, "*Backward bifurcation and global stability in epidemic model with treatment and vaccination*", Discr. Cont. Dyn. Sys. Ser. B 19(4) (2014), 999-1025.

[12] A. B. Gumel, C. C. McCluskey, and J. Watmough, "*An SVEIR modelfor assessing potential impact of an imperfect anti-SARS vaccine*", Math. Biosci. Eng. 3 (2006), 485-512.

[13] J.K. Hale, *Ordinary Differential Equations*, Wiley: New York, 1969.

[14] D. Hidayat , E. S. Nugraha, and N. Nuraini, "*A mathematical model of chagas disease transmission*", AIP Conf. Proc. 1937 (2018), 020008.

[15] https://www.unicef.org/immunization accessed in August 2, 2018.
[16] http://www.who.int/whr/1996/media_centre/press_release/en/ accessed on June, 26 2018.
[17] Z. Hu, W. Ma, and S. Ruan, "*Analysis of SIR epidemic models with non linear incidence rate and treatment*", Math. Biosci. 238(1) (2012), 12-20.
[18] R. M. May and R. M. Anderson, "*Transmission dynamics of HIV infection*", Nature 326(6109) (1987), 137-142.
[19] S. M. Moghadas, "*Modelling the effect of imperfect vaccines on disease epidemiology*", Discr. Cont. Dyn. Syst. Ser. B 4 (2004), 999-1012.
[20] WO. Kermack and A.G McKendrick, "*A contribution to the mathematical theory of epidemics*", Proc. R. Soc. B Biol. Sci. 115(772) (1927), 700-721.
[21] J. P. Tian and J. Wang, "*Global stability for cholera epidemic models*". Math. Biosci. 232(1)(2011), 31-41.
[22] P. Van den Driessche and J. Watmough, "*Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*", Math. Biosci. 180(1-2) (2002), 29-48.
[23] W. Wang and S. Ruan, "*Bifurcations in an epidemic model with constant removal rate of the infectives*", J. Math. Anal. Appl. 291(2) (2004), 775-793.
[24] W.D. Wang, "*Backward bifurcation of an epidemic model with treatment*", Math. Biosci. 201(1-2) (2006), 58-71.
[25] J.Y. Yang, F.Q. Zhang, and X.Y. Wang, "*SIV epidemic models with age of infection*", Int. J. Biomath. 2(01) (2009), 61-67.
[26] W. Yang, C. Sun, and J. Arino, "*Global analysis for a general epidemiological model with vaccination and varying population*", J. Math. Anal. Appl. 372(1) (2010), 208-223.
[27] J. Zhang and X.N. Liu, "*Backward bifurcation of and epidemic model with saturated treatment function*, J. Math. Anal. Appl. 348(1) (2008), 433-443.
[28] J. Zhang, J. Jia, and X. Song, "*Analysis of an SEIR epidemic model with saturated incidence and saturated treatment function*", Sci. World J. 2014 (2014), 1-11.

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