Advanced Studies in Contemporary Mathematics
29 (2019), No. 1, pp. 59 - 67
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.1.59

MULTI-DIRECTIONAL APPROACH OF PERFECTLY MATCHED LAYERS

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Abstract. One of the popular non-reflection boundary methods is perfectly matched layers (PML), in which the damping is introduced depending on only a single variable. There is no effective absorption for the waves with low-grazing incidence angles (high incidence angles) to the interface, which is one of the limitations of the classical PML method. In this paper, we introduce additional damping terms, which are positive functions of multi-variables and show the effectiveness of the absorption for the incident wave in the parallel to the interface when the wave propagates in a PML.

2010 Mathematics Subject Classification. 65M99, 35L20.

Keywords and phrases. Perfectly matched layers, Absorbing boundary condition.

1. Multi-Directional PML

A perfectly matched layer (PML) is widely used to model non-reflective boundary behavior of many different types of wave equations in different media [1, 2, 4, 5]. It has been found that a PML is often very effective and also simple to set, but several questions [5] are remained for the efficiency. For example, in the classical PML method, damping terms are introduced depending on only one variable for the exponential decay of wave propagation in the one direction. That there is no effective absorption for waves with low-grazing incidence angle (high incidence angle) to the interface is one of limitations mentioned in [5].

To deal with this shortcoming of the classical PML, we introduce additional damping terms, σ_x^y and σ_y^x, which are positive functions of x and y. This is the idea to impose the absorption of incident wave in the parallel to the interface when the wave propagate in a PML.Most of the results in this section and subsequent section are followed in [6].

Instead of the coordinate transformation in the Classical PML given by

$$\tilde{x} = x + i\frac{\int_a^x \sigma(s)ds}{\omega}, \quad i \in \mathbb{C}, \tag{1}$$

where w represents the angular frequency of the wave, we can consider the damping terms which depend on both variables x, y in the PML region $\{(x,y) \in \Omega \subset \mathbb{R}^2 \ : \ a \le |x| \le a + L_x, b \le |y| \le b + L_y, \}$.

For a detailed explanation, we consider the domain $\Omega := [-a - L_x, a + L_x] \times [-b - L_y, b + L_y] \subset \mathbb{R}^2$ and introduce the coordinate transform of variables in the frequency domain by the following,

$$\begin{aligned} \tilde{x}(x,y) :=& x + \frac{1}{iw}\left(\int_a^x \sigma_x(s)ds + \int_0^y \sigma_x^y(s)ds\right) := x + \frac{1}{iw}\sigma^x, \\ \tilde{y}(x,y) :=& y + \frac{1}{iw}\left(\int_b^y \sigma_y(s)ds + \int_0^x \sigma_y^x(s)ds\right) := y + \frac{1}{iw}\sigma^y, \end{aligned} \tag{2}$$

where $\sigma_x^y(x,y)$ and $\sigma_y^x(x,y)$ are non-negative functions in PML region and vanish in the computation area $[-a,a]\times[-b,b]$. We assume that

$$\sigma_x^y(x,y), \sigma_y^x(x,y) \in W^{1,\infty}(\Omega). \tag{3}$$

Differentiate $\tilde{x}, \tilde{y}$ with respect to x, y to obtain Jacobian matrix,

$$J := \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial y} \\ \frac{\partial \tilde{y}}{\partial x} & \frac{\partial \tilde{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{iw}\partial_x\sigma^x & \frac{1}{iw}\partial_y\sigma^x \\ \frac{1}{iw}\partial_x\sigma^y & 1 + \frac{1}{iw}\partial_y\sigma^y \end{bmatrix} = \begin{bmatrix} \frac{iw+\partial_x\sigma^x}{iw} & \frac{\partial_y\sigma^x}{iw} \\ \frac{\partial_x\sigma^y}{iw} & \frac{iw+\partial_y\sigma^y}{iw} \end{bmatrix},$$

which also gives the inverse of J, that is,

$$J^{-1} = \frac{iw}{D}\begin{bmatrix} iw + \partial_y\sigma^y & -\partial_y\sigma^x \\ -\partial_x\sigma^y & iw + \partial_x\sigma^x \end{bmatrix},$$

where $D := (iw)^2 + (\partial_x\sigma^x + \partial_y\sigma^y)iw + \partial_x\sigma^x\partial_y\sigma^y - \partial_x\sigma^y\partial_y\sigma^x$. Then, we have the partial derivatives of new coordinate systems,

$$\begin{cases} \frac{\partial}{\partial \tilde{x}} = \frac{iw(iw+\partial_y\sigma^y)}{D}\frac{\partial}{\partial x} - \frac{iw\partial_y\sigma^x}{D}\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \tilde{y}} = -\frac{iw\partial_x\sigma^y}{D}\frac{\partial}{\partial x} + \frac{iw(iw+\partial_x\sigma^x)}{D}\frac{\partial}{\partial y}. \end{cases} \tag{4}$$

We apply this new coordinate systems in the system of first order acoustic wave equation to obtain a new type of PML wave equations.

2. Multi-directional PML

2.1. Multi-directional un-split PML. Consider the system of first order acoustic wave equation with variable sound speed,

$$\begin{cases} \frac{1}{c^2}p_t + \nabla\cdot\vec{\mathbf{q}} = 0, & \text{in } \mathbb{R}^2\times(0,T], \\ \vec{\mathbf{q}}_t + \nabla p = \vec{\mathbf{0}}, & \text{in } \mathbb{R}^2\times(0,T], \end{cases} \tag{5}$$

with the initial condition $p(x,0) = p_0, \vec{\mathbf{q}}(x,0) = \vec{\mathbf{q}}_0$ and bounds of sound speed $0 < c_* \le c \le c^* < \infty$. Applying the new coordinates system (4) in the frequency domain of the system (5) after the even extension of solutions over $\mathbb{R}$, one can obtain the following system

(6)

$$\begin{cases} \frac{D}{c^2}\hat{p} + (iw+\partial_y\sigma^y)\frac{\partial\hat{\mathbf{q}}_x}{\partial x} - \partial_y\sigma^x\frac{\partial\hat{\mathbf{q}}_x}{\partial y} - \partial_x\sigma^y\frac{\partial\hat{\mathbf{q}}_y}{\partial x} + (iw+\partial_x\sigma^x)\frac{\partial\hat{\mathbf{q}}_y}{\partial y} = 0, \\ D\hat{\vec{\mathbf{q}}} + \begin{bmatrix} iw+\partial_y\sigma^y & -\partial_y\sigma^x \\ -\partial_x\sigma^y & iw+\partial_x\sigma^x \end{bmatrix}\begin{bmatrix} \frac{\partial\hat{p}}{\partial x} \\ \frac{\partial\hat{p}}{\partial y} \end{bmatrix} = 0. \end{cases}$$

Similarly to the classical PML wave system [3], auxiliary variables $\hat{p}^*$ and $\hat{\vec{\mathbf{q}}}^*$ are introduced by $\hat{p}^* = i\omega\hat{p}$ and $\hat{\vec{\mathbf{q}}}^* = i\omega\hat{\vec{\mathbf{q}}}$. The inverse *Fourier Transform* of (6) with respect to ω with the direct computations provides the following formulation,

$$(7)\qquad \begin{cases} \frac{1}{c^2}p_t + \frac{1}{c^2}\alpha p + \frac{1}{c^2}\beta p^* + \nabla\cdot\vec{\mathbf{q}} + M_\sigma\vec{\mathbf{q}}^* = 0,\\ \vec{\mathbf{q}}_t + \alpha\vec{\mathbf{q}} + \beta\vec{\mathbf{q}}^* + \nabla p + C_\sigma\nabla p^* = 0,\\ p_t^* = p,\\ \vec{\mathbf{q}}_t^* = \vec{\mathbf{q}}, \end{cases}$$

with the initial condition $(p,p^*) = (p_0,p_0^*)$, $(\vec{\mathbf{q}},\vec{\mathbf{q}}^*) = (\vec{\mathbf{q}}_0,\vec{\mathbf{q}}_0^*)$ and the boundary condition $(p,p^*)|_{\partial\Omega} = (0,0)$, where the coefficients are defined as $\alpha = \partial_x\sigma^x + \partial_y\sigma^y$, $\beta = \partial_x\sigma^x\partial_y\sigma^y - \partial_x\sigma^y\partial_y\sigma^x$,

$$M_\sigma = \begin{bmatrix} \partial_y\sigma^y\frac{\partial}{\partial x} - \partial_y\sigma^x\frac{\partial}{\partial y} & \partial_x\sigma^x\frac{\partial}{\partial y} - \partial_x\sigma^y\frac{\partial}{\partial x} \end{bmatrix},\ C_\sigma = \begin{bmatrix} \partial_y\sigma^y & -\partial_y\sigma^x \\ -\partial_x\sigma^y & \partial_x\sigma^x \end{bmatrix}.$$

We next introduce the regularized formulation of the system (7) with regularizing several terms in order to get regularity of weak solutions in (7), which derives a new formulation.

2.2. **Regularized multi-directional PML.** For the regularization, let us introduce a linear bounded operator $\delta_\varepsilon : H^{-1}(\Omega) \to L^2(\Omega)$ and the dual operator $\delta'_\varepsilon : L^2(\Omega) \to H_0^1(\Omega)$, which satisfies

$$\|\delta_\varepsilon(f) - f\|_{H^{-1}(\Omega)} \to 0 \text{ as } \varepsilon \to 0$$

and $\|\delta_\varepsilon(f)\|_{L^2(\Omega)} \le C_{\delta_\varepsilon}\|f\|_{H^{-1}(\Omega)}$ for some $C_{\delta_\varepsilon} > 0$. Note that δ_ε is a linear and bounded operator from $H^{-1}(\Omega)$ to $H^{-1}(\Omega)\cap L^2(\Omega)$. For more details on δ_ε, we refer the interested readers to [6].

We introduce a new formulation with the regularized term applying δ_ε and δ'_ε in (7),

$$(8)\qquad \begin{cases} \frac{1}{c^2}p_t + \frac{1}{c^2}\alpha p + \frac{1}{c^2}\beta p^* + \delta_\varepsilon\nabla\cdot\vec{\mathbf{q}} + \delta_\varepsilon\ M_\sigma\vec{\mathbf{q}}^* = 0,\\ \vec{\mathbf{q}}_t + \alpha\vec{\mathbf{q}} + \beta\vec{\mathbf{q}}^* + \nabla\delta'_\varepsilon p + C_\sigma\ \nabla\delta'_\varepsilon p^* = 0,\\ p_t^* =\ p,\\ \vec{\mathbf{q}}_t^* = \vec{\mathbf{q}} \end{cases}$$

with the initial conditions $(p(0),p^*(0)) = (p_0,p_0^*)$ and $(\vec{\mathbf{q}}(0),\vec{\mathbf{q}}^*(0)) = (\vec{\mathbf{q}}_0,\vec{\mathbf{q}}_0^*)$. Note that the zero *Dirichlet* boundary condition $\delta'_\varepsilon p|_{\partial\Omega} = 0$ is imposed in the system (8).

We define a weak solution of the system (8).

Definition 2.1. *We define*

$$(9)\qquad \{p,p^*\} \in L^2(0,T;L^2(\Omega)), \{\vec{\mathbf{q}},\vec{\mathbf{q}}^*\} \in L^2(0,T;\mathbb{L}^2(\Omega))$$

with

$$\{p_t,p_t^*\} \in L^2(0,T;L^2(\Omega)), \{\vec{\mathbf{q}}_t,\vec{\mathbf{q}}_t^*\} \in L^2(0,T;\mathbb{L}^2(\Omega))$$

is a weak solution of the initial-value boundary problem (8) *provided*

$$\begin{cases} (\frac{1}{c^2}p_t, r) + (\frac{1}{c^2}\alpha p, r) + (\frac{1}{c^2}\beta p^*, r) + (\delta_\varepsilon\nabla\cdot\vec{\mathbf{q}}, r) + (\delta_\varepsilon M_\sigma\vec{\mathbf{q}}^*, r) = 0,\\ (\vec{\mathbf{q}}_t, \vec{\mathbf{v}}) + (\alpha\vec{\mathbf{q}}, \vec{\mathbf{v}}) + (\beta\vec{\mathbf{q}}^*, \vec{\mathbf{v}}) + (\nabla\delta'_\varepsilon p, \vec{\mathbf{v}}) + (C_\sigma\nabla\delta'_\varepsilon p^*, \vec{\mathbf{v}}) = 0,\\ (p_t^*, r^*) - (p, r^*) = 0,\\ (\vec{\mathbf{q}}_t^*, \vec{\mathbf{v}}^*) - (\vec{\mathbf{q}}, \vec{\mathbf{v}}^*) = 0 \end{cases}$$

for all $r, r^* \in L^2(0,T;L^2(\Omega))$, $\vec{\mathbf{v}}, \vec{\mathbf{v}}^* \in L^2(0,T;\mathbb{L}^2(\Omega))$ *which satisfies the* Cauchy *initial data in a weak sense.*

We prove the existence and uniqueness of the weak solution (9) of the system.

Theorem 2.2. *We assume that the initial data* $(p_0, p_0^*, \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_0^*) \in [L^2(\Omega]^2 \times [\mathbb{L}^2(\Omega)]^2$. *The regularized system* (8) *admits a unique weak solution satisfying Definition 2.1, provided* (3) *holds true.*

Proof. We define the energy norm by

$$E := \|\frac{1}{c}p\|^2_{L^2(\Omega)} + \|p^*\|^2_{L^2(\Omega)} + \|\vec{\mathbf{q}}\|^2_{\mathbb{L}^2(\Omega)} + \|\vec{\mathbf{q}}^*\|^2_{\mathbb{L}^2(\Omega)}.$$

To show estimates of the energy, we apply the scalar product of all equations in (8) with p, p^* in $L^2(\Omega)$ and $\vec{\mathbf{q}}, \vec{\mathbf{q}}^*$ in $\mathbb{L}^2(\Omega)$ respectively, to obtain the identity,

$$\frac{dE}{dt} + F_1 + F_2 + F_3 + F_4 = 0, \tag{10}$$

where

$$\begin{aligned}
F_1 &= (\frac{1}{c^2}\alpha p, p) + (\frac{1}{c^2}\beta p^*, p) - (p, p^*),\\
F_2 &= (\alpha\vec{\mathbf{q}}, \vec{\mathbf{q}}) + (\beta\vec{\mathbf{q}}^*, \vec{\mathbf{q}}) - (\vec{\mathbf{q}}, \vec{\mathbf{q}}^*),\\
F_3 &= (\delta_\varepsilon \nabla\cdot\vec{\mathbf{q}}, p) + (\nabla\delta'_\varepsilon p, \vec{\mathbf{q}}),\\
F_4 &= (\delta_\varepsilon M_\sigma \vec{\mathbf{q}}^*, p) + (C_\sigma \nabla\delta'_\varepsilon p^*, \vec{\mathbf{q}}).
\end{aligned}$$

We have that $|F_1| + |F_2| \le CE$ a.e. in t for some $C > 0$ since $\alpha, \beta \in L^\infty(\tilde{\Omega})$ and the bounds of c in (5). It is allowed to have that $F_3 = 0$ by the duality of δ_ε and integration by parts,

$$\begin{aligned}
(\delta_\varepsilon\nabla\cdot\vec{\mathbf{q}}, p) + (\nabla\delta'_\varepsilon p, \vec{\mathbf{q}}) &= p(\delta_\varepsilon\nabla\cdot\vec{\mathbf{q}}) + (\nabla\delta'_\varepsilon p, \vec{\mathbf{q}}),\\
&= (\nabla\cdot)'\delta'_\varepsilon p(\vec{\mathbf{q}}) + (\nabla\delta'_\varepsilon p, \vec{\mathbf{q}}),\\
&= -\nabla\delta'_\varepsilon p(\vec{\mathbf{q}}) + \nabla\delta'_\varepsilon p(\vec{\mathbf{q}})\\
&= 0,
\end{aligned}$$

since $(\nabla\cdot)' = -\nabla$. The operators

$$\{\vec{\mathbf{q}}^*, p^*\} \to \left\{\delta_\varepsilon M_\sigma\vec{\mathbf{q}}^*, C_\sigma\nabla\delta'_\varepsilon p^*\right\}$$

are continuous from $[\mathbb{L}^2(\Omega)]^2 \times L^2(\Omega) \to [\mathbb{L}^2(\Omega)]^2 \times L^2(\Omega)$ since $\sigma_y^x(x,y)$ and $\sigma_x^y(x,y)$ are in $W^{1,\infty}(\Omega)$, which implies that

$$|F_4| \le CE \text{ a.e. in } t \in [0,T],$$

for some $C > 0$. Combining all bounds, it follows that from (10)

$$\frac{dE}{dt} \le C_T E \text{ a.e. } t \in [0,T]$$

for a suitable constant $C_T > 0$.

This is a standard *a priori* estimates, using this estimates we can obtain the existence part of Theorem 2.2 by the standard Galerkin method argument, and also uniqueness can be established by the estimates. We omit details here, since the detailed argument was presented in [6] □

Remark 2.3. *We don't present any stability analysis or numerical experiments in this section. Further investigation of the original system and the regularized one remains for further work. But, we introduce another simpler formulation with the same technique as in* (2) *to show that the multi-directional damping PML can be more effective than the classical one.*

3. Multi Directional Split PML in the parallel to y-axis

We apply the multi-directional damping (2) to the system of first order acoustic wave equation with Split PML techniques parallel to y-axis. Let the domain $\Omega = [-a - L_x, a + L_x] \times [-b, b]$ consist of the computational domain $[-a, a] \times [-b, b]$ with the PML only parallel to y-axis. The damping $\sigma^x(x, y)$ with $\sigma^y = 0$ in (2) is applied as follows:

$$\begin{cases} \tilde{x}(x, y(x)) = x + \frac{1}{iw}\sigma^x(x, y) = x + \frac{1}{iw}\left(\int_a^x \sigma_x(s)ds + \int_0^y \sigma_x^y(s)ds\right), \\ \tilde{y}(y) = y. \end{cases} \tag{11}$$

Note that the damping $\sigma_x^y(x)$ depends on both x and y in the PML. The coordinate change with the damping gives Jacobian matrix

$$J = \frac{\partial(\tilde{x}, \tilde{y})}{\partial(x, y)} = \frac{1}{iw}\begin{bmatrix} D & \sigma_x^y \\ 0 & 1 \end{bmatrix},$$

and

$$J^{-1} = \frac{1}{D}\begin{pmatrix} iw & -\sigma_x^y \\ 0 & D \end{pmatrix},$$

where $D = iw + \sigma_x + \frac{\partial}{\partial x}\int_0^y \sigma_x^y(s)ds$. Applying the Jacobian, one can have

$$\begin{cases} \frac{\partial}{\partial \tilde{x}} = \frac{iw}{D}\frac{\partial}{\partial x} - \frac{\sigma_x^y}{D}\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y}. \end{cases} \tag{12}$$

Following [8], we introduce the split system for the acoustic wave equation in order to apply the coordinate systems (12). Assume the solution p split into the two fields p^x and p^y satisfying $p = p^x + p^y$ and

$$p_t^x + c^2\frac{\partial}{\partial x}q_x = 0, \qquad p_t^y + c^2\frac{\partial}{\partial y}q_y = 0.$$

Then, we have the split system of acoustic wave equation,

$$\begin{cases} p_t^x + c^2\frac{\partial}{\partial x}q_x = 0, \\ p_t^y + c^2\frac{\partial}{\partial y}q_y = 0, \\ q_{xt} + \frac{\partial}{\partial x}(p^x + p^y) = 0, \\ q_{yt} + \frac{\partial}{\partial y}(p^x + p^y) = 0. \end{cases} \tag{13}$$

We apply (12) in the frequency space of (13) to obtain

$$\begin{cases} D\frac{1}{c^2}\hat{p}^x + \frac{\partial}{\partial x}\hat{q}_x - \sigma_x^y\frac{1}{iw}\frac{\partial}{\partial y}\hat{q}_x = 0, \\ iw\frac{1}{c^2}\hat{p}^y + \frac{\partial}{\partial y}\hat{q}_y = 0, \\ D\hat{q}_x + \frac{\partial}{\partial x}(\hat{p}^x + \hat{p}^y) - \sigma_x^y\frac{1}{iw}\frac{\partial}{\partial y}(\hat{p}^x + \hat{p}^y) = 0, \\ iw\hat{q}_y + \frac{\partial}{\partial y}(\hat{p}^x + \hat{p}^y) = 0. \end{cases} \tag{14}$$

We introduce an auxiliary variable $\hat{q}_x^* = -\frac{1}{iw}\frac{\partial}{\partial y}\hat{q}_x$ to obtain a new formulation after taking the inverse Fourier transform:

$$\begin{cases} \frac{1}{c^2}p_t^x + \frac{\bar{\sigma}}{c^2}p^x + \frac{\partial}{\partial x}q_x + \sigma_x^y q_x^* = 0, \\ \frac{1}{c^2}p_t^y + \frac{\partial}{\partial y}q_y = 0, \\ q_{xt} + \bar{\sigma}_x q_x + \frac{\partial}{\partial x}(p^x + p^y) + \sigma_x^y q_y = 0, \\ q_{yt} + \frac{\partial}{\partial y}(p^x + p^y) = 0, \\ q_{xt}^* + \frac{\partial}{\partial y}q_x \ = 0, \end{cases} \tag{15}$$

where $\bar{\sigma}_x = \sigma_x + \frac{\partial}{\partial x}\int_0^y \sigma_x^y(s)ds$.

3.1. **Numerical Results.** In this subsection, we show the system (15) is efficient and compare it with the classical Split PML [8] using centered differences and the staggered nodes in time and space.

For the description of the damping term following [7], in the absorbing layer, the choice of the damping functions can be constant, linear, or quadratic, etc. In our implementations, we use damping functions of the form;

(16)

$$\sigma_\alpha(\alpha) = \begin{cases} 0 & \text{for } |\alpha| < a_k, \quad \alpha = x, y, \\ \bar{\sigma}_0\left(\frac{|\alpha - a_k|}{L_k} - \frac{\sin(\frac{2\pi|\alpha - a_k|}{L_k})}{2\pi}\right) & \text{for } a_k \le |\alpha| \le a_k + L_k, k = 1, 2, \end{cases}$$

where $L_k, k = 1, 2$, are thickness of PML layers. Also, the components p^α, $q_x^*, \alpha = x, y$, are discretized at nodes (t^n, x_i, x_j) as $p^{\alpha}{}^n_{i,j}$, $q_x^*{}^n_{i,j}$, and q_α, are discretized at $(t^{n+\frac{1}{2}}, x_{i+\frac{1}{2}}, x_{j+\frac{1}{2}})$ as $q_\alpha{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$. This centered time stepping ensures a second order approximation in time. Denote by

$$\mathtt{A}_x^\pm := 1 \pm \bar{\sigma}_x\frac{\triangle t}{2}.$$

Step 1. Compute $q_\alpha{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}, \alpha = x, y$,

$$\mathtt{A}_x^+ q_x{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = \mathtt{A}_x^- q_x{}^{n-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \triangle t(\partial_x(u^x + u^y))^n_{i+\frac{1}{2},j+\frac{1}{2}} - \triangle t\sigma_y^x q_y{}^n_{i+\frac{1}{2},j+\frac{1}{2}},$$

$$q_y{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = q_y{}^{n-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - \triangle t(\partial_y(p^x + p^y))^n_{i+\frac{1}{2},j+\frac{1}{2}},$$

where

$$(\partial_x(p^x + p^y))^n_{i+\frac{1}{2},j+\frac{1}{2}} := (\partial_x p^x)^n_{i+\frac{1}{2},j+\frac{1}{2}} + (\partial_x p^y)^n_{i+\frac{1}{2},j+\frac{1}{2}},$$

$$(\partial_x p^x)^n_{i+\frac{1}{2},j+\frac{1}{2}} := \frac{(p^{x}{}^n_{i+1,j+1} - p^{x}{}^n_{i,j+1} + p^{x}{}^n_{i+1,j} - p^{x}{}^n_{i,j})}{2\triangle x},$$

$$(\partial_x p^y)^n_{i+\frac{1}{2},j+\frac{1}{2}} := \frac{(p^{y}{}^n_{i+1,j+1} - p^{y}{}^n_{i,j+1} + p^{y}{}^n_{i+1,j} - p^{y}{}^n_{i,j})}{2\triangle x},$$

$$q_y{}^n_{i+\frac{1}{2},j+\frac{1}{2}} := \frac{q_y{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + q_y{}^{n-\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}}{2},$$

and $(\partial_x(p^x + p^y))^n_{i+\frac{1}{2},j+\frac{1}{2}}$ is similarly defined.

Step 2. Compute $q_{x i,j}^{*n+1}$,

$$q_{x i,j}^{*n+1} = q_{x i,j}^{*n} - \frac{\triangle t}{2\triangle y}\left(q_{x\,i+\frac{1}{2},j+\frac{1}{2}}^{\;n+\frac{1}{2}} - q_{x\,i+\frac{1}{2},j-\frac{1}{2}}^{\;n+\frac{1}{2}} + q_{x\,i-\frac{1}{2},j+\frac{1}{2}}^{\;n+\frac{1}{2}} - q_{x\,i-\frac{1}{2},j-\frac{1}{2}}^{\;n+\frac{1}{2}} \right).$$

Step 3. Compute $u^{\alpha n}_{i,j}$, $\alpha = x, y$,

$$\frac{1}{c^2}\mathtt{A}_x^+ p^{xn+1}_{i,j} = \frac{1}{c^2}\mathtt{A}_x^- p^{xn}_{i,j} - \triangle t(\partial_x q_x)_{i,j}^{n+\frac{1}{2}} - \triangle t\sigma_y^x\left(\frac{q_{x i,j}^{n+1} + q_{x i,j}^{n}}{2}\right),$$

$$\frac{1}{c^2}p^{yn+1}_{i,j} = \frac{1}{c^2}p^{yn}_{i,j} - \frac{\triangle t}{2\triangle y}\left(q_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{\;n+\frac{1}{2}} - q_{y\,i+\frac{1}{2},j-\frac{1}{2}}^{\;n+\frac{1}{2}} + q_{y\,i-\frac{1}{2},j+\frac{1}{2}}^{\;n+\frac{1}{2}} - q_{y\,i-\frac{1}{2},j-\frac{1}{2}}^{\;n+\frac{1}{2}} \right).$$

We impose smooth variable sound speed $c(x, y) \in [-0.5, 0.5]$ (see Figure 1),

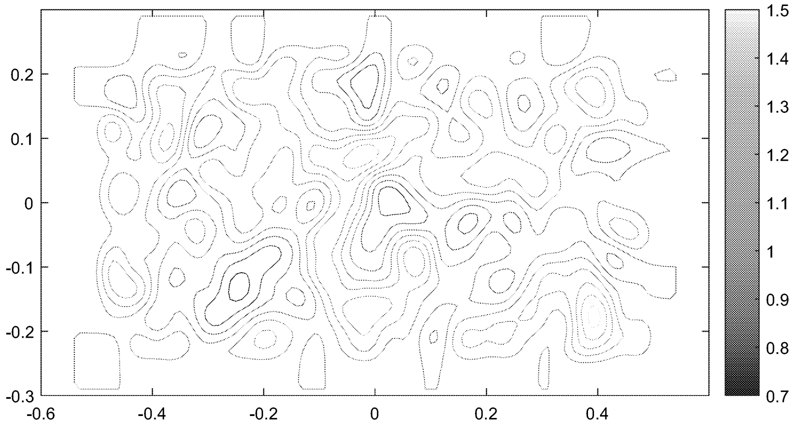

FIGURE 1. Variable sound speed in the computational domain= $[-0.3, 0.3] \times [-0.6, 0.6]$

and set the damping σ_x^y as

$$\sigma_x^y(x, y) = \sigma_y(y)\int_a^x \sigma_x(s)ds,$$

where σ_x and σ_y are defined as in (16) with various maximum damping coefficients $\bar{\sigma}_0$. We consider the computational domain $[-0.3, 0.3] \times [-0.6, 0.6]$ with the PML region $[-0.4, -0.3] \cup [0.3, 0.4]$ parallel to y-axis. We compare the numerical solution obtained with a reference solution, computed with the same numerical scheme on a very large domain $[-0.6, 0.6] \times [-0.6, 0.6]$. In Figure 2, it is shown the discrete L^2-*error* of the classical Split PML and Multi-Directional Split PML on the computational domain and the difference of the errors with two different damping. Similarly, Figure 3 shows the maximum error of two PMLs in the computational domain. In this case, the multi-directional Split PML leads to a smaller error than that for the classical Split PML for both L^2-*error* and *maximum error*.

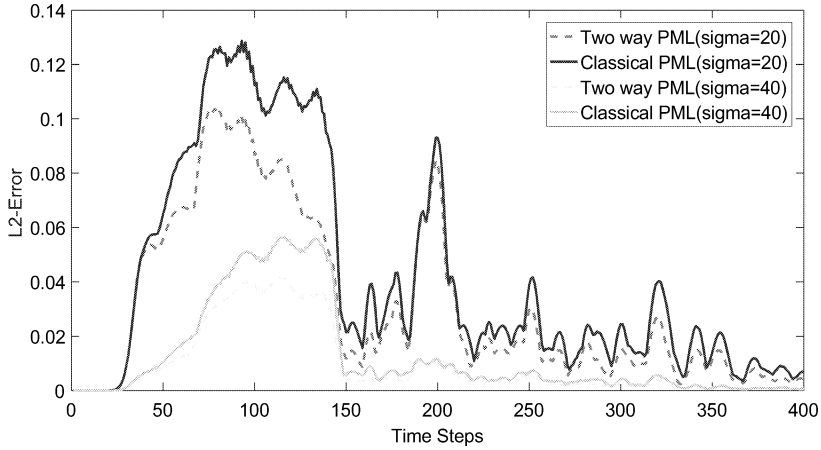

FIGURE 2. L^2-error of classical Split PML and Multi-Directional Split PML in the computational domain

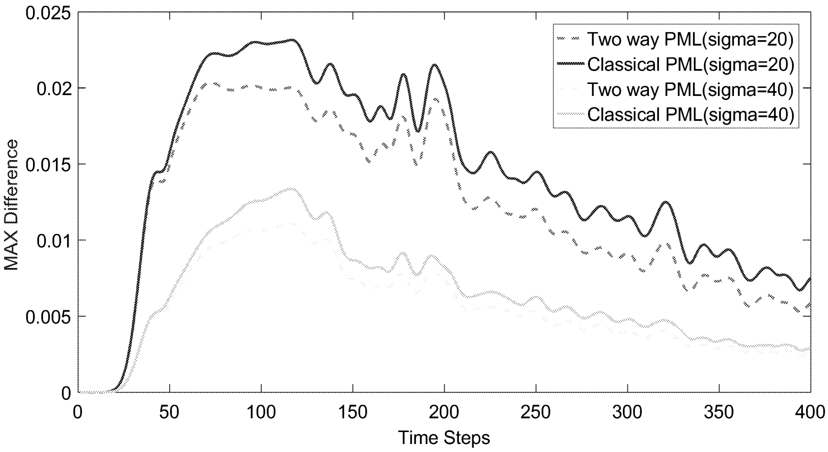

FIGURE 3. Maximum error of classical Split PML and Multi-Directional Split PML in the computational domain

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