ATTRACTOR DIMENSIONS OF THREE-DIMENSIONAL NAVIER-STOKES-α MODEL FOR FAST ROTATING FLUIDS ON GENERIC-PERIOD DOMAINS: COMPARISON WITH NAVIER-STOKES EQUATIONS

BONG-SIK KIM

ABSTRACT. We study the fractal and Hausdorff dimensions of the global attractor for the three-dimensional Navier-Stokes-α model for fast rotating geophysical fluids. The Navier-Stokes-α model is a nonlinear dispersive regularization of the exact Navier-Stokes equations obtained by Lagrangian averaging and tend to the Navier-Stokes equations as $\alpha \to 0^+$. We estimate upper bounds for the dimensions of the global attractor and study the dependence of the dimensions on the parameter α. All the estimates are uniform in α, and our estimates of attractor dimensions remain finite when $\alpha \to 0^+$.

1. INTRODUCTION

We consider the three-dimensional rotating Navier-Stokes-α equations (RNS-α) with periodic boundary conditions in a torus $\mathbb{T}^3 = [0, 2\pi a_1] \times [0, 2\pi a_2] \times [0, 2\pi a_3]$:

$$
\begin{aligned}
(1) \qquad \frac{\partial v}{\partial t} + (u \cdot \nabla)v + v_j \nabla u^j + \Omega e_3 \times u &= -\nabla p + \nu \Delta v + f \\
\nabla \cdot v = \nabla \cdot u = 0 \quad \text{and} \quad v(t,x)|_{t=0} &= v_0 \\
u &= (I - \alpha^2 \Delta)^{-1} v
\end{aligned}
$$

where $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, $v = v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ is the velocity field, $p = p(x,t)$ is the pressure of a homogeneous incompressible fluid, ν is the viscosity, and $f = f(x)$ is a divergence free body force. Ω is the Coriolis parameter, which is twice the angular velocity of the rotation around the vertical unit vector $e_3 = (0, 0, 1)$. The system (1) reduces to the exact rotating Navier-Stokes equations (RNS) when $\alpha \to 0^+$.

Kim and Nicolaenko [8] established the existence and global regularity of solutions of the system (1) and proved the existence of its global attractor. In this paper, we estimate the fractal and Hausdorff dimensions of the global attractor for the system (1) and give special attention to the limiting case when $\alpha \to 0^+$, that is, when the rotating Navier-Stokes-α equations (RNS-α) tend to the exact rotating Navier-Stokes equations (RNS). We focus on generic-period domains and eliminate *nontrivial resonant parts* (strict three-wave resonant interactions). The generic domain parameters (a_1, a_2, a_3)

form a dense set of full Lebesque measure in $\mathbb{R}^3$ for which the strict three-wave resonant interactions are effectively annihilated [1].

Ilyin and Titi [7] estimated attractor dimensions for two-dimensional navier-Stokes-α equations. Their estimates, however, blow up as $\alpha \to 0^+$. Gibbon and Holm [6] obtained length-scale estimates for NS-α equations in terms of the Reynolds number, which blow up in the limit when $\alpha \to 0^+$, too. Several other time-averaged estimates related to NS-α equations don't remain finite in the Navier-Stokes limit, except the cases where there are known equivalent upper bounds for the Navier-Stokes equations (see Table 1 in [6]). They analyzed the system in the context of the filtered velocity $u = (I - \alpha^2 \Delta)v$. Instead, we study the system from the perspective of the non-filtered velocity $v = (I - \alpha^2 \Delta)^{-1}u$. The Helmholtz inverse operator $\mathcal{R}_\alpha = (I - \alpha^2 \Delta)^{-1}$ plays a crucial role in the process, leading to uniform estimates that remain finite in the Navier-Stokes limit as $\alpha \to 0^+$.

We consider domain parameters, a_1, a_2, and a_3, bounded away from both zero and infinity. v and u are vector fields defined on $D = \mathbb{T}^3 \times [0, T]$ for any $T > 0$. Periodicity of the boundary conditions leads naturally to a Fourier representation of the fields,

$$v = \sum_n v_n e^{i(n_1 x_1/a_1 + n_2 x_2/a_2 + n_3/a_3)} = \sum_n v_n e^{i\check{n}\cdot x},$$

where v_n's are Fourier Coefficients, $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, and $\check{n} = (\check{n}_1, \check{n}_2, \check{n}_3)$ are wave numbers with $\check{n}_j = n_j/a_j$ for $j = 1, 2, 3$. We set $a_1 = 1$ without loss of generality and define the Fourier-Sobolev space of divergence free periodic vector fields as follows:

$$H^s = \left\{ v \in [L^2(\mathbb{T}^3)]^3 \mid v = \sum_{n \in \mathbb{Z}^3} v_n e^{i\check{n}\cdot x}, \; v_n^* = v_{-n}, \; v_0 = 0, \; \check{n} \cdot v_n = 0, \; ||v||_s^2 < \infty \right\},$$

with the norm

$$||v||_s^2 = \sum_{n \in \mathbb{Z}^3} |\check{n}|^{2s} |v_n|^2.$$

Here v_n^* is the complex conjugate of v_n. The corresponding inner product is denoted by $< \cdot, \cdot >_s$. We set $H^0 = H$ when $s = 0$. Also, $< \cdot, \cdot >_0 = < \cdot, \cdot >$, $|| \cdot ||_0 = | \cdot |$, and $|| \cdot ||_1 = || \cdot ||$. We assume that

$$\int_{\mathbb{T}^3} v(x, 0)\, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} f(x)\, dx = 0 \quad \text{for all } t \geq 0.$$

This yields $\int_{\mathbb{T}^3} v(x, t)\, dx = 0$ for all $t \geq 0$, and allows for the use of the Poincaré inequality.

We denote P_L as the usual Leray projection onto the divergence free subspace and introduce the Helmholtz inverse operator, $\mathcal{R}_\alpha = (I - \alpha^2 \Delta)^{-1}$, which is given by

$$\mathcal{R}_\alpha v = (I - \alpha^2 \Delta)^{-1} v.$$

A bilinear operator B_α on divergence free vector fields [8] is define by

$$B_\alpha(u, v) = P_L \left[(\mathcal{R}_\alpha u \cdot \nabla)v + v_j \nabla(\mathcal{R}_\alpha u)_j \right] = -P_L \left[\mathcal{R}_\alpha v \times \operatorname{curl} v \right].$$

This bilinear operator has a connection with the classical Navier-Stokes bilinear operator

$$B(u, v) = P_L[(u \cdot \nabla)v].$$

Lemma 1.1. *For every* $u, v, w \in H^1$

$$< B_\alpha(u,v), \ w > = < B(\mathcal{R}_\alpha u, v), \ w > - < B(w,v), \ \mathcal{R}_\alpha u >$$

Proof.

$$
\begin{aligned}
< B_\alpha(u,v), w > &= \ < -P_L[\mathcal{R}_\alpha u \times (\nabla \times v)], \ w > \\
&= \ < P_L[(\mathcal{R}_\alpha u \cdot \nabla)v + \sum_{j=1}^{3} v_j \cdot \nabla(\mathcal{R}_\alpha u)_j], \ w > \\
&= \ < P_L[(\mathcal{R}_\alpha u \cdot \nabla)v], \ w > + < P_L[\sum_{j=1}^{3} v_j \cdot \nabla(\mathcal{R}_\alpha u)_j], \ w > \\
&= \ < B(\mathcal{R}_\alpha u, v), \ w > + < \sum_{j=1}^{3} v_j \cdot \nabla(\mathcal{R}_\alpha u)_j, \ w > \\
&= \ < B(\mathcal{R}_\alpha u, v), \ w > - < B(w,v), \ \mathcal{R}_\alpha u >
\end{aligned}
$$

For the second equality we use the identity, $(a \cdot \nabla)b = \nabla(a \cdot b) - (b \cdot \nabla)a - a \times \operatorname{curl} b - b \times \operatorname{curl} a$, to get

$$\nabla(v \cdot \mathcal{R}_\alpha u) - \mathcal{R}_\alpha u \times \operatorname{curl} v = (\mathcal{R}_\alpha u \cdot \nabla)v + (v \cdot \nabla)\mathcal{R}_\alpha u + v \times \operatorname{curl} \mathcal{R}_\alpha u.$$

Noticing that $(v \cdot \nabla)\mathcal{R}_\alpha u + v \times \operatorname{curl} \mathcal{R}_\alpha u = \sum_{j=1}^{3} v_j \nabla(\mathcal{R}_\alpha u)_j$, we can get the equality. For the last equality, we directly calculate the second inner product such as

$$
\begin{aligned}
< \sum_{j=1}^{3} v_j \nabla(\mathcal{R}_\alpha u)_j, \ w > &= \ \sum_{i,j=1}^{3} \int_Q v_j \frac{\partial(\mathcal{R}_\alpha u)_j}{\partial x_i} w_i \, dx \\
&= \ \sum_{i,j} \int_Q w_i \frac{\partial(\mathcal{R}_\alpha u)_j}{\partial x_i} v_j \, dx \\
&= \ -\sum_{i,j} \int_Q w_i \frac{\partial v_j}{\partial x_i}(\mathcal{R}_\alpha u)_j \, dx \\
&= \ - < (w \cdot \nabla)v, \ \mathcal{R}_\alpha u > \\
&= \ - < P_L[(w \cdot \nabla)v], \ \mathcal{R}_\alpha u > \\
&= \ < -B(w,v), \ \mathcal{R}_\alpha u > .
\end{aligned}
$$

Therefore, the result follows. ∎

Now we rewrite Eq(1) in terms of the unfiltered velocity v:

(2) $$\frac{\partial v}{\partial t} + \Omega \, P_L J P_L \mathcal{R}_\alpha v + \nu A v + B_\alpha(v,v) = f,$$

where $A = -P_L \Delta P_L$ and J is a rotation matrix given by

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In Fourier-Sobolev space, the action P_L on n-th Fourier component of a vector field is given by $P_L v = \sum_n (P_n v_n) e^{i\check{n}\cdot x}$ and $P_n v_n = (v_n - \frac{\check{n}\cdot v_n}{|\check{n}|^2}\check{n})$ with

$$P_n = I - \frac{1}{|\check{n}|^2}\begin{pmatrix} n_1^2 & \frac{n_1 n_2}{a_2} & \frac{n_1 n_3}{a_3} \\ \frac{n_1 n_2}{a_2} & \frac{n_2^2}{a_2^2} & \frac{n_2 n_3}{a_2 a_3} \\ \frac{n_1 n_3}{a_3} & \frac{n_2 n_3}{a_2 a_3} & \frac{n_3^2}{a_3^2} \end{pmatrix}.$$

The Helmholtz inverse operator $\mathcal{R}_\alpha$ commutes with curl and, for each wave number n,

$$(\mathcal{R}_\alpha)_n = \frac{1}{1 + \alpha^2 |\check{n}|^2}.$$

Then, for each wave number $n \in \mathbb{Z}^3$, the RNS-α equations have the form

$$\frac{\partial v_n}{\partial t} + \frac{1}{1 + \alpha^2 |\check{n}|^2}\Omega\, P_n J P_n v_n + \nu |\check{n}|^2 v_n + B_\alpha(v, v)_n = f_n,$$

where

$$B_\alpha(v, v)_n = -iP_n \sum_{k+m=n} \frac{1}{1 + \alpha^2 |\check{k}|^2}(v_k \times (\check{m} \times v_m)).$$

The existence of unique regular solutions for all Ω greater than some threshold Ω_0 has been proved in [2] for $\alpha = 0$ and in [8] for $\alpha > 0$.

Theorem 1.2 ([2]). *For every triplet of positive real numbers (a_1, a_2, a_3), the following result holds. Let $s > 1/2$ and $v_0 \in H^s(\mathbb{T}^3)$ a divergence-free vector field. Then there exits a constant $\Omega_0 > 0$, depending on $\|v_0\|_s$, $\|f\|_{s-1}, \nu$, and the domain parameter (a_1, a_2, a_3), such that for all $\Omega \geq \Omega_0$, there is a unique global solution*

$$v(t) \in C([0, \infty) : H^s(\mathbb{T}^3)) \cap L^2((0, \infty) : H^{s+1}(\mathbb{T}^3))$$

to the three-dimensional rotating Navier-Stokes equations ($\alpha = 0$ in (1)). Furthermore, if f is independent of t, then there exists a global attractor for the three-dimensional rotating Navier-Stokes equations bounded in H^s; such an attractor has a finite fractal dimension and attracts every weak Leray solution as $t \to +\infty$.

Theorem 1.3 ([8]). *For every triplet of positive real numbers (a_1, a_2, a_3), the following result holds. Let $s > 5/2$ and $v_0 \in H^s(\mathbb{T}^3)$ a divergence-free vector field. Then there exits a constant $\Omega_\alpha > 0$, depending on $\|v_0\|_s$, $\|f\|_{s-1}, \nu$, and the domain parameter (a_1, a_2, a_3), such that for all $\Omega \geq \Omega_\alpha$, there is a unique global solution*

$$v(t) \in C([0, \infty) : H^s(\mathbb{T}^3)) \cap L^2((0, \infty) : H^{s+1}(\mathbb{T}^3))$$

to the equation for any $\alpha \geq 0$. Moreover, all the estimates are uniform in α (i.e., the estimates don't blow up as $\alpha \to 0^+$). If f is independent of t, then there exists a global attractor for the system (1) bounded in H^s and its fractal dimension is finite.

Remark: The solutions of the three-dimensional rotating Naiver-Stokes-α equations uniformly converge in L^2 to those of the three-dimensional rotating Navier-Stokes equations as $\alpha \to 0^+$ (see Section 8 of [8]).

We consider the Eq. (1) in the limit as $\Omega \to +\infty$, which gives *resonant limit α-equations*. Working with the resonant limit α-equations on specific

periodic domains (generic periods), we obtain upper estimates of the dimensions of the global attractor for the resonant limit α-equations, which approximate the dimensions of the global attractor for three-dimensional RNS-α equations on generic-period domains:

Theorem 1.4. (Main Result). *Let $\mathcal{A}_\alpha$ be the global attractor of the Eq.(4) in H. Then its Hausdorff dimension $d_H(\mathcal{A}_\alpha)$ and the fractal dimension $d_F(\mathcal{A}_\alpha)$ are finite and satisfy, for an absolute constant K_α, the estimate*

$$d_H(\mathcal{A}_\alpha) < K_\alpha \left(\frac{\rho_V}{\nu_\alpha}\right)^2 \quad and \quad d_F(\mathcal{A}_\alpha) \leq 2d_H(\mathcal{A}_\alpha)$$

where $c_l, \tilde{c}, c_1,$ and d are absolute constants, $K(\alpha) = \left(\frac{c_l}{d}\right)^{3/2}(24c(\alpha)+1)^{3/2}\tilde{c}^{3/2}$, $c^2(\alpha) = \frac{1}{1+\alpha^2 c_1}\left[\frac{1}{1+\alpha^2 c_1} + \frac{1}{\alpha^2 c_1}\right]$, $\rho_V^2 = 2|f|^2/(\nu^2\lambda_1^2)$, and λ_1 the first eigenvalue of $A = -P_L\Delta P_L$. In particular, for $\alpha = 0$, the global attractor has a sharp upper bound

$$d_H(\mathcal{A}_0) < \tilde{K}\left(\frac{\rho_V}{\nu_0}\right)^{6/5}$$

Observe that $\lim_{\alpha\to 0^+} K(\alpha) = \left(\frac{(24\sqrt{2}+1)c_l\tilde{c}}{d}\right)^{3/2} \equiv K_0 < \infty$ so that the estimates doesn't blow up when $\alpha \to 0^+$. In general, $K_0 \neq \tilde{K}$.

Accepting the point of view that the dimension of a global attractor for Navier-Stokes equations is associated with the number of degree of freedom in turbulent flows [4], then these finite dimension estimates gives a rigorous justification that asymptotic dynamics of turbulent rotating fluids can be described by two-dimensional and three-component (2D-3C) non-steady Navier-Stokes equations when Ω is large enough. Also, the attractor dimensions are related to the fundamental length scale ℓ of turbulent flows, below which wave interactions do not affect its dynamics. For α equations, the fundamental length scale in terms of the fractal dimension $d_F(\mathcal{A}_\alpha)$ is

$$\ell_\alpha^2 \sim \left(\frac{|\mathbb{T}^3|}{d_F(\mathcal{A}_\alpha)}\right) \leq \ell_0^2 \sim \left(\frac{|\mathbb{T}^3|}{d_F(\mathcal{A}_0)}\right).$$

As shown in [1, 2, 8], the estimates depend crucially on the period of the torus, a_1, a_2, a_3. If we omit the nonlinearity in (1) and the viscous and forcing terms, we end up with the system

$$\partial_t v + (I - \alpha^2\Delta)^{-1}\Omega\, Jv = -\nabla p, \quad \nabla \cdot v = 0,$$

which describes the propagation of waves, called *Poincaré waves* or *inertial waves*. The corresponding dispersion law relating the pulsation ω to the wavenumber $\xi \in \mathbb{R}^3$ is

$$\omega(\xi) = \pm\Omega\frac{\xi_3}{|\xi|}.$$

Therefore, the two-dimensional part of the initial data evolves according to two-dimensional Euler or Navier-Stokes equations, and the three-dimensional part generates waves, which propagate very rapidly in the domain with a speed of Ω. Chemin *et al.* [3] detailed well about the propagation of high-speed waves in the fluid, and we brief their explanation here. The time average of these waves vanishes, but they carry a non-zero energy. The

wavenumbers of these waves are bounded as $\Omega \to \infty$ and a priori no short wavelengths are created. On periodic flows, Poincaré waves persist for long times, and interact not only with the limit two-dimensional flow, but also with themselves. A wave ξ interacts with a wave ξ' and generates another wave ξ'' provided

$$\xi + \xi' = \xi'' \quad \text{and} \quad \frac{\xi_3}{|\tilde{\xi}|} + \frac{\xi'_3}{|\tilde{\xi}'|} = \frac{\xi''_3}{|\tilde{\xi}''|}, \quad \text{where} \quad \tilde{\xi} = \left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \frac{\xi_3}{a_3} \right),$$

which are the usual resonance conditions in the three-wave interaction problem. In the periodic case, all the components of ξ, ξ' and ξ'' are integers, and the above conditions turn out to be Diophantine equations which do not have integer solutions for almost all (a_1, a_2, a_3) except the trivial solutions given by symmetries. Therefore, generically in the sizes of the periodic box, the waves do not interact with themselves and only interact with the two-dimensional underlying flow. Mathematically to handle these resonant wave interactions, Babin, Mahalov and Nicolaenko [1, 2] introduced the Poincaré group operator (see Section 2). Then they utilized methods of small denominators and Diophantine incommensurability conditions on the domain geometrical parameters a_1, a_2, a_3 to investigate the fast singular oscillating limits of Eq.(1) as $\Omega \to \infty$. In that approach, the collective contribution to the dynamics made by fast Poincaré waves is accounted for by rigorous estimates of wave resonances and quasi-resonances via small divisor analysis. We start off the next section by applying such an approach to derive the resonant limit α-equations.

2. Resonant limit α-equations

Poincaré propagator $E_\alpha(\Omega t) = e^{\Omega t P_L J P_L \mathcal{R}_\alpha}$ is defined as the unitary group solution $E_\alpha(-\Omega t)\Phi_0 = \Phi(t)$ $(E_\alpha(0) = I$ is the identity) to the linear Poincaré problem:

$$\partial_t \Phi + \Omega P_L J P_L \mathcal{R}_\alpha \Phi = 0, \quad \Phi|_{t=0} = \Phi_0, \text{ with } \nabla \cdot \Phi_0 = 0.$$

Denote $M_\alpha = P_L J P_L \mathcal{R}_\alpha$ and $M_{\alpha n} = (M_\alpha)_n = \frac{1}{1+\alpha^2|\check{n}|^2} P_n J P_n$ for each wavenumber n. The matrix $M_{\alpha n}$ has the eigenvalues, $\pm i\omega_\alpha(n)$, where

$$\omega_\alpha(n) = \frac{\check{n}_3}{(1 + \alpha^2|\check{n}|^2)|\check{n}|} = \omega_{\alpha n}, \quad |\check{n}| = \sqrt{\theta_1 n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2}, \quad \theta_j = \frac{1}{a_j^2}.$$

In Fourier space,

$$\begin{aligned} E_\alpha(\Omega t)_n &= \cos(\Omega \, \omega_{\alpha n} t) I + \frac{1}{|\check{n}|} \sin(\Omega \, \omega_{\alpha n} t) R_n, \\ &= \frac{1}{2} \left[e^{i\Omega \omega_{\alpha n} t} \left(I - i\frac{1}{|\check{n}|} R_n \right) + e^{-i\Omega \omega_{\alpha n} t} \left(I + i\frac{1}{|\check{n}|} R_n \right) \right], \end{aligned}$$

where the matrix iR_n is the Fourier transform of the curl vector; $(\text{curl}\, v)_n = iR_n v_n = i\check{n} \times v_n$ with

$$R_n = \begin{pmatrix} 0 & -\check{n}_3 & \check{n}_2 \\ \check{n}_3 & 0 & -n_1 \\ -\check{n}_2 & n_1 & 0 \end{pmatrix}.$$

Next, we set $V(t) := E_\alpha(\Omega t)v(t)$. Under this transformation, the equation (2) becomes

$$(3) \qquad \frac{\partial V}{\partial t} + \nu AV = B_\alpha(\Omega t, V, V) + E_\alpha(\Omega t)f,$$

where

$$\begin{aligned} B_\alpha(\Omega t, V, V) &= E_\alpha(\Omega t)P_L\{[\mathcal{R}_\alpha E_\alpha(-\Omega t)V] \times [\text{curl}(E_\alpha(-\Omega t)V)]\} \\ &= -E_\alpha(\Omega t)B_\alpha(E_\alpha(-\Omega t)V, E_\alpha(-\Omega t)V). \end{aligned}$$

For each wave number $n \in \mathbb{Z}^3$,

$$B_\alpha(\Omega t, V_k, V_m)_n = i\frac{1}{1+\alpha^2|\check{k}|^2}E_\alpha(\Omega t)_n P_n[E_\alpha(-\Omega t)_k V_k \times (\check{m} \times E_\alpha(-\Omega t)_m V_m)],$$

which is explicitly time-dependent with rapidly varying coefficients. This suggests that, for $\Omega >> 1$, the dynamic mechanisms of (3) evolve over two different time scales; the first one being induced by the fast Poincaré waves and the second given by the evolution of the Poincaré "slow envelope" $V(t)$. $B_\alpha(\Omega t, V, V)$ contains resonant terms (Ωt-independent terms) and nonresonant terms (Ωt-dependent terms), and we can decompose it as

$$B_\alpha(\Omega t, V, V) = \tilde{B}_\alpha(V, V) + B_\alpha^{osc}(\Omega t, V, V).$$

Observe that $B_\alpha^{osc}(\Omega t, V, V)$ contains all nonresonant terms and $\tilde{B}_\alpha(V, V)$, "the resonant bilinear operator", contains all resonant terms. Averaging over fast time scale in the limit $\Omega \to \infty$ removes the nonresonant operator:

$$\lim_{\Omega \to \infty} \frac{1}{2\pi} \int_0^{2\pi} B_\alpha^{osc}(\Omega s, V, V)_n \, ds = 0,$$

and we arrive at the *resonant limit α-equations*;

$$(4) \qquad \begin{aligned} \frac{\partial w}{\partial t} + \nu Aw &= \tilde{B}_\alpha(w, w) + \tilde{f} \\ w(0) &= v(0) \end{aligned}$$

where

$$\begin{aligned} \tilde{B}_\alpha(w, w) &= \lim_{\Omega \to \infty} \frac{1}{2\pi} \int_0^{2\pi} B_\alpha(\Omega s, w, w) \, ds \\ \tilde{f} &= \lim_{\Omega \to \infty} \frac{1}{2\pi} \int_0^{2\pi} E_\alpha(\Omega s)f \, ds. \end{aligned}$$

The existence of regular solutions of the resonant limit α-equations (4) was established in [2, 8] based on rigorous a priori estimates of the (bilinear) resonant limit operator $\tilde{B}_\alpha(w, w)$. The estimates were uniformly in α. Bootstrapping from global regularity of the resonant limit α-equations, the existence of a global regular solution of the full 3D RNS-α for large Ω (Theorem 1.2 and 1.3) was proved. The convergence of the solutions to those of the exact RNS equations as $\alpha \to 0^+$ was also proved in the context of attractors [8].

3. Resonant Set and Operator Splitting

Let H be any Hilbert space and $\overline{u}$ the x_3-averaging of $u \in H$:

$$\overline{u}(t, x_1, x_2) = \frac{1}{2\pi a_3} \int_0^{2\pi a_3} u(t, x_1, x_2, x_3) \, dx_3.$$

Denote $\overline{H} = \{\overline{u}(t, x_1, x_2) | u \in H\}$. Then $\overline{H}$ is a closed subspace of H, and any $u \in H$ has a unique representation $u = \overline{u} + u^\perp \in \overline{H} \bigoplus H^\perp$. Note that $\overline{u^\perp} = 0$. This defines orthogonal projections P_b and $P_b^\perp = I - P_b$ on H as

$$P_b u = \overline{u} \quad \text{and} \quad P_b^\perp u = u^\perp$$

We call P_b a *barotropic projection* and $P_b^\perp$ a *baroclinic projection*, which make an orthogonal decomposition $H = \overline{H} \bigoplus H^\perp$ with $\overline{H} = P_b H$ and $H^\perp = P_b^\perp H$ (For more details, see §3.4, [8]).

Lemma 3.1. *For any finite dimensional subspace $S \subset H$ with $dim(S) = d$, there exists an orthonormal basis $\{\phi_i\}_{1 \leq i \leq d}$ in H, such that $\{\phi_i\}_{1 \leq i \leq d_1}$ and $\{\phi_i\}_{d_1+1 \leq i \leq d}$ orthonormally span $P_b S$ and $P_b^\perp S$, respectively. (Lemma 4.2.2, [9]).*

Let's set $D_l(k, m, n) = \pm\omega_{\alpha k} \pm \omega_{\alpha m} \pm \omega_{\alpha n}$, where $l = 1, 2, ..., 8$ is the combination of signs $\pm$. The resonant nonlinear interactions of Poincaré waves for $\tilde{B}_\alpha(w, w)$ in (4) are present when the Poincaré frequencies satisfy the resonant relation $D_l(k, m, n) = 0$, and we define the corresponding *resonant set* K by

$$K = \left\{ (k, m, n) \in \mathbb{Z}^3 : \pm\omega_{\alpha k} \pm \omega_{\alpha m} \pm \omega_{\alpha n} = 0, \quad n = k + m \right\}.$$

We can decompose the resonant set K into three groups for further analysis; pure 2D interactions (K_{2D}), two wave interactions ($\tilde{K}$), and three wave interactions (K^*).

(i) $K_{2D} = \{(k, m, n) \in K | k_3 = m_3 = n_3 = 0\}$ corresponds to pure two dimensional horizontal interactions (i.e., depends on x_1, x_2 and does not depend on x_3 in physical space.)

(ii) $\tilde{K} = \{(k, m, n) \in K | k_3 m_3 n_3 = 0, \quad k_3^2 + m_3^2 + n_3^2 \neq 0\}$ is the set of two wave resonances. Here $k_3 m_3 n_3 = 0$ represents that one or two of k_3, m_3 and n_3 would be zero. But, if two of them are zero, we have 1-wave interaction which are excluded. It requires the second condition $k_3^2 + m_3^2 + n_3^2 \neq 0$. This is the case when one of the three frequencies ω_α equals zero and two remaining ω_α are nonzero; for example, $\{(k, m, n) \in K | \xi_{\alpha n} = 0, \omega_{\alpha k} + \omega_{\alpha m} = 0, \omega_{\alpha k} \neq 0 \neq \omega_{\alpha m}\} = K_{14} = (K_1 \cap K_4) \setminus K_{2D}$. $\tilde{K}$ can be expressed in the way of

$$\tilde{K} = K_{14} \cup K_{24} \cup K_{34},$$

where $K_{j4} = (K_j \cap K_4) \setminus K_{2D}$ for $j = 1, 2, 3$ and

$$
\begin{aligned}
K_{14} &= \{(k, m, n) \in K | n_3 = 0, \check{k}_3 = -\check{m}_3 \neq 0, |\check{m}| = |\check{k}|\} \\
K_{24} &= \{k_3 = 0, \check{m}_3 = \check{n}_3 \neq 0, |\check{m}| = |\check{n}|\} \\
K_{34} &= \{m_3 = 0, \check{k}_3 = \check{n}_3 \neq 0, |\check{k}| = |\check{n}|\}
\end{aligned}
$$

Formally there exist three more 2-wave cones, but they are empty sets [1].

(iii) $K^* = \{(k, m, n) \in K | k_3 m_3 n_3 \neq 0\}$ is the set of strict three wave resonances.

Then the resonant limit operator $\tilde{B}_\alpha(w, w)$ on K has the following representation:

$$\tilde{B}_\alpha(w, w) = \tilde{B}_I^\alpha(\overline{w}, \overline{w}) + \tilde{B}_{II}^\alpha(\overline{w}, w^\perp) + \tilde{B}_{III}^\alpha(w^\perp, w^\perp)$$

where

$$
\begin{aligned}
\tilde{B}_I^\alpha(\overline{w}, \overline{w}) &= i \sum_{K_{2D}} \frac{1}{1 + \alpha^2 |\check{k}|^2} P_n[\overline{w}_k \times (\check{m} \times \overline{w}_m)] \\
&= -i \sum_{K_{2D}} \frac{1}{1 + \alpha^2 |\check{k}|^2} P_n[(\check{m} \cdot \overline{w}_k)\overline{w}_m] \\
\tilde{B}_{II}^\alpha(\overline{w}, w^\perp) &= -i \sum_{K_{24}} \frac{1}{1 + \alpha^2 |\check{k}|^2} P_n[(\check{m} \cdot \overline{w}_k)w_m^\perp] \\
&\quad + \frac{i}{2} \sum_{K_{24}} \frac{1}{1 + \alpha^2 |\check{k}|^2} P_n[(\overline{w}_k \cdot w_m^\perp)\check{m} + (\check{k} \cdot w_m^\perp)\overline{w}_k] \\
&\quad + \frac{i}{2} \sum_{K_{34}} \frac{1}{1 + \alpha^2 |\check{n}|^2} P_n[(w_k^\perp \cdot \overline{w}_m)\check{m} - (\check{m} \cdot w_k^\perp)\overline{w}_m] \\
&\quad - \frac{i}{2} \sum_{K_{34}} \frac{1}{1 + \alpha^2 |\check{n}|^2} \frac{1}{|\check{n}|^2} P_n \left[\check{n} \times \left\{(\check{k} \times w_k^\perp) \times (\check{m} \times \overline{w}_m)\right\}\right] \\
\tilde{B}_{III}^\alpha(w^\perp, w^\perp) &= \sum_{(k,m,n) \in K^*} Q_{kmn}(w_k^\perp, w_m^\perp),
\end{aligned}
$$

where $Q_{kmn}(v_k, v_m)$ is a bilinear form in $v_k, v_m \in \mathbb{C}^3$ with the estimate

$$|Q_{kmn}(v_k, v_m)| \leq |\check{m}||v_k||v_m|.$$

In this paper, we consider the catalytic α-limit system, that is, the resonant α-limit equations not including the strict three-wave resonant operator B_{III}^α. As pointed out in [1], there exist a generic set of domain parameters (a_1, a_2, a_3), dense in $\mathbf{R}_3^+$ and of full Lebesque measure, for which B_{III}^α is identically zero. Defining

$$\tilde{B}_c^\alpha(w, w) = B_I^\alpha(\bar{w}, \bar{w}) + B_{II}^\alpha(\bar{w}, w^\perp)$$

the resonant limit α-equations (4) take the form

(5)
$$
\begin{aligned}
\frac{dw}{dt} + \nu Aw + \tilde{B}_c^\alpha(w, w) &= \tilde{f} \\
w(0) = w_0, \quad \nabla \cdot w &= 0,
\end{aligned}
$$

where $\tilde{B}_c^\alpha$ is a bilinear operator of the catalytic system;

$$
\begin{aligned}
B_I^\alpha(\bar{w}, \bar{w}) &= P_L(\mathcal{R}_\alpha \bar{w} \cdot \nabla_h)\bar{w} = P_L[\mathcal{R}_\alpha \overline{w} \times \operatorname{curl} \overline{w}] \\
B_{II}^\alpha(\bar{w}, w^\perp) &= P_L(\mathcal{R}_\alpha \bar{w} \cdot \nabla)w^\perp = P_L[\mathcal{R}_\alpha \overline{w} \times \operatorname{curl} w^\perp].
\end{aligned}
$$

Eqs.(5) has a unique solution $w \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ for $\alpha \geq 0$ with $T < \infty$ [8, 9]. Hence the semigroup $S_\alpha(t) : H \to H$ is defined: $S_\alpha(t)w_0 = w(t)$, where $w(t)$ is the solution of (5). The semigroup $S_\alpha(t)$ has an absorbing ball in H and a global attractor $\mathcal{A}_\alpha \subset H$ [8].

4. Attractor Dimensions

We now estimate the dimension of the global attractor $\mathcal{A}_\alpha$. Attractor dimension is associated with the number of degrees of freedom for turbulent rotating fluids for the Coriolis parameter Ω large. The focus is on the estimation of the dimensions uniform in α, no blow up as $\alpha \to 0^+$.

We consider the variational equation corresponding to Eq. (5):

$$(6) \qquad \begin{aligned} \frac{d\Phi}{dt} &= L_c(w)\Phi \\ \Phi(0) &= \xi \end{aligned}$$

with

$$\begin{aligned} L_c(w)\Phi &= -\nu A\Phi - \tilde{B}_c^\alpha(w,\Phi) - \tilde{B}_c^\alpha(\Phi,w) \\ &= -\nu A\Phi - B_I^\alpha(\bar{w},\bar{\Phi}) - B_I^\alpha(\bar{\Phi},\bar{w}) - B_{II}^\alpha(\bar{w},\Phi^\perp) - B_{II}^\alpha(\bar{\Phi},w^\perp) \end{aligned}$$

Following the standard procedure as in section 13.4, [10], we can show that the equations (6) have a unique solution:

Lemma 4.1. *If w is a solution of Eq. (5), then Eq. (6) possesses a unique solution*

$$\Lambda(t,w_0)\xi = \Phi(t) \in L^2(0,T;H^1) \cap C([0,T];H), \ \forall T > 0.$$

Furthermore, for every $t > 0$, the flow $w_0 \to S_{\alpha t}(w_0)$ generated by the limit α-equations (5) is uniformly differentiable on $\mathcal{A}_\alpha$ with the differential

$$DS_{\alpha t}(w_0) = \Lambda(t,w_0) : \xi \in \mathcal{A}_\alpha \to \Phi(t) \in H,$$

where Φ is the solution of (6). In particular, the linear operator $\Lambda(t,w_0)$ is compact for all $t > 0$.

Now we are ready to apply the following theorem to estimate the fractal dimension of the global attractor. Set

$$\mathcal{TR}_n(\mathcal{A}_\alpha) = \sup_{w_0 \in \mathcal{A}_\alpha} \ \sup_{P^{(n)}(0)} \ \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{Tr}\left(L(s;w_0)P^{(n)}(s)\right) ds,$$

where $\mathrm{Tr}(M)$ is the trace of M.

Theorem 4.2. *([10], Theorem 13.16) Suppose that $S_{\alpha t}(t)$ is uniformly differentiable on $\mathcal{A}_\alpha$ and that there exists a t_0 such that $\Lambda(t,w_0)$ is compact for all $t \geq t_0$. If $\mathcal{TR}_n(\mathcal{A}_\alpha) < 0$ then $\dim_f(\mathcal{A}_\alpha) \leq n$.*

Let $\Phi_1,...,\Phi_N$ be solutions of the linearized system (6) with corresponding initial conditions $\xi_1,...,\xi_N$. Let $\phi_1,...,\phi_N$ be the orthonormal system spanning $\{\Phi_1,...,\Phi_N\}$. Then the trace at time $t \geq 0$ can be sought like

$$(7) \qquad Tr\left(L_c(t)P_N(\Phi_1(t),...,\Phi_N(t))\right) = \sum_{i=1}^N \langle L_c(t)\phi_i(t),\phi_i(t)\rangle$$

Using $\phi = \phi + \phi^\perp$ and bilinearity of the operators B_I^α and B_{II}^α, the inner product on the right-handed side of (7) can be expressed as

$$
\begin{aligned}
-\langle L_c(t)\phi_i, \phi_i \rangle \;=\;& \left\langle \nu A\bar{\phi}_i + B_I^\alpha(\bar{w}, \bar{\phi}_i) + B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle \\
& + \left\langle \nu A\phi_i^\perp + B_{II}^\alpha(\bar{w}, \phi_i^\perp) + B_{II}^\alpha(\bar{\phi}_i, w^\perp), \bar{\phi}_i \right\rangle \\
& + \left\langle \nu A\bar{\phi}_i + B_I^\alpha(\bar{w}, \bar{\phi}_i) + B_I^\alpha(\bar{\phi}_i, \bar{w}), \phi_i^\perp \right\rangle \\
& + \left\langle \nu A\phi_i^\perp + B_{II}^\alpha(\bar{w}, \phi_i^\perp) + B_{II}^\alpha(\bar{\phi}_i, w^\perp), \phi_i^\perp \right\rangle
\end{aligned}
$$

$\tilde{B}_\alpha$ is the skew-symmetric limit bilinear form. We collect the following properties from [1, 2, 8, 9].

Lemma 4.3. *For any $w, \phi \in H^1$*

(1) $\left\langle \nu A\bar{\phi}_i, \phi_i^\perp \right\rangle = \nu \left\langle A\bar{\phi}_i, \phi_i^\perp \right\rangle = \nu \left\langle P_b(A\bar{\phi}_i), P_b^\perp(A\phi_i^\perp) \right\rangle = 0.$

(2) $\left\langle \nu A\phi_i^\perp, \bar{\phi}_i \right\rangle = 0.$ *Similar to 1.*

(3) $B_I^\alpha(\bar{w}, \bar{\phi}_i) = P_b\tilde{B}_\alpha(w, \phi_i).$

(4) $\left\langle B_I^\alpha(\bar{w}, \bar{\phi}_i), \phi_i^\perp \right\rangle = \left\langle B_I^\alpha(\bar{w}, \bar{\phi}_i), P_b^\perp \phi_i \right\rangle = \left\langle P_b^\perp B_I^\alpha(\bar{w}, \bar{\phi}_i), \phi_i \right\rangle = \left\langle 0, \phi_i \right\rangle = 0,$ *since P_b and $P_b^\perp$ are self-adjoint.*

(5) *Similarly,* $\left\langle B_I^\alpha(\bar{\phi}_i, \bar{w}), \phi_i^\perp \right\rangle = 0.$

With Lemma 4.3 we can reduce the above inner product.

$$
\begin{aligned}
-\left\langle L_c(t)\vec{\phi}_i, \vec{\phi}_i \right\rangle \;=\;& \left\langle \nu A\bar{\phi}_i + B_I^\alpha(\bar{w}, \bar{\phi}_i) + B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle \\
& + \left\langle B_{II}^\alpha(\bar{w}, \phi_i^\perp) + B_{II}^\alpha(\bar{\phi}_i, w^\perp), \bar{\phi}_i \right\rangle \\
& + \left\langle \nu A\phi_i^\perp + B_{II}^\alpha(\bar{w}, \phi_i^\perp) + B_{II}^\alpha(\bar{\phi}_i, w^\perp), \phi_i^\perp \right\rangle
\end{aligned}
$$

Corollary 4.4. *([9], Cor 3.2.13, p45) If $f \in P_b H, h \in \bar{H}$ and $g \in P_b^\perp H \cap V$, then*

$$\langle B_{II}^\alpha(f, g), h \rangle = 0.$$

Proof. From Lemma 3.4 in [8] with the self-adjoint property of P_b. ($h \in \bar{H}$ so that $h = P_b h$). $\blacksquare$

From this corollary we obtain

$$\left\langle B_{II}^\alpha(\bar{w}, \phi_i^\perp) + B_{II}^\alpha(\bar{\phi}_i, w^\perp), \bar{\phi}_i \right\rangle = 0.$$

From Theorem 4.2 of [8]

$$\left\langle B_{II}^\alpha(\bar{w}_i, \phi_i^\perp), \phi_i^\perp \right\rangle = 0.$$

Thus, by the construction of $(\phi_i)_{1 \le i \le N}$, for each i either $\phi_i = 0$ or $\phi_i^\perp = 0$

$$\left\langle B_{II}^\alpha(\bar{\phi}_i, w^\perp), \phi_i^\perp \right\rangle = 0.$$

([9], p58 with Lemma 4.2.2), and

$$-\left\langle L_c(t)\vec{\phi}_i, \vec{\phi}_i \right\rangle = \left\langle \nu A\bar{\phi}_i + B_I^\alpha(\bar{w}, \bar{\phi}) + B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle + \left\langle \nu A\phi_i^\perp, \phi_i^\perp \right\rangle.$$

In summary,

Lemma 4.5. *Let* $\Phi_1, ..., \Phi_N$ *be solutions of the linearized system (6) with corresponding initial conditions* $\xi_1, ..., \xi_N$. *Let* $\phi_1, ..., \phi_N$ *be the orthonormal system spanning* $\{\Phi_1, ..., \Phi_N\}$ *in* H. *Then the trace at time* $t \geq 0$ *is given by*

$$Tr\left(L_c(t)P_N(\Phi_1(t), ..., \Phi_N(t))\right) = \sum_{i=1}^{N} \langle L_c(t)\phi_i, \phi_i \rangle$$

$$= -\sum_{i=1}^{N} \left[\left\langle \nu A\bar{\phi}_i + B_I^\alpha(\bar{w}, \bar{\phi}) + B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle + \left\langle \nu A\phi_i^\perp, \phi_i^\perp \right\rangle \right]$$

Now let's estimate for the Time-Averaged Trace in Lemma 4.5.

4.1. A Priori Estimates.

4.1.1. *Estimate 1.* Refer to Theorem 4.2.4 (p58) in [9]. We may use

$$\sum_{i=1}^{N_1} \left\langle B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle = \int_{T^2} \sum_{i=1}^{N_1} \left(\mathcal{R}_\alpha \bar{\phi}_i \times \mathrm{curl}\bar{w}\right) \cdot \bar{\phi}_i \, dx.$$

Instead, we will follow the line below. Denote $\bar{\phi}_i = <\bar{\phi}_{i1}, \bar{\phi}_{i2}, \bar{\phi}_{i3}>$. By Lemma 1.1,

$$\sum_{i=1}^{N_1} \left\langle B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle = \sum_{i=1}^{N_1} \left[\left\langle B(\mathcal{R}_\alpha \bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle - \left\langle B(\bar{\phi}_i, \bar{w}), \mathcal{R}_\alpha \bar{\phi}_i \right\rangle \right]$$

$$= \sum_{i=1}^{N_1} \left[\left\langle P_L\left[(\mathcal{R}_\alpha \bar{\phi}_i \cdot \nabla)\bar{w}\right], \bar{\phi}_i \right\rangle - \left\langle P_L\left[(\bar{\phi}_i \cdot \nabla)\bar{w}\right], \mathcal{R}_\alpha \bar{\phi}_i \right\rangle \right]$$

$$= \sum_{i=1}^{N_1} \left[\left\langle (\mathcal{R}_\alpha \bar{\phi}_i \cdot \nabla)\bar{w}, \bar{\phi}_i \right\rangle - \left\langle (\bar{\phi}_i \cdot \nabla)\bar{w}, \mathcal{R}_\alpha \bar{\phi}_i \right\rangle \right]$$

$$\text{(since } P_L \text{ is self-adjoint and } P_L\bar{\phi}_i = \bar{\phi}_i.)$$

$$= \sum_{i=1}^{N_1} \left[\int_{T^2} \sum_{j=1}^{2}\sum_{k=1}^{3} \left(\mathcal{R}_\alpha \bar{\phi}_i\right)_j D_j \bar{w}_k \bar{\phi}_{ik} \, dx - \int_{T^2} \sum_{j=1}^{2}\sum_{k=1}^{3} \bar{\phi}_{ij} D_j \bar{w}_k \left(\mathcal{R}_\alpha \bar{\phi}_i\right)_k \, dx \right]$$

$$= \int_{T^2} \sum_{i=1}^{N_1}\sum_{j=1}^{2}\sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i\right)_j D_j \bar{w}_k \bar{\phi}_{ik} - \bar{\phi}_{ij} D_j \bar{w}_k \left(\mathcal{R}_\alpha \bar{\phi}_i\right)_k \right] dx$$

Then,

$$\left| \int_{T^2} \sum_{i=1}^{N_1}\sum_{j=1}^{2}\sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i\right)_j D_j \bar{w}_k \bar{\phi}_{ik} - \bar{\phi}_{ij} D_j \bar{w}_k \left(\mathcal{R}_\alpha \bar{\phi}_i\right)_k \right] dx \right| \leq$$

$$\left| \int_{T^2} \sum_{i=1}^{N_1}\sum_{j=1}^{2}\sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i\right)_j D_j \bar{w}_k \bar{\phi}_{ik} \right] dx \right| + \left| \int_{T^2} \sum_{i=1}^{N_1}\sum_{j=1}^{2}\sum_{k=1}^{3} \left[\bar{\phi}_{ij} D_j \bar{w}_k \left(\mathcal{R}_\alpha \bar{\phi}_i\right)_k \right] dx \right|$$

We only need to estimate the first term on the right-handed side because the second term can be estimates similarly.

$$\left| \int_{T^2} \sum_{i=1}^{N_1} \sum_{j=1}^{2} \sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i \right)_j D_j \bar{w}_k \bar{\phi}_{ik} \right] dx \right| \leq \int_{T^2} \left| \sum_{i=1}^{N_1} \sum_{j=1}^{2} \sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i \right)_j D_j \bar{w}_k \bar{\phi}_{ik} \right] \right| dx$$

Here,

$$\left| \sum_{i=1}^{N_1} \sum_{j=1}^{2} \sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i \right)_j D_j \bar{w}_k \bar{\phi}_{ik} \right] \right| \leq \sum_{i=1}^{N_1} \left| \sum_{j=1}^{2} \sum_{k=1}^{3} \left[\left(\mathcal{R}_\alpha \bar{\phi}_i \right)_j D_j \bar{w}_k \bar{\phi}_{ik} \right] \right|$$

$$\leq \sum_{i=1}^{N_1} \left\{ \left[\sum_{j=1}^{2} \sum_{k=1}^{3} (D_j \bar{w}_k(x))^2 \right]^{1/2} \left[\sum_{j=1}^{2} \sum_{k=1}^{3} \left(\mathcal{R}_\alpha \bar{\phi}_i(x) \right)_j^2 \bar{\phi}_{ik}^2(x) \right]^{1/2} \right\}$$

$$= |\nabla \bar{w}(x)| \sum_{i=1}^{N_1} \left[\sum_{j=1}^{2} \left(\mathcal{R}_\alpha \bar{\phi}_i(x) \right)_j^2 \sum_{k=1}^{3} \bar{\phi}_{ik}^2(x) \right]^{1/2}$$

$$\leq |\nabla \bar{w}(x)| \sum_{i=1}^{N_1} \left[\sum_{j=1}^{3} \left(\mathcal{R}_\alpha \bar{\phi}_i(x) \right)_j^2 \sum_{k=1}^{3} \bar{\phi}_{ik}^2(x) \right]^{1/2}$$

$$= |\nabla \bar{w}(x)| \sum_{i=1}^{N_1} \left\{ \left[\sum_{j=1}^{3} \left(\mathcal{R}_\alpha \bar{\phi}_i(x) \right)_j^2 \right]^{1/2} \left[\sum_{k=1}^{3} \bar{\phi}_{ik}^2(x) \right]^{1/2} \right\}$$

$$\leq |\nabla \bar{w}(x)| \left[\sum_{i=1}^{N_1} \sum_{j=1}^{3} \left(\mathcal{R}_\alpha \bar{\phi}_i(x) \right)_j^2 \right]^{1/2} \left[\sum_{i=1}^{N_1} \sum_{k=1}^{3} \bar{\phi}_{ik}^2(x) \right]^{1/2}$$

$$= |\nabla \bar{w}(x)| \left[\sum_{i=1}^{N_1} \left| \mathcal{R}_\alpha \bar{\phi}_i(x) \right|^2 \right]^{1/2} \left[\sum_{i=1}^{N_1} \left| \bar{\phi}_i(x) \right|^2 \right]^{1/2}$$

$$\leq |\nabla \bar{w}(x)| \left[\sum_{i=1}^{N_1} \left| \mathcal{R}_\alpha \bar{\phi}_i(x) \right|^2 \right]^{1/2} \left(\sum_{i=1}^{N_1} \left| \bar{\phi}_i(x) \right|^2 \right)$$

(Why do we need the last inequality? It is to get the estimate $||\rho||_0^2 \leq c_l \sum ||\bar{\phi}_i||^2$ so that we can combine two nonlinear-term estimates to have $||\bar{w}||^2$ and be able to estimate $\int_t^{t+\tau} ||\bar{w}(s)||^2 \, ds$. Otherwise, we will have $||\rho^{1/2}||_0 = N_1$ and get additional $||\bar{w}||$ term and have to estimate $\int_t^{t+\tau} ||\bar{w}(s)|| \, ds$: $||\rho^{1/2}||_0^2 = \int_\Omega |\rho^{1/2}(x)|^2 \, dx = \sum_{i=1}^{N_1} \int_\Omega |\bar{\phi}_i(x)|^2 dx = \sum_{i=1}^{N_1} ||\bar{\phi}_i||_0^2 = N_1$).

Integrating both sides and using Hölder's inequality, we obtain

$$\int_{T^2} |\nabla \bar{w}(x)| \left[\sum_{i=1}^{N_1} |\mathcal{R}_\alpha \bar{\phi}_i(x)|^2 \right]^{1/2} \left[\sum_{i=1}^{N_1} |\bar{\phi}_i(x)|^2 \right] dx \leq ||\rho_\alpha||_\infty \int_{T^2} |\nabla \bar{w}(x)| \rho(x) \, dx$$

$$\leq ||\rho_\alpha||_\infty ||\bar{w}|| |\rho|_{L^2},$$

where $\rho_\alpha(x) = \left[\sum_{i=1}^{N_1} |\mathcal{R}_\alpha \bar{\phi}_i(x)|^2\right]^{1/2}$ and $\rho(x) = \sum_{i=1}^{N_1} |\bar{\phi}_i(x)|^2$ with $||\rho_\alpha||_\infty = \sup_{x \in \mathbb{T}^3} |\rho_\alpha(x)|$. Therefore, we get the estimate

$$(8) \qquad \left|\sum_{i=1}^{N_1} \left\langle B_I^\alpha(\bar{\phi}_i, \bar{w}), \bar{\phi}_i \right\rangle\right| \leq 2||\rho_\alpha||_\infty |\rho|_{L^2} ||\bar{w}||$$

4.1.2. *Estimate 2 ($||\rho_\alpha||_\infty$ Estimate).* Let $\theta_i = \bar{\phi}_i$ and $v_i = (1 - \alpha^2\Delta)^{-1}\theta_i$. Then,

$$\rho_\alpha(x) = \left[\sum_{i=1}^{N_1} \left|(1 - \alpha^2\Delta)^{-1}\theta_i\right|^2\right]^{1/2} = \left[\sum_{i=1}^{N_1} |v_i|^2\right]^{1/2}$$

For $\theta = \theta_i \in H$, we have $v = v_i \in H^3$. With the Sobolev embedding theorem, we can infer that the existence of a dimensionless constant $c(\alpha)$ depending on α such that

$$||v||_\infty \leq C||v||_2 = C||(1 - \alpha^2\Delta)^{-1}\theta||_2 \leq c(\alpha)||\theta||_0.$$

Now we will compute the conatnat $c(\alpha)$, following the same process as in [7]. Suppose that $\xi_1, \xi_2, ..., \xi_m \in \mathbb{R}$ and $\sum_{j=1}^{m} \xi_j^2 = 1$. Then, using the orthonormality of the θ_j $(j = 1, 2, ..., m = N_1)$, and the above inequality we obtain

$$
\begin{aligned}
\left|\sum_{j=1}^{m} \xi_j v_j(x)\right| &\leq c(\alpha)||\sum_{j=1}^{m} \xi_j \theta_j|| \\
&= c(\alpha)\left[\int \left(\sum_{j=1}^{m} \xi_j \theta_j(x)\right)^2 dx\right]^{1/2} \\
&= c(\alpha)\left(\sum_{i,j=1}^{m} \xi_i \xi_j \delta_{ij}\right)^{1/2} = c(\alpha)\left(\sum_{j=1}^{m} \xi_j^2\right)^{1/2} = c(\alpha)
\end{aligned}
$$

Using the representation $v_j = v_{j1} \cdot e_1 + v_{j2} \cdot e_2$ we find that

$$\left[\sum_{j=1}^{m} \xi_j v_j(x)\right]^2 \leq \left(\sum_{j=1}^{m} \xi_j v_{j1}(x)\right)^2 + \left(\sum_{j=1}^{m} \xi_j v_{j2}(x)\right)^2 \leq c(\alpha)^2$$

First, we set

$$\xi_j = \frac{v_{j1}(x)}{(\sum_{j=1}^{m}(v_{j1}(x))^2)^{1/2}}$$

and later set

$$\xi_j = \frac{v_{j2}(x)}{(\sum_{j=1}^{m}(v_{j2}(x))^2)^{1/2}}.$$

Substituting these into the above inequality one after another, we obtain

$$\rho_\alpha^2(x) = \sum_{j=1}^{m} |v_j(x)|^2 = \sum_{j=1}^{m}(v_{j1}(x))^2 + \sum_{j=1}^{m}(v_{j2}(x))^2 \leq 2c(\alpha)^2.$$

To compute $c(\alpha)$, we use the Fourier Series

$$\theta_j(x) = \sum_{k \in Z_0^2} a_{jk} e^{ik \cdot x}, \quad Z_0^2 = \mathbb{Z}^2 \{0\}, \quad x = (x_1, x_2), \quad k = (k_1, k_2)$$

so that

$$(1 - \alpha^2 \Delta)^{-1} \theta_j(x) = \sum_{k \in Z_0^2} \frac{a_{jk}}{1 + \alpha^2 |k|^2} e^{ik \cdot x}.$$

Thus,

$$
\begin{aligned}
|(1 - \alpha^2 \Delta)^{-1} \theta_j(x)| &\leq \sum_{k \in Z_0^2} \frac{|a_{jk}|}{1 + \alpha^2 |k|^2} \\
&\leq \left(\sum_{k \in Z_0^2} \frac{1}{(1 + \alpha^2 |k|^2)^2} \right)^{1/2} \left(\sum_{k \in Z_0^2} |a_{jk}|^2 \right)^{1/2} \\
&= c_j(\alpha) \|\theta_j\|_0 \\
&= c_j(\alpha), \quad (\text{since } \|\theta_j\|_0 = 1),
\end{aligned}
$$

where

$$
\begin{aligned}
c_j^2(\alpha) &= \sum_{k \in Z_0^2} \frac{1}{(1 + \alpha^2 |k|^2)^2} \\
&= \sum_{p=1}^{\infty} \frac{1}{(1 + \alpha^2 \lambda_p)^2}, \quad \text{where} \\
&\{\lambda_p, p = 1, 2, ...\} = \{\hat{k}^2 = \hat{k}_1^2 + \hat{k}_2^2, \hat{k}_i = k_i / a_i \text{ for } i = 1, 2, (k_1, k_2) \in Z_0^2\} \\
&\leq \sum_{p=1}^{\infty} \frac{1}{(1 + \alpha^2 c_1 p)^2} \quad \text{for an absolute constant } c_1 \\
&= \frac{1}{(1 + \alpha^2 c_1)^2} + \sum_{p=2}^{\infty} \frac{1}{(1 + \alpha^2 c_1 p)^2} \\
&\leq \frac{1}{(1 + \alpha^2 c_1)^2} + \int_1^{\infty} \frac{dx}{(1 + \alpha^2 c_1 x)^2} \\
&= \frac{1}{1 + \alpha^2 c_1} \left[\frac{1}{1 + \alpha^2 c_1} + \frac{1}{\alpha^2 c_1} \right]
\end{aligned}
$$

Then we can set

$$\sum_{j=1}^{N_1} c_j^2(\alpha) \leq \frac{N_1}{1 + \alpha^2 c_1} \left[\frac{1}{1 + \alpha^2 c_1} + \frac{1}{\alpha^2 c_1} \right].$$

Therefore,

$$\|\rho_\alpha\|_\infty^2 \leq 2 N_1 c^2(\alpha),$$

where

$$c^2(\alpha) = \frac{1}{1 + \alpha^2 c_1} \left[\frac{1}{1 + \alpha^2 c_1} + \frac{1}{\alpha^2 c_1} \right].$$

Observe that $\lim_{\alpha \to 0^+} c^2(\alpha) = 2 < \infty$.

4.1.3. *Estimate 3.* This is a new term that the exact Navier-Stokes equations don't have.

$$
\sum_{i-1}^{N_1} \left\langle B_I^\alpha(\bar{w}, \bar{\phi}_i, \bar{\phi}_i) \right\rangle \;=\; \sum_{i=1}^{N_1} \left[\left\langle B(\mathcal{R}_\alpha \bar{w}, \bar{\phi}_i), \bar{\phi}_i \right\rangle - \left\langle B(\bar{\phi}_i, \bar{\phi}_i), \mathcal{R}_\alpha \bar{w} \right\rangle \right]
$$

$$
\left(\text{Note that } \left\langle B(\mathcal{R}_\alpha \bar{w}, \bar{\phi}_i), \bar{\phi}_i \right\rangle = 0 \text{ by skew-symmetry} \right)
$$

$$
=\; \sum_{i=1}^{N_1} \left\langle B(\bar{\phi}_i, \mathcal{R}_\alpha \bar{w}), \bar{\phi}_i \right\rangle \quad \text{by skew-symmetry}
$$

$$
=\; \sum_{i=1}^{N_1} \left[\int_{T^2} \sum_{j=1}^{2} \sum_{k=1}^{3} \bar{\phi}_{ij} D_j(\mathcal{R}_\alpha \bar{w}_k) \bar{\phi}_{ik}\, dx \right]
$$

Then,

$$
\left| \sum_{i=1}^{N_1} \left\langle B_I^\alpha(\bar{w}, \bar{\phi}_i, \bar{\phi}_i) \right\rangle \right| \;=\; \left| \int_{T^2} \sum_{i=1}^{N_1} \sum_{j=1}^{2} \sum_{k=1}^{3} \bar{\phi}_{ij} D_j(\mathcal{R}_\alpha \bar{w}_k) \bar{\phi}_{ik}\, dx \right|
$$

$$
\leq\; \int_{T^2} |\nabla(\mathcal{R}_\alpha \bar{w}_k)| \left[\sum_{j=1}^{2} \sum_{k=1}^{3} \left(\sum_{i=1}^{N_1} \bar{\phi}_{ij}(x) \bar{\phi}_{ik}^2(x) \right)^2 \right]^{1/2} dx
$$

$$
\leq\; \int_{T^2} |\nabla(\mathcal{R}_\alpha \bar{w}_k)| \left[\sum_{j=1}^{3} \sum_{k=1}^{3} \left(\sum_{i=1}^{N_1} \bar{\phi}_{ij}(x) \bar{\phi}_{ik}^2(x) \right)^2 \right]^{1/2} dx
$$

$$
\leq\; \int_{T^2} |\nabla(\mathcal{R}_\alpha \bar{w}_k)|\, \rho(x)\, dx, \qquad \text{where} \quad \rho(x) = \sum_{i=1}^{N_1} |\bar{\phi}_i(x)|^2
$$

$$
\leq\; ||\mathcal{R}_\alpha \bar{w}||\, |\rho|_{L^2}
$$

$$
\leq\; ||\bar{w}||\, |\rho|_{L^2}
$$

$$
\leq\; ||\bar{w}|| \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{1/2}
$$

(by the Lieb-Thirring inequality; see p59 [9] for detail),

where c_l is an absolute constant and

$$
|\nabla(\mathcal{R}_\alpha \bar{w}_k)| = \left(\sum_{j=1}^{2} \sum_{k=1}^{3} |D_j(\mathcal{R}_\alpha \bar{w}_k)| \right)^{1/2}.
$$

4.1.4. *Collection of Estimate 1, 2, and 3.*

$Tr(L_1(t)P_N(\bar{\Phi}_1(t), ..., \bar{\Phi}_N(t))$

$$\leq -\nu \sum_{i=1}^{N_1} ||\bar{\phi}_i(t)||^2 + 2||\rho_\alpha||_\infty \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{\frac{1}{2}} ||\bar{w}|| + ||\bar{w}|| \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{\frac{1}{2}}$$

$$\leq -\nu \sum_{i=1}^{N_1} ||\bar{\phi}_i(t)||^2 + 2\sqrt{2N_1}c(\alpha)||\bar{w}|| \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{\frac{1}{2}} + ||\bar{w}|| \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{\frac{1}{2}}$$

$$= -\nu \sum_{i=1}^{N_1} ||\bar{\phi}_i(t)||^2 + \left(2\sqrt{2N_1}c(\alpha) + 1 \right) ||\bar{w}|| \left(c_l \sum_{i=1}^{N_1} ||\bar{\phi}_i||^2 \right)^{1/2}$$

$$\leq -\frac{\nu}{2} \sum_{i=1}^{N_1} ||\bar{\phi}_i(t)||^2 + \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu} ||\bar{w}||^2 \ \text{ by Young's inequality}$$

$$\leq -c_0 \nu \lambda_1 \frac{N_1(N_1 + 1)}{4} + \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu} ||\bar{w}||^2$$

Hence,

$$\frac{1}{t} \int_0^t Tr(L_1(s)P_N(\bar{\Phi}_1(s), ..., \bar{\Phi}_N(s)) \, ds \leq$$

$$-c_0 \nu \lambda_1 \frac{N_1(N_1 + 1)}{4} + \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu} \frac{1}{t} \int_0^t ||\bar{w}(s)||^2 \, ds$$

The estimate of the remaining trace follows [9, p 60]:

$$\frac{1}{t} \int_0^t Tr(AP_N(\Phi_1^\perp(s), ..., \Phi_N^\perp(s)) \, ds \geq \frac{3}{10} c_0 \lambda_1 \nu N_2^{5/3}.$$

Therefore,

$$\frac{1}{t} \int_0^t Tr(L_c(s)P_N(\vec{\Phi}_1(s), ..., \vec{\Phi}_N(s)) \, ds \leq -\nu \lambda_1 c_0 \frac{N_1^2 + N_2^{5/3}}{4}$$

$$+ \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu} \frac{1}{t} \int_0^t ||\bar{w}(s)||^2 \, ds.$$

4.2. **Dimensions.** Let $q_N(t) = \frac{1}{t} \int_0^t Tr(L_c(s)P_N(\vec{\Phi}_1(s), ..., \vec{\Phi}_N(s)) \, ds$. Then,

$$q_N(t) \ \leq \ \nu \lambda_1 \left(-\frac{c_0}{4}(N_1^2 + N_2^{5/3}) + \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu^2} \frac{1}{t\lambda_1} \int_0^t ||\bar{w}(s)||^2 \, ds \right)$$

$$\limsup_{t \to \infty} q_N(t) \ \leq \ \nu \lambda_1 \left(-\frac{c_0}{4}(N_1^2 + N_2^{5/3}) + \frac{\left(2\sqrt{2N_1}c(\alpha) + 1 \right)^2 c_l}{2\nu^2} \epsilon \right),$$

where $\epsilon = \nu \lambda_1 \limsup_{t \to \infty} \sup_{w_0 \in X} \frac{1}{t\lambda_1} \int_0^t ||\bar{w}(s)||^2 \, ds$ with $X = \mathcal{A}_\alpha$.

To estimate q_N in terms of N ($= N_1 + N_2$), a technical lemma is needed.

Lemma 4.6. *([9]) Let $q \geq 0$. Then there exists a constant $c = c(q) > 0$ such that*

$$x^q + y^q \geq c(x + y)^q$$

for all $x, y \geq 0$.

Proof. The result is true when $q = 0$ with $c = 1$ for all $x, y \geq 0$. It is also valid when $x = y = 0$ for any $q > 0$. Thus it will be sufficient to prove that, for any $q > 0$, there exists a constant $c > 0$ such that

$$\frac{x^q + y^q}{(x+y)^q} = \frac{1 + (y/x)^q}{(1 + (y/x))^q} \geq c,$$

whenever $x, y > 0$. By setting $z = y/x$, we can define a strictly positive and continuous function on $(0, \infty)$

$$f(z) = \frac{1 + z^q}{(1+z)^q}$$

Notice that

$$\lim_{z \to 0^+} f(z) = \lim_{z \to \infty} f(z) = 1,$$

and we can continuously extend f on $(0, \infty)$ to f_e on $[0, \infty)$. Clearly $f_e(0) = 1$ and since $\lim_{z \to 0^+} f_e(z) = 1$, there exists $z_0 > 0$ such that

$$f_e(z) \geq \frac{1}{2}$$

whenever $z \geq z_0$. Since $[0, z_0]$ is compact, f_e has an absolute minimum $m_0 = f_e(z_1) > 0$ for some $z_1 \in [0, z_0]$. Choosing $c = c(q) = \min\{1/2, m_0\}$ proves the lemma. ∎

We can make the constant c in the Lemma 4.6 more precisely.

Corollary 4.7.

$$c(q) = \begin{cases} 1 & if \quad 0 < q \leq 1 \\ \frac{1}{2^{q-1}} & if \quad q > 1. \end{cases}$$

Proof. We can assume $z_0 > 1$ without loss of generality. To find a minimum $m_0 > 0$ of f_e on $[0, z_0]$,

$$\begin{aligned} f_e'(z) = 0 \quad &\Leftrightarrow \quad qz^{q-1}(1+z)^q - q(1+z^q)(1+z)^{q-1} = 0 \\ &\Leftrightarrow \quad q(1+z)^{q-1}(z^{q-1} - 1) = 0 \\ &\Leftrightarrow \quad z^{q-1} - 1 = 0, \text{ since } q > 0 \text{ and } (1+z)^{q-1} \neq 0 \\ &\Leftrightarrow \quad z^{q-1} = 1. \end{aligned}$$

- Case 1. When $q = 1$
  $z^0 = 1$ for all $z \in (0, z_0)$; i.e., $f_e'(z) = 0$ so that $f_e(z) = 1$ for all $z \in (0, z_0)$. Since $f_e(0) = 1$ and $m_0 = f_e(z_0) = 1$ by the continuity of f_e on $[0, z_0]$. It implies $c = 1$.
- Case 2. When $0 < q < 1$

$$z^{q-1} = 1 \Leftrightarrow \frac{1}{z^{1-q}} = 1 \Leftrightarrow z^{1-q} = 1$$

Taking the logarithm of both sides yields

$$(1-q)\ln z = 0 \Leftrightarrow z = 1 \text{ since } 1 - q > 0.$$

By the first derivative test in calculus, f_e has a local maximum at $z = 1$ on $[0, z_0]$, with $f_e(1) = 2/2^q > 1$, so that the minimum must be $f_e(0) = 1$, which is less than $f_e(z_0)$. Thus $c = 1$.

- Case 3. When $q > 1$
  Similar to Case 2 with $q > 1$. f_e has an absolute minimum at $z = 1$ on $[0, z_0]$, with $f_e(1) = 2/2^q = 2^{1-q} < 1$. Thus, $c = 2^{1-q}$.

This proves the corollary. ∎

By Corollary 4.7,

$$N_1^2 + N_2^{5/3} \geq N_1^{5/3} + N_2^{5/3} \geq c \left(\frac{5}{3} \right) N^{5/3} = \left(\frac{1}{4} \right)^{1/3} N^{5/3}.$$

So,

$$
\begin{aligned}
\lim_{t \to \infty} q_N(t) &\leq \nu \lambda_1 \left(-dN^{5/3} + \frac{(2\sqrt{2}\sqrt{N}c(\alpha) + 1)^2 c_l}{2\nu^2} \epsilon \right), \quad \text{where } d = \frac{c_0}{4\sqrt[3]{c}} \\
&= \nu \lambda_1 \left(-dN^{5/3} + \frac{(8Nc^2(\alpha) + 4\sqrt{2}\sqrt{N}c(\alpha) + 1)c_l}{2\nu^2} \epsilon \right) \\
&\leq \nu \lambda_1 \left(-dN^{5/3} + \frac{8\sqrt{2}Nc(\alpha) + 4\sqrt{2}Nc(\alpha) + 1}{\nu^2} c_l \epsilon \right) \\
&\quad \text{since } c^2(\alpha) < 2 \text{ and } N \geq 1 \\
&\leq \nu \lambda_1 \left(-dN^{5/3} + \frac{24Nc(\alpha) + 1}{\nu^2} c_l \epsilon \right) \\
&\leq \nu \lambda_1 \left(-dN^{5/3} + \frac{N(24c(\alpha) + 1)}{\nu^2} c_l \epsilon \right)
\end{aligned}
$$

We want to find the smallest $N > 0$ such that

$$-dN^{5/3} + \frac{N(24c(\alpha) + 1)}{\nu^2} c_l \epsilon < 0.$$

Setting the nonlinear equation

$$(9) \qquad N^{5/3} = \frac{N(24c(\alpha) + 1)}{d\nu^2} c_l \epsilon$$

yields

$$
\begin{aligned}
N &= \frac{1}{\nu^3} \sqrt{\frac{c_l^3 \epsilon^3 (24c(\alpha) + 1)^3}{d^3}} \\
&= \left(\frac{c_l}{d} \right)^{3/2} \left(\frac{(24c(\alpha) + 1)\epsilon}{\nu^2} \right)^{3/2} \\
&= \left(\frac{c_l}{d} \right)^{3/2} (24c(\alpha) + 1)^{3/2} \left(\frac{\epsilon}{\nu^2} \right)^{3/2}
\end{aligned}
$$

so that

$$N \geq \left(\frac{c_l}{d} \right)^{3/2} (24c(\alpha) + 1)^{3/2} \left(\frac{\epsilon}{\nu^2} \right)^{3/2}.$$

Hence,

$$
\begin{aligned}
d_H(\mathcal{A}_\alpha) \quad &< \quad \left(\frac{c_l}{d}\right)^{3/2} (24c(\alpha)+1)^{3/2} \left(\frac{\epsilon}{\nu^2}\right)^{3/2} \\
&< \quad \left(\frac{c_l}{d}\right)^{3/2} (24c(\alpha)+1)^{3/2}\tilde{c}^{3/2} \left(\frac{\rho_V}{\nu^2}\right)^3 \quad \text{since} \ \ \epsilon < \tilde{c}\rho_V^2 \\
&= \quad K(\alpha) \left(\frac{\rho_V}{\nu^2}\right)^3
\end{aligned}
$$

where

$$
K(\alpha) = \left(\frac{c_l}{d}\right)^{3/2} (24c(\alpha)+1)^{3/2}\tilde{c}^{3/2} \to \left(\frac{(24\sqrt{2}+1)c_l\tilde{c}}{d}\right)^{3/2} \equiv K_0 \ \text{as} \ \alpha \to 0^+
$$

In particular, when $\alpha = 0$, the exact rotating Navier-Stokes equations don't have the second term of the first inequality on page 17, and we get better estimate:

$$
d_H(\mathcal{A}_0) < \tilde{K} \left(\frac{\rho_{v_0}}{\nu_0}\right)^{6/5}
$$

This completes the proof of our main result Theorem 1.4.

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P.O. Box 10021, Department of Mathematics & Natural Sciences, American University of Ras Al Khaimah, United Arab Emirates
Email address: bkim@aurak.ac.ae