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A NOTE ON THE (h, q)-CHANGHEE NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we introduce a new q-analogue of the Changhee numbers and polynomials of the first kind and the second kind of order r, which are called the Witt-type formula for the q-analogue of Changhee polynomials of order r. We can derive some new interesting identities related to the q-Changhee polynomials of order r.

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1. INTRODUCTION

In recent years, the various special polynomials have been defined by using p-adic q-integral on $\mathbb{Z}_p$ and introduced by T. Kim (see [9, 10, 11, 12]). Let p be chosen as a fixed odd prime number. Throughout this paper, we make use of the following notations. $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = p^{-1}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic q-invariant integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

(1)
$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [1, 2, 6, 13]).}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1), we get

$$qI_{-q}(f_1) + I_q(f) = [2]_q f(0). \tag{2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l, \tag{3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$(e^t-1)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!}, \text{ (see [3, 5, 15, 16, 17]).} \tag{4}$$

Unsigned Stirling numbers of the first kind is given by

$$x^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n} |S_1(n,l)|x^l. \tag{5}$$

Note that if we replace x to $-x$ in (3), then

$$\begin{aligned}(-x)_n =&(-1)^n x^{(n)} = \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l\\ =&(-1)^n\sum_{l=0}^{n}|S_1(n,l)|x^l.\end{aligned} \tag{6}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

Recently, D. S. Kim et. al. introduced the *Changhee polynomials of the first kind of order* r are defined by the generating function to be

$$\left(\frac{2}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \text{ (see [8]),} \tag{7}$$

and the *Changhee polynomials of the second kind of order* r are given by

$$\left(\frac{2(1+t)}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{Ch}_n^{(r)}(x)\frac{t^n}{n!}, \text{ (see [8]),}$$

and E. -J. Moon et. al. defined the *q-Changhee polynomials of order* r as follows.

$$\left(\frac{1+q}{q(1+t)+1}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [14]).}$$

In recent years, several authors have studies the various generalization of Changhee polynomials (see [4, 7, 8, 13, 14]), and in [1], authors give new q-analogue of Changhee numbers and polynomials.

In this paper, we introduce a new q-analogue of the Changhee numbers and polynomials of the first kind and the second kind of order r, which are called the Witt-type formula for the q-analogue of Changhee polynomials of order r. We can derive some new interesting identities related to the q-Changhee polynomials of order r.

2. On a q-analogue of Changhee numbers and polynomials of order r

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. First, we consider the following integral representation associated with the Pochhammer symbol :

$$(8)\qquad \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} q^{\sum_{i=1}^{r} h_i y_i}(x+y_1+\cdots+y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r),$$

where $n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$, $h_1,\ldots,h_r\in\mathbb{Z}$ and $r\in\mathbb{N}$. By (8),

(9)

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i}(x+y_1+\cdots+y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!}\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i}\left(\sum_{n=0}^{\infty}\binom{x+y_1+\cdots+y_r}{n}t^n\right)d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i}(1+t)^{x+y_1+\cdots+y_r}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r),\end{aligned}$$

where $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$. By (2) and (9), we have

$$(10)\qquad \begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i}(x+y_1+\cdots+y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!}\\ =&\prod_{i=1}^{r}\left(\frac{[2]_q}{q^{h_i+1}(1+t)+1}\right)(1+t)^x.\end{aligned}$$

If we put

$$F_q^{(h_1,\ldots,h_r)}(x,t)=\prod_{i=1}^{r}\left(\frac{[2]_q}{q^{h_i+1}(1+t)+1}\right)(1+t)^x,$$

then

$$F_q^{(-1,\ldots,-1)}(x,t)=\left(\frac{[2]_q}{2+t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}Ch_{n,q}^{(r)}(x)\frac{t^n}{n!},$$

and

$$\lim_{q\to1}F_q^{(-1,\ldots,-1)}(x,t)=\left(\frac{2}{2+t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}Ch_{n}^{(r)}(x)\frac{t^n}{n!}.$$

Thus, $F_q^{(h_1,\ldots,h_r)}(x,t)$ seems to be a new q-extension of the generating function for the Changhee polynomials of the first kind of order r. Thus, by (7) and (9), we obtain the following definition.

Definition 2.1. A (h,q)-*Changhee polynomials of the first kind* is defined by the generating function to be

$$\sum_{n=0}^{\infty}Ch_{n,q}^{(h_1,\ldots,h_r)}(x)\frac{t^n}{n!}=\prod_{i=1}^{r}\left(\frac{[2]_q}{q^{h_i+1}(1+t)+1}\right)(1+t)^x.$$

Moreover,

$$Ch_{n,q}^{(h_1,\ldots,h_r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i}(x+y_1+\cdots+y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r).$$

In the special case $x=0$ in Definition 2.1, $Ch_{n,q}^{(h_1,\dots,h_r)}(0|q) = Ch_{n,q}^{(h_1,\dots,h_r)}(0)$ is called the *nth* (h,q)*-Changhee numbers of the first kind of order* r. Note that, by (7) and Definition 2.1,

$$Ch_{n,q}^{(-1,\dots,-1)}(x) = \left(\frac{q+1}{2}\right)^r Ch_n^{(r)}(x). \tag{11}$$

The equation (11) shows that the (h,q)-Changhee polynomials of the first kind of order r is closely related the nth Changhee polynomials of order r.

It is easy to show that

$$\begin{aligned}&\prod_{i=1}^{r}\left(\frac{[2]_q}{q^{h_i+1}(1+t)+1}\right)(1+t)^x\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Ch_{n-m,q}^{(h_1,\dots,h_r)}(x)_m\right)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

By Definition 2.1 and (12), we have

$$\begin{aligned}Ch_{n,q}^{(h_1,\dots,h_r)}(x) &= \sum_{m=0}^{n}\binom{x}{m}Ch_{n-m,q}^{(h_1,\dots,h_r)}\frac{n!}{n-m!}\\ &= \sum_{m=0}^{n}\binom{x}{n-m}Ch_{m,q}^{(h_1,\dots,h_r)}\frac{n!}{m!}.\end{aligned} \tag{13}$$

Since

$$\begin{aligned}(x+y_1+\cdots+y_r)_n &= \sum_{l=0}^{n}S_1(n,l)(x+y_1+\cdots+y_r)^l\\ &= \sum_{l=0}^{n}S_1(n,l)\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\dots,l_r\ge 0}} y_1^{l_1}y_2^{l_2}\cdots(x+y_r)^{l_r},\end{aligned} \tag{14}$$

by Definition 2.1 and (13), we have

$$\begin{aligned}&Ch_{n,q}^{(h_1,\dots,h_r)}(x)\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}\sum_{l=0}^{n}S_1(n,l)\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\dots,l_r\ge 0}} y_1^{l_1}y_2^{l_2}\cdots(x+y_r)^{l_r}\\ =&\sum_{l=0}^{n}S_1(n,l)\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\dots,l_r\ge 0}} E_{l_1,q}^{(h_1)}\cdots E_{l_{r-1},q}^{(h_{r-1})}E_{l_r,q}^{(h_r)}(x),\end{aligned} \tag{15}$$

where $E_{n,q}^{(h)}(x)$ are the (h,q)-Euler polynomials derived from

$$E_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{hy}(x+y)^n d\mu_{-q}(y),\ \text{(see [?])}.$$

Thus, by (13) and (15), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\begin{aligned} Ch_{n,q}^{(h_1,\ldots,h_r)}(x) &= \sum_{m=0}^{n} \binom{x}{n-m} Ch_{m,q}^{(h_1,\ldots,h_r)} \frac{n!}{m!} \\ &= \sum_{l=0}^{n} \sum_{\substack{l_1+\cdots+l_r=l \\ l_1,\ldots,l_r \geq 0}} S_1(n,l) E_{l_1,q}^{(h_1)} \cdots E_{l_{r-1},q}^{(h_{r-1})} E_{l_r,q}^{(h_r)}(x). \end{aligned}$$

Note that, by (1), the generating function of (h,q)-Euler polynomials are

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy} e^{(x+y)t} d\mu_{-q}(y) \\ &= \frac{[2]_q}{q^{h+1} e^t + 1} e^{xt}. \end{aligned} \tag{16}$$

By replacing t by $e^t - 1$ in Definition 2.1,

$$\begin{aligned} &\sum_{n=0}^{\infty} Ch_{n,q}^{(h_1,\ldots,h_r)}(x) \frac{1}{n!} \left(e^t - 1\right)^n \\ =& \sum_{n=0}^{\infty} Ch_{n,q}^{(h_1,\ldots,h_r)}(x) \frac{1}{n!} n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!} \\ =& \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Ch_{m,q}^{(h_1,\ldots,h_r)}(x) S_2(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{17}$$

and, by (16),

$$\begin{aligned} &\prod_{i=1}^{r} \left(\frac{[2]_q}{q^{h_i+1} e^t + 1} \right) e^{xt} \\ =& \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} E_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} E_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \\ =& \sum_{n=0}^{\infty} \sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\ldots,l_r \geq 0}} \binom{n}{l_1,\ldots,l_r} E_{l_1,q}^{(h_1)} \cdots E_{l_{r-1},q}^{(h_{r-1})} E_{l_r,q}^{(h_r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

Thus, by (17) and (18), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\ldots,l_r \geq 0}} \binom{n}{l_1,\ldots,l_r} E_{l_1,q}^{(h_1)} \cdots E_{l_{r-1},q}^{(h_{r-1})} E_{l_r,q}^{(h_r)}(x) = \sum_{m=0}^{n} Ch_{m,q}^{(h_1,\ldots,h_r)}(x) S_2(n,m).$$

Let us define the (h,q)-*Changhee polynomials of the second kind* is defined as follows:

$$\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} (-x-y_1-\cdots-y_r)_n d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \tag{19}$$

where n is a nonnegative integer. In particular, $\widehat{Ch}_{n}^{(h_1,\ldots,h_r)}(0)=\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}$ are called the *nth* (h,q)*-Changhee numbers of the second kind.*

By (3) and (19), it leads to

(20)

$$\begin{aligned}
&\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}(-x-y_1-\cdots-y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}(-1)^n(x+y_1+\cdots+y_r)^{(n)} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\sum_{l=0}^{n}|S_1(n,l)|(-1)^n\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}E_{l_1,q}^{(h_1)}\cdots E_{l_r,q}^{(h_r)}E_{l_{r-1},q}^{(h_r)}(x).
\end{aligned}$$

Thus, we state the following theorem.

Theorem 2.4. *For* $n\geq 0$*, we have*

$$\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x)=\sum_{l=0}^{n}\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}|S_1(n,l)|(-1)^nE_{l_1,q}^{(h_1)}\cdots E_{l_r,q}^{(h_r)}E_{l_{r-1},q}^{(h_r)}(x).$$

Let us now consider the generating function of the (h,q)-Changhee polynomials of the second kind as follows:

(21)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_1}(-x-y_1-\cdots-y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!}\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}\binom{-x-y_1-\cdots-y_r}{n}t^n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}(1+t)^{-x-y_1-\cdots-y_r} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}+1+t}\right)(1+t)^{r-x}.
\end{aligned}$$

By replacing t by e^t-1, we have

$$\begin{aligned}
\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}e^{-t}+1}\right)e^{-xt}&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x)\frac{(e^t-1)^n}{n!}\\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{x^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x)S_2(n,m)\right)\frac{x^n}{n!},
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
&\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}e^{-t}+1}\right)e^{-xt}\\
&=\left(\prod_{i=1}^{r-1}\left(\sum_{n=0}^{\infty}(-1)^n E_{n,q}^{(h_i)}\frac{t^n}{n!}\right)\right)\left(\sum_{n=0}^{\infty}(-1)^n E_{n,q}^{(h_r)}(x)\frac{t^n}{n!}\right) \qquad (23)\\
&=\sum_{n=0}^{\infty}(-1)^n\left(\sum_{\substack{l_1+\cdots+l_r=n\\ l_1,\ldots,l_r\geq 0}}\binom{n}{i_1,\ldots,l_r}E_{l_1,q}^{(h_1)}\cdots E_{l_{r-1},q}^{(h_{r-1})}E_{l_r,q}^{(h_r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

By (22) and (23), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$\begin{aligned}
&\sum_{m=0}^{n}\widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x)S_2(n,m)\\
=&(-1)^n\sum_{l_1+\cdots+l_r=n}\binom{n}{i_1,\ldots,l_r}E_{l_1,q}^{(h_1)}\cdots E_{l_{r-1},q}^{(h_{r-1})}E_{l_r,q}^{(h_r)}(x).
\end{aligned}$$

By Theorem 2.3 and Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}Ch_{m,q}^{(h_1,\ldots,h_r)}(x)S_2(n,m)=(-1)^n\sum_{m=0}^{n}\widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x)S_2(n,m).$$

By Definition 2.1,

(24)

$$\begin{aligned}
&(-1)^n\frac{Ch_{n,q}^{(h_1,\ldots,h_r)}(x)}{n!}\\
=&(-1)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\binom{x+y_1+\cdots+y_r}{n}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\binom{-x-y_1-\cdots-y_r+n-1}{n}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\binom{-x-y_1-\cdots-y_r}{m}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
=&\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x)}{m!},
\end{aligned}$$

and

(25)

$$
\begin{aligned}
&(-1)^n \frac{\widehat{Ch}_{n,q}^{(h_1,\dots,h_r)}(x)}{n!} \\
=&(-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x-y_1-\cdots-y_r}{n} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\
=&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x+y_1+\cdots+y_r+n-1}{n} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\
=&\sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x+y_1+\cdots+y_r}{m} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\
=&\sum_{m=1}^{n} \binom{n-1}{m-1} \frac{Ch_{m,q}^{(h_1,\dots,h_r)}(x)}{m!}.
\end{aligned}
$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$(-1)^n \frac{Ch_{n,q}^{(h_1,\dots,h_r)}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{Ch}_{m,q}^{(h_1,\dots,h_r)}(x)}{m!},$$

and

$$(-1)^n \frac{\widehat{Ch}_{n,q}^{(h_1,\dots,h_r)}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{Ch_{m,q}^{(h_1,\dots,h_r)}(x)}{m!}.$$

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