

A NOTE ON DEGENERATE CENTRAL FACTORIAL POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In a recent work, the degenerate central factorial numbers of the second kind were introduced. In this paper, we study the degenerate central factorial polynomials of the second kind and give some identities for these polynomials associated with special numbers and polynomials.

1. Introduction

The falling factorial sequences are defined by

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1). \quad (1.1)$$

The Stirling number of the first kind is defined by the falling factorial sequence to be

$$(x)_n = \sum_{k=0}^n S_1(n, k)x^k, \quad (n \geq 0), \quad (\text{see}[1-10]). \quad (1.2)$$

It is well known that the Stirling number of the second kind is given by

$$x^n = \sum_{k=0}^n S_2(n, k)(x)_k, \quad (n \geq 0), \quad (\text{see}[9, 10]). \quad (1.3)$$

For $n \geq 0$, the central factorial is defined by

$$x^{[0]} = 1, \quad x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x - \frac{n}{2} + 1\right), \quad (n \geq 1). \quad (1.4)$$

As is known, the central factorial numbers of the second kind are defined by

$$x^n = \sum_{k=0}^n T(n, k)x^{[k]}, \quad (n \geq 0), \quad (\text{see}[1, 6, 7]). \quad (1.5)$$

From (1.5), we note that the generating function of $T(n, k)$ is given by

$$\frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!}, \quad (\text{see}[7]). \quad (1.6)$$

For $\lambda \in \mathbb{R}$, the λ -analogue of falling factorial sequences, $(x)_{n,\lambda}$, ($n \geq 0$), is defined by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x - \lambda) \cdots (x - (n-1)\lambda), \quad (n \geq 1). \quad (1.7)$$

In [6], the degenerate central factorial numbers of the second kind are defined by the generating function to be

$$\frac{1}{k!} \left((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}} \right)^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n, k) \frac{t^n}{n!}. \quad (1.8)$$

Note that $\lim_{\lambda \rightarrow 0} T_{2,\lambda}(n, k) = T(n, k)$, ($n, k \geq 0$).

In this paper, we study the degenerate central factorial polynomials of the second kind and give some identities for these polynomials associated with special numbers and polynomials.

2. Degenerate central factorial polynomials of the second kind

For $x \in \mathbb{R}$, we consider the degenerate central factorial polynomials of the second kind which are given by the generating function to be

$$\frac{1}{k!} \left((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}} \right)^k (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=k}^{\infty} T_{2,\lambda}(n, k|x) \frac{t^n}{n!}. \quad (2.1)$$

When $x = 0$, $T_{2,\lambda}(n, k) = T_{2,\lambda}(n, k|0)$ are the degenerate central factorial numbers of the second kind.

From (2.1), we easily get

$$\begin{aligned} \sum_{n=k}^{\infty} T_{2,\lambda}(n, k|x) \frac{t^n}{n!} &= \frac{1}{k!} \left((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}} \right)^k (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!} \right) \\ &= \sum_{n=k}^{\infty} \left(\sum_{l=k}^n \binom{n}{l} T_{2,\lambda}(l, k) (x)_{n-l,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

By comparing the coefficients on the both sides of (2.2), we obtain

$$T_{2,\lambda}(n, k|x) = \sum_{l=k}^n \binom{n}{l} T_{2,\lambda}(l, k) (x)_{n-l,\lambda}, \quad (2.3)$$

where $n, k \geq 0$.

For $r \in \mathbb{N}$, the r -central factorial numbers of the second kind are defined by the

generating function to be

$$\frac{1}{k!}(1+\lambda t)^{\frac{r}{\lambda}}((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^k = \sum_{n=k}^{\infty} T_{2,\lambda}^{(r)}(n+r, k+r) \frac{t^n}{n!}. \quad (2.4)$$

Then, by (2.4), we get

$$\begin{aligned} \sum_{n=k}^{\infty} T_{2,\lambda}^{(r)}(n+r, k+r) \frac{t^n}{n!} &= \frac{1}{k!} ((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^k (1+\lambda t)^{\frac{r}{\lambda}} \\ &= \left(\sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} r^m \lambda^{-m} \frac{1}{m!} (\log(1+\lambda t))^m \right) \\ &= \left(\sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} r^m \lambda^{-m} \sum_{i=m}^{\infty} S_1(i, m) \lambda^i \frac{t^i}{i!} \right) \\ &= \left(\sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \frac{t^l}{l!} \right) \left(\sum_{i=0}^{\infty} \left(\sum_{m=0}^i r^m \lambda^{i-m} S_1(i, m) \right) \frac{t^i}{i!} \right) \\ &= \sum_{n=k}^{\infty} \left(\sum_{l=k}^n \sum_{m=0}^{n-l} \binom{n}{l} r^m \lambda^{n-l-m} S_1(n-l, m) T_{2,\lambda}(l, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

Comparing the coefficients on the both sides of (2.5), we have

$$T_{2,\lambda}^{(r)}(n+r, k+r) = \sum_{l=k}^n \sum_{m=0}^{n-l} \binom{n}{l} r^m \lambda^{n-l-m} S_1(n-l, m) T_{2,\lambda}(l, k), \quad (2.6)$$

where $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$.

Now, we observe that

$$\begin{aligned} \frac{1}{k!}(1+\lambda t)^{\frac{r}{\lambda}}((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^k &= \frac{1}{k!} ((1+\lambda t)^{\frac{1}{\lambda}} - 1 + 1)^r ((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^k \\ &= \sum_{m=0}^{\infty} \binom{r}{m} \frac{1}{k!} (1+\lambda t)^{\frac{m}{2\lambda}} ((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^{m+k} \\ &= \sum_{m=0}^{\infty} \binom{r}{m} m! \binom{m+k}{m} (1+\lambda t)^{\frac{m}{2\lambda}} \frac{1}{(m+k)!} \\ &\quad \times ((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}})^{m+k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \binom{r}{m} m! \binom{m+k}{m} \sum_{n=m+k}^{\infty} T_{2,\lambda}(n, m+k | \frac{m}{2}) \frac{t^n}{n!} \\
&= \sum_{n=k}^{\infty} \left(\sum_{m=0}^{n-k} \binom{r}{m} m! \binom{m+k}{m} T_{2,\lambda}(n, m+k | \frac{m}{2}) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

Therefore, by comparing the coefficients on the both sides of (2.7), we get

$$T_{2,\lambda}^{(r)}(n+r, k+r) = \sum_{m=0}^{n-k} \binom{m+k}{m} \binom{r}{m} m! T_{2,\lambda}(n, m+k | \frac{m}{2}), \tag{2.8}$$

where $n, k \geq 0$ with $n \geq k$.

From (1.7), we have

$$\begin{aligned}
(1 + \lambda t)^{\frac{x+r}{\lambda}} &= (1 + \lambda t)^{\frac{r}{\lambda}} ((1 + \lambda t)^{\frac{1}{\lambda}} - 1 + 1)^x \\
&= (1 + \lambda t)^{\frac{r}{\lambda}} \sum_{k=0}^{\infty} (x)_k \frac{1}{k!} ((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}})^k \\
&\quad \times (1 + \lambda t)^{\frac{k}{2\lambda}} \\
&= \sum_{k=0}^{\infty} (x)_k \frac{1}{k!} ((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}})^k (1 + \lambda t)^{\frac{1}{\lambda}(\frac{k}{2}+r)} \\
&= \sum_{k=0}^{\infty} (x)_k \sum_{n=k}^{\infty} T_{2,\lambda}(n, k | \frac{k}{2} + r) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (x)_k T_{2,\lambda}(n, k | \frac{k}{2} + r) \right) \frac{t^n}{n!},
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
(1 + \lambda t)^{\frac{x+r}{\lambda}} &= \sum_{n=0}^{\infty} \binom{\frac{x+r}{\lambda}}{n} \lambda^n t^n = \sum_{n=0}^{\infty} \left(\frac{x+r}{\lambda} \right)_n \lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (x+r)_{n,\lambda} \frac{t^n}{n!}.
\end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10), we get

$$(x+r)_{n,\lambda} = \sum_{k=0}^n (x)_k T_{2,\lambda}(n, k | \frac{k}{2} + r), \tag{2.11}$$

where $n, k \geq 0$.

On the other hand,

$$\begin{aligned} (1 + \lambda t)^{\frac{x+r}{\lambda}} &= e^{\frac{x+r}{\lambda} \log(1+\lambda t)} = \sum_{k=0}^{\infty} \left(\frac{x+r}{\lambda}\right)^k \frac{1}{k!} (\log(1+t))^k \\ &= \sum_{k=0}^{\infty} \left(\frac{x+r}{\lambda}\right)^k \sum_{n=k}^{\infty} S_1(n, k) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \lambda^{n-k} (x+r)^k S_1(n, k)\right) \frac{t^n}{n!}. \end{aligned} \quad (2.12)$$

From (2.10) and (2.12), we note that

$$(x+r)_{n,\lambda} = \sum_{k=0}^n \lambda^{n-k} (x+r)^k S_1(n, k), \quad (2.13)$$

where $n, k \geq 0$.

From (2.1), we can derive the following equation:

$$\begin{aligned} \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k &= \sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \frac{1}{l!} \lambda^{-l} (e^{\lambda t} - 1)^l \\ &= \sum_{l=k}^{\infty} T_{2,\lambda}(l, k) \lambda^{-l} \sum_{n=l}^{\infty} S_2(n, l) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty} \left(\sum_{l=k}^n T_{2,\lambda}(l, k) S_2(n, l) \lambda^{n-l}\right) \frac{t^n}{n!}. \end{aligned} \quad (2.14)$$

By (1.6) and (2.14), we get

$$T(n, k) = \sum_{l=k}^n T_{2,\lambda}(l, k) S_2(n, l) \lambda^{n-l}, \quad (2.15)$$

where $n, k \geq 0$ with $n \geq k$.

From (2.4), we have

$$\begin{aligned} \sum_{n=k}^{\infty} T_{2,\lambda}^{(r)}(n+r, k+r) \frac{t^n}{n!} &= \frac{1}{k!} (1 + \lambda t)^{\frac{r}{\lambda}} \left((1 + \lambda t)^{\frac{1}{2\lambda}} - (1 + \lambda t)^{-\frac{1}{2\lambda}}\right)^k \\ &= \frac{1}{k!} (1 + \lambda t)^{\frac{1}{\lambda}(r-\frac{k}{2})} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} (1 + \lambda t)^{\frac{l}{\lambda}} \\ &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{\frac{1}{\lambda}(l+r-\frac{k}{2}) \log(1+\lambda t)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{\infty} \lambda^{-m} \left(l+r-\frac{k}{2}\right)^m \frac{1}{m!} (\log(1+\lambda t))^m \\
&= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{\infty} \lambda^{-m} \left(l+r-\frac{k}{2}\right)^m \sum_{n=m}^{\infty} S_1(n, m) \lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \lambda^{n-m} S_1(n, m) \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left(l+r-\frac{k}{2}\right)^m \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \lambda^{n-m} S_1(n, m) \left(\frac{1}{k!} \delta^k r^m\right) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

Comparing the coefficients on the both sides of (2.16), we obtain

$$T_{2,\lambda}^{(r)}(n+r, k+r) = \sum_{m=0}^n \lambda^{n-m} S_1(n, m) \left(\frac{1}{k!} \delta^k r^m\right), \tag{2.17}$$

where $r \in \mathbb{N}$, $n, k \in \mathbb{Z}$ with $n \geq k \geq 0$, and $\delta f(x) = f(x + \frac{1}{2}) - (x - \frac{1}{2})$. For $m, k \geq 0$, we have

$$\begin{aligned}
&\frac{1}{m!} (1+\lambda t)^{\frac{r}{\lambda}} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^m \frac{1}{k!} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k \\
&= \frac{1}{m!k!} (1+\lambda t)^{\frac{r}{\lambda}} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^{m+k} \\
&= \frac{(m+k)!}{m!k!} \frac{1}{(m+k)!} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^{m+k} (1+\lambda t)^{\frac{r}{\lambda}} \\
&= \binom{m+k}{m} \sum_{n=m+k}^{\infty} T_{2,\lambda}^{(r)}(n+r, m+k+r) \frac{t^n}{n!}.
\end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{m!} (1+\lambda t)^{\frac{r}{\lambda}} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^m \frac{1}{k!} \left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k \\
&= \left(\sum_{l=m}^{\infty} T_{2,\lambda}^{(r)}(l+r, m+r) \frac{t^l}{l!} \right) \left(\sum_{j=k}^{\infty} T_{2,\lambda}(j, k) \frac{t^j}{j!} \right) \\
&= \sum_{n=k+m}^{\infty} \left(\sum_{l=m}^{n-k} \binom{n}{l} T_{2,\lambda}^{(r)}(l+r, m+r) T_{2,\lambda}(n-l, k) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

Therefore, by (2.18) and (2.19), we get

$$\binom{m+k}{m} T_{2,\lambda}^{(r)}(n+r, m+k+r) = \sum_{l=m}^{n-k} \binom{n}{l} T_{2,\lambda}^{(r)}(l+r, m+r) T_{2,\lambda}(n-l, k). \quad (2.20)$$

References

1. P. L. Butzer, M. Schmidt, E. L. Stark, L. Vogt, *Central factorial numbers: their main properties and some applications*. Number. Funct. Anal. Optim. 10 (1989), no. 5-6, 419–488
2. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian number*, Utilitas Math. 15 (1979), 51-88.
3. L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions. Revised and enlarged edition*. D. Reidel Publishing Co., Dordrecht, 1974. xi+343 pp. ISBN:90-277-0441-4
4. T. Kim, D. S. Kim, *Degenerate Bernstein polynomials*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas RACSAM (2018). <https://doi.org/10.1007/s13398-018-0594-9>
5. T. Kim, D. S. Kim, *Degenerate Laplace transform and degenerate gamma function*. Russ. J. Math. Phys. 24 (2017), no. 2, 241-248
6. T. Kim, D. S. Kim, *Degenerate central factorial numbers of the second kind* (communicated)
7. T. Kim, *A note on central factorial numbers*, Proc. Jangjeon Math. Soc. 21(2019), no. 4, 575–588.
8. M. Merca, *Connections between central factorial numbers and Bernoulli polynomials*. Period. Math. Hungar. 73 (2016), no. 2, 259–264.
9. R. Riordan, *Combinatorial identities*. John Wiley & Sons, Inc., New York-London-Sydney 1968 xiii+256 pp.
10. S. Roman, *The umbral calculus*. Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. x +193 pp. ISBN:0-12-594380-6

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