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NOT NECESSARILY CONTINUOUS LOCALLY BOUNDED FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF CONNECTED LIE GROUPS

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Abstract. We obtain an explicit form for every (not necessarily continuous) locally bounded finite-dimensional representation of a connected Lie group in terms of representations of the radical and a Levi subgroup of the group.

§ 1. Introduction

The explicit form of continuous irreducible finite-dimensional representations of connected Lie groups in terms of the radical and a Levi subgroup is well known (see, e.g., [1], Ch. 8, § 7).

In the present note, we show that the same explicit form holds for the not necessarily continuous irreducible locally bounded finite-dimensional representations of connected Lie groups in terms of the radical and a Levi subgroup.

§ 2. Preliminaries

We begin with recalling a lemma used in [2] in the proof of a generalization of the well-known Lie theorem concerning continuous finite-dimensional representations of any solvable Lie group to the case of not necessary continuous representations (Lemma 2.1 of [2]). For the convenience of the reader, we present this lemma with a proof.

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Lemma. *Let G be a group, let N be a normal subgroup of G, let π be an irreducible representation of G in a finite-dimensional vector space E, and let there be a one-dimensional subspace $L \subset E$ invariant with respect to the restriction σ of the representation π to N. If there is a divisible subset $X \subset G/N$ generating G/N, then all operators of the representation σ are multiples of the identity operator on the space E.*

Proof. By assumption, the representation σ has a one-dimensional subrepresentation, and thus a common eigenvector $\xi \in E$. Thus, in particular,

$$\sigma(n)\xi = \lambda(n)\xi \quad \text{for any} \quad n \in N,$$

where $\lambda(n) \in \mathbb{C}^*$, $n \in N$. In this case,

$$\sigma(n)\pi(g)\xi = \pi(g)\sigma(g^{-1}ng)\xi = \lambda(g^{-1}ng)\pi(g)\xi, \qquad n \in N, \quad g \in G, \tag{1}$$

and therefore all vectors of the form $\pi(g)\xi$, $g \in G$, are common eigenvectors of the representation σ. All vectors of this kind span a finite-dimensional subspace $F \subset E$ (the linear span of all vectors of the form $\pi(g)\xi$, $g \in G$, in the subspace E) and, by the construction of the subspace F, this subspace has a basis of the eigenvectors of the representation σ. Certainly, every set of linearly independent vectors of the form $\sigma(g)\xi$, $g \in G$, is finite. It follows from formula (1) that the nonzero subspace F is invariant with respect to the representation π of the group G. By assumption, the representation π is irreducible, and thus $F = E$. Thus, the set of diverse functions of the form λ_g, $g \in G$, is finite, where

$$\lambda_g(n) = \lambda(g^{-1}ng), \qquad g \in G, \quad n \in N.$$

Since λ_g, $g \in G$, are complex-valued characters of the group N, these functions are invariant with respect to the inner automorphisms of the group N. Therefore, $\lambda_n = \lambda$, $n \in N$. Thus, the group G/N acts transitively on the (nonempty) finite set $\{\lambda_g \mid g \in G\}$ by permutations of this finite set and, if the number of elements in this finite set is equal to m, then the image of the group G/N is a subgroup of the symmetric group S_m, and therefore the order of this subgroup is a divisor of the number $m!$. By assumption, for any $x \in X$, there is an element $y \in X$ such that $y^{m!} = x$. Hence, the permutation corresponding to the element x is the $m!$th power of the permutation corresponding to the element y, and therefore this is the identity permutation.

Since X generates the group G/N by assumption, it follows that all permutations defining the transitive action of the group G/N on $\{\lambda_g \mid g \in G\}$ are identity permutations. Consequently, the set $\{\lambda_g \mid g \in G\}$ is a singleton, we have $\lambda_g = \lambda$ for any $g \in G$, and thus all operators of the representation σ are scalar multiples of the identity operator on E (operators of multiplication by a number, namely, $\sigma(n) = \lambda(n)1_E$, $n \in N$). Recall that $\lambda(gng^{-1}) = \lambda(n)$ for all $g \in G$ and $n \in N$.

§ 3. Main theorem

Let us cite the theorem claiming a generalization of the well-known Lie theorem concerning continuous finite-dimensional representations of any solvable Lie group to the case of not necessary continuous representations (see Theorem 2.1 of [2]).

Theorem 1. *Let G be a solvable group, and let every commutative quotient group H in the composition series of G be divisible. Let π be a representation of G in a finite-dimensional complex linear space E. If the representation π is irreducible, then the representation space E of π is one-dimensional.*

This makes it possible to prove our first main result.

Theorem 2. *Let G be a connected Lie group, let R be the radical of G, let L be a Levi subgroup of G, and let π be a (not necessarily continuous) irreducible locally bounded finite-dimensional representation of G in a space E. Then there is a (not necessarily continuous) character χ of R, satisfying the condition*

$$\chi(grg^{-1}) = \chi(r) \quad \text{for every} \quad g \in G, r \in R, \tag{2}$$

and an (automatically continuous) irreducible representation ρ of L on E such that

$$\pi(g) = \pi(lr) = \chi(r)\rho(l) \quad \text{for every} g = lr, l \in L, r \in R. \tag{3}$$

Proof. Let σ be the restriction of π to the closed normal subgroup R. The representation σ has an irreducible subrepresentation. By Theorem 1, this irreducible subrepresentation is one-dimensional, and hence there is a one-dimensional subspace $L \subset E$ invariant with respect to the restriction σ of the representation π to R. Since G/R is a Lie group, it is generated by a neighborhood $U \subset G/R$ of the identity element, and U may be assumed to be so small that the exponential mapping is a homeomorphism of the local preimage of U in the Lie algebra of G/R onto U. Therefore, U is a divisible generating set for G/R. Thus, all conditions of the above lemma are satisfied in our situation, and therefore, by the lemma, $\sigma(R)$ is the family of multiples of the identity operator defined by the character χ. It follows from the proof of the lemma that formula (2) holds, which completes the proof of the theorem.

§ 4. Concluding remarks

Thus, we have proved that formula (3) gives the general form of every (not necessarily continuous) locally bounded finite-dimensional representation of a connected Lie group. This enables us to prove the following corollary.

Corollary. *A connected Lie group G has a (not necessarily continuous) finite-dimensional irreducible unitary representation of dimension exceeding one if and only if the Levi subgroup L of G contains a nontrivial compact normal subgroup.*

Proof. Let us apply Theorem 2. The representation ρ of L can be simultaneously irreducible, non-one-dimensional, and unitary if and only if the list of simple factors of the group L contains a nontrivial simple compact group, because noncompact simple Lie groups have no nontrivial finite-dimensional unitary representations.

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