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ON I-CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND MODULUS FUNCTIONS

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Abstract. In this paper we introduce and study I-convergent double sequence spaces ${}_2S_0^I(F,p)$, ${}_2S^I(F,p)$ and ${}_2S_\infty^I(F,p)$ with the help of compact operator T on the real space $\mathfrak{R}$ and a sequence of modulus functions $F=(f_{ij})$. We investigate some topological and algebraic properties, and also prove some inclusion relations on these spaces.

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1. Introduction

The initial works on double sequences is found in Bromwich [3]. Later on, it was studied by Hardy [4], Moricz [13], Moricz and Rhoades [14], Başarir and Sonalcan [2] and many others. Hardy [4] introduced the notion of regular convergence for double sequences. Mursaleen and Edely [15] have recently introduced the statistical convergence in double sequence spaces. The notion of ideal convergence in double sequences was introduced by Tripathy and Tripathy [23].

Throughout the paper $\mathbb{N}$, $\mathfrak{R}$ and $\mathbb{C}$ denote the sets of positive integers, real numbers and complex numbers, respectively. A complex double sequence is a function x from $\mathbb{N}\times\mathbb{N}$ into $\mathbb{C}$ and briefly denoted by $\{x_{ij}\}$ and we denote all double sequence spaces with ${}_2\omega$. By the convergence of a double sequence we mean the convergence in Pringsheim sense i.e., a double sequence $x=(x_{ij})$ has Pringsheim limit L (denoted by $P-\lim x=L$) provided that given $\epsilon>0$ there exists $n\in\mathbb{N}$ such that $|x_{ij}-L|<\epsilon$ whenever $i,j>n$ [17]. The double sequence $x=(x_{ij})$ is said to be bounded if there exists a positive number K such that $|x_{ij}|<K$ for all i and j.

Let X and Y be two normed linear spaces. An operator $T:X\to Y$ is said to be a compact linear operator (or completely continuous linear operator), if

(i) T is linear.
(ii) T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operators $C(X,Y)$ is a closed subspace of $B(X,Y)$ and $C(X,Y)$ is a Banach space if Y is a Banach space. Throughout the paper we denote by ${}_2\ell_\infty$, ${}_2c$ and ${}_2c_0$ the Banach spaces of bounded, convergent and null double sequences of reals respectively, with the norm

$$\|x\| = \sup_{ij \in \mathbb{N}} |x_{ij}| \, .$$

The idea of modulus was structured by Nakano in 1953 [16].
A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(iii) f is non decreasing, and
(iv) f is continuous from the right at zero.

Ruckle [18-20] used the idea of a modulus function f to construct the sequence space

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

This space is an FK -space, and Ruckle [20] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Ruckle [18-20] further proved that, for any modulus f, $X(f) \subset \ell_1$ and $X(f)^\alpha \subset \ell_\infty$, where

$$X(f)^\alpha = \left\{ y = (y_k) : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty \right\}.$$

The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$ [20]. Later on Kolk [8, 9] gave an extension of $X(f)$ by considering a sequence of modulus $F = f_k$ and defined the sequence space

$$X(F) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f_k(|x_k|) \in X \right\}.$$

The following well known inequality will be used throughout the article. Let $p = (p_{ij})$ be any sequence of positive real numbers with $0 \leq p_{ij} \leq \sup_{ij} p_{ij} = H$, $D = \max\{1, 2^{X-1}\}$ then

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq D \left(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}} \right) \tag{1}$$

for all $a_{ij}, b_{ij} \in \mathbb{C}$ and $(i, j) \in \mathbb{N} \times \mathbb{N}$. Also $|a|^{p_i} \leq \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

In the next section we give some basic definitions that are used throughout the paper.

2. Definitions and preliminaries

Definition 2.1 Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of all X) is called an ideal in X if

(i) $\phi \in I$,
(ii) I is additive i.e., $A, B \in I \Rightarrow A \cup B \in I$,
(iii) I is hereditary i.e., $A \in I$ and $B \subset A \Rightarrow B \in I$.

A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if $\Phi \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if each $I \neq \Phi$ and $X \notin I$. Evidently $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial $J \neq I$ containing I as a subset. For each ideal I there is a filter $\mathfrak{J}(I)$ corresponding to I i.e.,

$$\mathfrak{J}(I) = \{K \subseteq I : K^C \in I \quad \text{where} \quad K^C = \mathbb{N} - K\}.$$

Definition 2.2 A double sequence $(x) = (x_{ij}) \in_2 \omega$ is said to be I-convergent to a number L (denoted by $I - \lim x = L$) if for every $\epsilon > 0$, we have

$$\{i, j \in \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

Definition 2.3 A double sequence $(x_{ij}) \in_2 \omega$ is said to be I-null if number $L = 0$. In this case we write $I - \lim x = 0$.

Definition 2.4 A double sequence $(x_{ij}) \in_2 \omega$ is said to be I-Cauchy if for every $\epsilon > 0$, there exist numbers $m = m(\epsilon)$, $n = n(\epsilon)$ such that

$$\{i, j \in \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I.$$

Definition 2.5 A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies that $(\alpha_{ij} x_{ij}) \in E$ for all sequences of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $i, j \in \mathbb{N}$.

Definition 2.6 A double sequence space E is said to be symmetric if $(x_{\pi(ij)}) \in E$ whenever $(x_{ij}) \in E$ where (π_i) and (π_j) is a permutation on $\mathbb{N}$.

Definition 2.7 A double sequence space E is said to be a sequence algebra if $(x_{ij} \cdot y_{ij}) \in E$ whenever $(x_{ij}) \in E$, $(y_{ij}) \in E$.

Definition 2.8 A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 2.9 Let $K = \{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 \ldots \text{ and } k_1 < k_2 < k_3 \ldots\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space. A K-step of E is a sequence space

$$\lambda_K^E = \left\{x = (x_{n_i k_j}) \in_2 \omega : (x_{ij}) \in E\right\}.$$

Definition 2.10 A canonical pre-image of a sequence $(x_{n_i k_j}) \in E$ is a sequence $(b_{nk}) \in E$ defined as follows:

$$b_{nk} = \begin{cases} x_{nk} & \text{for } n, k \in K \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.11 A double sequence space E is said to be monotone if it contains the canonical preimages of its step spaces.

The notion of ideal convergence (I-convergence) was first introduced by Kostyrko et al. [10] as a generalization of statistical convergence of sequences in a metric space and studied some properties of such convergence. Since then many researchers have studied these subjects and obtained various interesting results using ideal convergence see ([11, 5,6, 21]).

We use the following lemmas for proving some results of this paper.

Lemma 2.1 (24, 25)**.** *Let E be a sequence space. If E is solid then E is monotone.*

Lemma 2.2 (25)**.** *Let $K \notin \mathfrak{J}(I)$, and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 2.3 (10, Lemma 5.1)**.** *If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.*

Following Basar and Altay [1], Malkowsky [12] and Sengonul [22], Khan et. al. [7] introduced the sequence spaces $S^I(f)$, $S_0^I(f)$ and $S_\infty^I(f)$ as follows:

$$S^I(f) = \{x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\left(|T(x_k) - L| \geq \epsilon\right) \in I\}, \text{ for some } L \in \mathbb{C}\},$$

$$S_0^I(f) = \{x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\left(|T(x_k)| \geq \epsilon\right) \in I\}\},$$

$$S_\infty^I(f) = \{x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : \exists K > 0 \; f\left(|T(x_k)| \geq K\right) \in I\}\}.$$

3. Construction of new double sequence spaces

This section brings to limelight new I-convergent double sequence spaces with the help of compact operator T and a sequence of modulus functions $F = (f_{ij})$.

Let $F = (f_{ij})$ be a sequence of modulus functions and $p = (p_{ij})$ be a sequence of positive real numbers we introduce the following sequence spaces:

$$_2S^I(F,p) = \{x = (x_{ij}) \in_2 \omega : \{f_{ij}\left(|T(x_{ij}) - L|\right)^{p_{ij}} \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{C}\},$$

$$_2S_0^I(F,p) = \{x = (x_{ij}) \in_2 \omega : \{f_{ij}\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I\},$$

$$_2S_\infty^I(F,p) = \{x = (x_{ij}) \in_2 \omega : \exists K > 0 : \{f_{ij}\left(|T(x_{ij})|\right)^{p_{ij}} \geq K\} \in I\},$$

$$_2S_\infty(F,p) = \left\{x = (x_{ij}) \in_2 \omega : \left\{\sup_{ij} f_{ij}\left(|T(x_{ij})|\right)^{p_{ij}} < \infty\right\}\right\}.$$

We also denote by

$$_2\mathfrak{M}_S^I(F,p) =_2 S_\infty^I(F,p) \cap_2 S^I(F,p)$$

and

$$_2\mathfrak{M}_{S_0}^I(F,p) =_2 S_\infty^I(F,p) \cap_2 S_0^I(F,p)$$

We now examine some topological properties and establish some inclusion relations on these new spaces.

Theorem 3.1. *For any sequence of modulus functions $F = (f_{ij})$ and let X denote any of the spaces ${}_2S_0^I(F,p)$, ${}_2S^I(F,p)$, ${}_2\mathfrak{M}_S^I(F,p)$ and ${}_2\mathfrak{M}_{S_0}^I(F,p)$, then X is a linear space.*

Proof. We prove the assertion only for ${}_2S^I(F,p)$, the others can be proved similarly. Let $x = (x_{ij}), y = (y_{ij}) \in_2 S^I(F,p)$ and let α, β be scalars. Then there exists positive numbers $\epsilon > 0$ such that

$$(2)\quad \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij}) - L_1|\right)^{p_{ij}} \geq \frac{\epsilon}{2},\ \text{ for some }\ L_1 \in \mathbb{C}\right\} \in I,$$

and

$$(3)\quad \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij}) - L_2|\right)^{p_{ij}} \geq \frac{\epsilon}{2},\ \text{ for some }\ L_2 \in \mathbb{C}\right\} \in I.$$

Since $F = (f_{ij})$ is a modulus function, then from (1) we have,

$$\begin{aligned} F\left(|T(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)|\right)^{p_{ij}} &= F\left(|T(\alpha x_{ij} - \alpha L_1) + (\beta y_{ij} - \beta L_2)|\right)^{p_{ij}} \\ &\leq D\left(M_\alpha\right)^H F\left(|T(x_{ij} - L_1)|\right)^{p_{ij}} + D\left(M_\beta\right)^H F\left(|T(y_{ij} - L_2)|\right)^{p_{ij}} \end{aligned}$$

where M_α and M_β are positive integers such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq M_\beta$. From the above inequality, we get

$$\begin{aligned} &\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)|\right)^{p_{ij}} \geq \epsilon\} \\ &\qquad \subseteq \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(\alpha x_{ij} - \alpha L_1)|\right)^{p_{ij}} \geq \frac{\epsilon}{2D\left(M_\alpha\right)^H}\right\} \\ &\qquad \bigcup \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(\beta y_{ij} - \beta L_2)|\right)^{p_{ij}} \geq \frac{\epsilon}{2D\left(M_\beta\right)^H}\right\}. \end{aligned}$$

By using (2) and (3) the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)\right)^{p_{ij}} \geq \epsilon\} \in I.$$

This completes the proof. □

Theorem 3.2. *A sequence $x = (x_{ij}) \in_2 \mathfrak{M}_S^I(F,p)$ is I-convergent if and only if for every $\epsilon > 0$ there exists $N_\epsilon, M_\epsilon \in \mathbb{N}$ such that*

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij}\left(|T(x_{ij}) - T(x_{N_\epsilon, M_\epsilon})|\right)^{p_{ij}} < \epsilon\} \in\ {}_2\mathfrak{M}_S^I(F,p).$$

Proof. Suppose that $L = I - \lim x$. Then we have

$$A_\epsilon = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij}\left(|T(x_{ij}) - L|\right)^{p_{ij}} < \frac{\epsilon}{2}\right\} \in\ {}_2\mathfrak{M}_S^I(F,p)\quad \text{for all } \epsilon > 0.$$

Next fix $N_\epsilon, M_\epsilon \in A_\epsilon$, then we have

$$\begin{aligned} f_{ij}\left(|T(x_{ij}) - T(x_{N_\epsilon, M_\epsilon})|\right)^{p_{ij}} &\leq f_{ij}\left(|T(x_{ij}) - L|\right)^{p_{ij}} + f_{ij}\left(|T(x_{N_\epsilon, M_\epsilon}) - L|\right)^{p_{ij}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

for all $i, j \in A_\epsilon$.
Hence $\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij}\left(|T(x_{ij}) - T(x_{N_\epsilon, M_\epsilon})|\right)^{p_{ij}} < \epsilon\} \in\ {}_2\mathfrak{M}_S^I(F,p)$.
Conversely suppose that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_{ij}\left(|T(x_{ij}) - T(x_{N_\epsilon, M_\epsilon})|\right)^{p_{ij}} < \epsilon\} \in\ {}_2\mathfrak{M}_S^I(F,p),$$

that is $\{(i,j)\in\mathbb{N}\times\mathbb{N}:(|T(x_{ij})-T(x_{N_\epsilon,M_\epsilon})|)^{p_{ij}}<\epsilon\}\in\ {}_2\mathfrak{M}_S^I(F,p)$ for all $\epsilon>0$. Then the set

$B_\epsilon=\{(i,j)\in\mathbb{N}\times\mathbb{N}:T(x_{ij})\in[T(x_{N_\epsilon,M_\epsilon})-\epsilon\,,\,T(x_{N_\epsilon,M_\epsilon})+\epsilon]\}\in_2\mathfrak{M}_S^I(F,p)$.

for all $\epsilon>0$. Let $R_\epsilon=[T(x_{N_\epsilon,M_\epsilon})-\epsilon\,,\,T(x_{N_\epsilon,M_\epsilon})+\epsilon]$. If we fix $\epsilon>0$, then we have $B_\epsilon\in\ {}_2\mathfrak{M}_S^I(F,p)$ as well as $B_{\frac{\epsilon}{2}}\in\ {}_2\mathfrak{M}_S^I(F,p)$. Hence $B_\epsilon\cap B_{\frac{\epsilon}{2}}\in\ {}_2\mathfrak{M}_S^I(F,p)$ which implies that $R=R_\epsilon\cap R_{\frac{\epsilon}{2}}\neq\phi$, that is $\{(i,j)\in\mathbb{N}\times\mathbb{N}:T(x_{ij})\in R\}\in\ {}_2\mathfrak{M}_S^I(F,p)$ that is diam $R\leq$ diam $R_{\frac{\epsilon}{2}}$, where diam R denotes the length of interval R.
In this way by principal of induction we found the sequence of closed intervals $R_\epsilon=I_0\supseteq I_1\supseteq\ldots\supseteq I_{ij}\supseteq\ldots$ with the property that
diam $I_{ij}\leq\frac{1}{2}$diam I_{i-1j-1} for $(i,j=1,2,3,\ldots)$ and
$\{(i,j)\in\mathbb{N}\times\mathbb{N}:T(x_{ij})\in I_{ij}\}\in\ {}_2\mathfrak{M}_S^I(F,p),\ (i,j=1,2,3,\ldots)$.
Then there exists a interval $\xi\in\bigcap I_{ij}$ where $(i,j)\in\mathbb{N}\times\mathbb{N}$ such that
$\xi=I-\lim T(x_{ij})$ so that $F(\xi)=I-\lim F\,[T(x_{ij})]$ therefore $L=I-\lim F\,[T(x_{ij})]$.
This completes the proof of the theorem. □

Theorem 3.3. *If $F=(f_{ij})$ is a sequence of modulus function, then the inclusion ${}_2S_0^I(F,p)\subset\ {}_2S^I(F,p)\subset\ {}_2S_\infty^I(F,p)$ holds.*

Proof. The inclusion $S_0^I(F,p)\subset\ {}_2S^I(F,p)$ is obvious. Now let
$x=(x_{ij})\in\ {}_2S^I(F,p)$, then there exists $L\in\mathbb{C}$ such that

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}:F\,(|T(x_{ij})-L|)^{p_{ij}}\geq\epsilon\}\in I.$$

Now ,we have

$$\begin{aligned}F\,(|T(x_{ij})|)^{p_{ij}}&=F\,(|T(x_{ij})-L+L|)^{p_{ij}}\\&\leq D\left\{\frac{1}{2}F\,(|T(x_{ij})-L|)^{p_{ij}}+\frac{1}{2}F\,(|L|)^{p_{ij}}\right\}\\&\leq D\left\{\frac{1}{2}F\,(|T(x_{ij})-L|)^{p_{ij}}+\max\left(1,\left[\frac{1}{2}F\,(|L|)\right]^H\right)\right\}.\end{aligned}$$

Taking supremum over i,j on both sides, we get $x=(x_{ij})\in\ {}_2S_\infty^I(F,p)$.
Hence ${}_2S_0^I(F,p)\subset\ {}_2S^I(F,p)\subset\ {}_2S_\infty^I(F,p)$. □

Theorem 3.4. *Let F,G be sequences of modulus functions satisfying Δ_2-condition , then*

(i) $X(G,p)\subseteq X(F\circ G,p)$.
(ii) $X(F,p)\cap X(G,p)\subseteq X(F+G,p)$.

for $X={}_2S^I$, ${}_2S_0^I$, ${}_2\mathfrak{M}_{S_0}^I$ and ${}_2\mathfrak{M}_S^I$.

Proof. Let $x=(x_{ij})\in_2S_0^I(G,p)$, then there exists $\rho>0$ such that

(4) $$\{(i,j)\in\mathbb{N}\times\mathbb{N}:G\,(|T(x_{ij})|)^{p_{ij}}\geq\epsilon\}\in I$$

Let $\epsilon>0$ and choose $0<\delta<1$, such that $F(t)<\epsilon$ for $0\leq t\leq\delta$.
Put $y_{ij}=G\,(|T(x_{ij})|)$ and consider

(5) $$\lim_{i,j}\,[F(y_{ij})]^{p_{ij}}=\lim_{y_{ij}\leq\delta,i,j\in\mathbb{N}}[F(y_{ij})]^{p_{ij}}+\lim_{y_{ij}>\delta,i,j\in\mathbb{N}}[F(y_{ij})]^{p_{ij}}\,.$$

Since F is a sequence of modulus functions so we have $F(\lambda x) \leq \lambda F(x)$, $0 < \lambda < 1$.
Therefore we have

$$\lim_{y_{ij} \leq \delta, i,j \in \mathbb{N}} [F(y_{ij}]^{p_{ij}} = [F(2)]^H + \lim_{y_{ij} \leq \delta, i,j \in \mathbb{N}} [(y_{ij})]^{p_{ij}} . \tag{6}$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now since F is non-decreasing it follows that,

$$F(y_{ij}) < F\left(1 + \frac{y_{ij}}{\delta}\right) < \frac{1}{2}F(2) + \frac{1}{2}F\left(\frac{2y_{ij}}{\delta}\right). \tag{7}$$

Again, since F satisfies Δ_2-condition, we have

$$F(y_{ij}) < \frac{1}{2}K\left(\frac{y_{ij}}{\delta}\right)F(2) + \frac{1}{2}KF\left(\frac{2y_{ij}}{\delta}\right) \tag{8}$$

$$\frac{1}{2}K\left(\frac{y_{ij}}{\delta}\right)F(2) + \frac{1}{2}K\left(\frac{y_{ij}}{\delta}\right)F(2) \tag{9}$$

$$= K\left(\frac{y_{ij}}{\delta}\right)F(2). \tag{10}$$

Hence, we have

$$\lim_{y_{ij} > \delta, i,j \in \mathbb{N}} [F(y_{ij}]^{p_{ij}} \leq \max\left\{1, \left(k\delta^{-1}F(2)\right)^G\right\} \lim_{y_{ij} > \delta, i,j \in \mathbb{N}} [(y_{ij}]^{p_{ij}} . \tag{11}$$

Therefore from (4), (5) and (11) it follows that

$$\{(x_{ij}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(G\left(|T(x_{ij})|\right)^{p_{ij}}\right) \geq \epsilon\} \in I\} . \tag{12}$$

Hence $X(G,p) \subseteq X(F \circ G, p)$.
(ii) Let $x = (x_{ij}) \in {}_2S_0^I(F,p) \cap {}_2S_0^I(G,p)$. Let $\epsilon > 0$ be given, then we have

$$\{(x_{ij}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I\}$$

and

$$\{(x_{ij}) : \{(i,j) \in \mathbb{N} \times \mathbb{N} : G\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I\} .$$

Therefore, the inclusions

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : (F+G)\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\}$$
$$\subseteq [\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \cup \{(i,j) \in \mathbb{N} \times \mathbb{N} : G\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\}] \in I$$

implies that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : (F+G)\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I.$$

Thus $x = (x_{ij}) \in {}_2S_0^I(F+G,p)$.
For $X = {}_2S^I(F,p)$, ${}_2S_0^I(F,p)$, ${}_2\mathfrak{M}_{S_0}^I(F,p)$ and ${}_2\mathfrak{M}_S^I(F,p)$ the inclusion is similar. □

Corollary 3.5. $(X,p) \subseteq X(F,p)$ *for* $X = {}_2S^I$, ${}_2S_0^I$, ${}_2\mathfrak{M}_{S_0}^I$ *and* ${}_2\mathfrak{M}_S^I$.

Theorem 3.6. *For any sequence of modulus functions* $F = (f_{ij})$ *the spaces* ${}_2S_0^I(F,p)$ *and* ${}_2\mathfrak{M}_{S_0}^I(F,p)$ *are solid and monotone.*

Proof. We shall prove the theorem for ${}_2S_0^I(F,p)$. Let $x = (x_{ij}) \in_2 S_0^I(F,p)$ then there exists $\epsilon > 0$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I. \tag{13}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| < 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, then the result follows from (13) and the following inequality

$$F\left(|T(\alpha_{ij}x_{ij})|\right)^{p_{ij}} \leq |\alpha_{ij}| F\left(|T(x_{ij})|\right)^{p_{ij}} \leq F\left(|T(x_{ij})|\right)^{p_{ij}}$$

for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.
The space is monotone follows from Lemma 2.1 □

Theorem 3.7. *The spaces ${}_2S^I(F,p)$ and ${}_2\mathfrak{M}_S^I(F,p)$ are neither solid nor monotone in general.*

Proof. The proof of this theorem follows from the following example. Let $I = I_f$, $F(x) = x^2$, $p = (p_{ij}) = 1$ for all $x = (x_{ij}) \in [0, \infty)$ and T be an identity operator on $\mathfrak{R}$. Consider the K-step space of $X_K(F)$ of $X(F)$ defined as follows:
Let $x = (x_{ij}) \in X(F)$ and $y = (y_{ij}) \in X_K(F)$ be such that

$$y_{ij} = \begin{cases} x_{ij}, & i+j \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij} \in_2 S^I(F,p)$, but its K-step space pre image does not belong to ${}_2S^I(F,p)$. Thus ${}_2S^I(F,p)$ is not monotone and hence not solid by Lemma 2.1. □

Theorem 3.8. *The spaces ${}_2S^I(F,p)$ and ${}_2S_0^I(F,p)$ are sequence algebras.*

Proof. We prove the result for ${}_2S_0^I(F,p)$. For the space ${}_2S^I(F,p)$ the result can be proved similarly.
Let $x = (x_{ij})$ and $y = (y_{ij})$ be in ${}_2S_0^I(F,p)$ then, we have

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I,$$

and

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(y_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I.$$

Therefore,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : F\left(|T(x_{ij})T(y_{ij})|\right)^{p_{ij}} \geq \epsilon\} \in I.$$

Thus $(x_{ij} \cdot y_{ij}) \in {}_2S_0^I(F,p)$.
Hence ${}_2S_0^I(F,p)$ is a sequence algebra. □

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