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CONTINUITY CONDITIONS FOR LOCALLY BOUNDED FINITE-DIMENSIONAL REPRESENTATIONS OF ALMOST CONNECTED LOCALLY COMPACT GROUPS

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Abstract. We obtain a criterion for the continuity of the restriction of a locally bounded finite-dimensional representation of an almost connected locally compact group to the commutator subgroup of the group.

§ 1. Introduction

Continuity conditions for locally bounded finite-dimensional representations of connected locally compact groups were found in [1–4]. These results make it possible to obtain continuity conditions for locally bounded finite-dimensional representations of almost connected locally compact groups.

§ 2. Preliminaries

Recall the Dong Hoon Lee's supplement theorem (Theorem 2.13 of [5]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0 is compact) admits a totally disconnected compact subgroup D such that $G = G_0 D$.

We also recall the following notion.

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Lemma 1. *Let G be an almost connected locally compact group, and let $\mathcal{N}$ be the family of compact normal subgroups $N \neq \{e\}$ such that G/N is a* (*not necessarily connected*) *Lie group. Then $\mathcal{N}$ is a nontrivial filter basis convergent to $\{e\}$.*

Proof. This follows immediately from the Gleason–Montgomery–Zippin–Yamabe theorem.

Definition 1. Let G be an almost connected locally compact group and let π be a locally bounded finite-dimensional representation of G in a normed vector space E. The set

$$\mathrm{FDG}(\pi) = \bigcap_{N \in \mathcal{N}} \overline{\pi(N)}$$

where $\overline{\pi(N)}$ stands for the closure of the π-image $\pi(N)$ of N, is called the *final discontinuity group* of π and is denoted by $\mathrm{FDG}(\pi)$.

Lemma 2. *Let G be an almost connected locally compact group and let π be a locally bounded finite-dimensional representation of G in a normed vector space E. Then* $\mathrm{FDG}(\pi)$ *is a compact normal subgroup of the closure $\overline{\pi(G)}$ of the π-image $\pi(G)$ of G in the algebra of linear operators on the space E.*

Moreover, the following conditions are equivalent:

(1) $\mathrm{FDG}(\pi) = \{1_E\}$;

(2) $\mathrm{FDG}(\pi)$ *is contained in a ball in the space of linear operators on E* (*equipped with the standard operator norm*) *centered at 1_E and of radius less than $\sqrt{3}$.*

Proof. The proof is just like that of the similar assertion for totally disconnected groups (see Lemma 2 in [6]).

§ 3. Main results

Theorem. *Let G be an almost connected locally compact group and let π be a locally bounded finite-dimensional representation of G in a normed vector space E. If* $\mathrm{FDG}(\pi) = \{1_E\}$, *then the representation π is continuous on the commutator subgroup of G.*

Proof. Let G' be the commutator subgroup of G. Since G' and the component of the identity element, G_0, are normal subgroups of G, it follows that $G' \cap G_0$ is a normal subgroup of G containing the commutator subgroup $G_0' = [G_0, G_0]$ of G_0. Since $\mathrm{FDG}(\pi) = \{1_E\}$, it follows that there is an

$N \in \mathcal{N}$ for which the π-image $\pi(N)$ is contained in a ball centered at 1_E and with a radius less than $\sqrt{3}$, which contains no nontrivial subgroups of the group of invertible operators in E, and therefore $\pi(N) = 1$, which implies that π can be viewed as a (not necessarily continuous) representation of some (not necessarily connected) Lie group $M = G/N$.

Since D (see the preliminaries) is compact and totally disconnected, it follows that the image $d = D/(N \cap D)$ of D in G/N is finite. Therefore, $G' \cap G_0$ is generated by a finite group d and the commutator subgroup G_0' of G_0.

As is known, the restriction of π to G_0' is continuous (see, e.g., [1]), and, because of the above generation, the restriction of π to $(G' \cap G_0)/N$ is also continuous.

Since G'/N is a finite extension of $(G' \cap G_0)/N$, it follows that the restriction of π (regarded as a representation of G/N) to the commutator subgroup $(G/N)'$ of G/N is continuous on $(G/N)'$.

Finally, since the image G'/N of the commutator subgroup G' of G is obviously contained in the commutator subgroup $(G/N)'$ of G/N, we see immediately that the original representation π is continuous on G', as was to be proved.

§ 4. Comments

Since every locally compact group contains an open almost connected subgroup. the above theorem enables us to pose the following question.

Question. How one can describe the class of locally compact groups G all of whose locally bounded finite-dimensional representations π (in normed linear spaces E_π) for which there is an open almost compact subgroup $O \subset G$ such that $\mathrm{FDG}(O) = 1_{E_\pi}$ have continuous restriction to the commutator subgroup G' of G?

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